Abstracts

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Residues on Affine Grassmannians

(joint work with Mathieu Florence)

The compact Lie groups play an essential role in the theory of Lie groups and it makes sense to generalize the notion of compacity for a smooth affine group G over a base field k, that is a closed k-subgroup of some $\mathrm{G}L_n$ (e.g. the orthogonal group of a quadratic form). We consider the fourth following candidates.

(I) (rank one subgroups) G does not carry any k–subgroup isomorphic to the additive group $\mathbb{G}_a$ nor the multiplicative group $\mathbb{G}_m$;

(II) (Boundedness property) $G\big(k((t))\big)$ is bounded for the valuation topology.

(III) $G\big(k[[t]]\big) = G\big(k((t))\big)$;

(IV) (No point at infinity) There exists a (projective) compactification X of G such that $G(k) = X(k)$.

We have the easy implications (IV) $\Longrightarrow$ (III) $\Longrightarrow$ (II) $\Longrightarrow$ (I). If k is a perfect field and G is smooth, Borel and Tits have shown in 1965 the implication (I) $\Longrightarrow$ (IV) so that all conditions agree [1, th. 8.2]. Furthermore in the case of the real numbers (and for p-adic fields), this is equivalent to say that the group $G(k)$ of points is compact (*ibid*, 9.3). For unipotent subgroups over imperfect fields, the equivalence (I) $\Longleftrightarrow$ (III) is due to J. Tits, see [3, Appendice B.2].

For k imperfect and G reductive, we have that (I) $\Longrightarrow$ (II) according to a result of Bruhat-Tits-Rousseau, we refer to Prasad's elementary proof [7]; actually (IV) holds as well by using nice compactifications of G starting with the wonderful compactification in the adjoint case.

The next step is Gabber's talk [6] in Oberwolfach in 2012. Using the theory of pseudo-reductive groups, Gabber proved (among other things) that the four conditions are equivalent in the general case. The main result of today generalizes (partly) Gabber's statement over rings in a quite elementary manner.

Theorem 1 [5]**.** Let A be a ring (commutative, unital) and let G be a closed A–subgroup scheme of $\mathrm{SL}_{N,A}$ for some N. Then the following are equivalent:

(I) $\mathrm{Hom}_{A-gp}(\mathbb{G}_a, G) = 1$ and $\mathrm{Hom}_{A-gp}(\mathbb{G}_m, G) = 1$;

(III) $G\big(A[[t]]\big) = G\big(A((t))\big)$ where $A((t)) = A[[t]][\frac{1}{t}]$.

We call that property *wound* (ployé in French). The proof goes by associating to an element $g \in G\big(A((t))\big) \setminus G\big(A[[t]]\big)$ its residue $\mathrm{res}(g) : \mathbb{G}_a \to G$ or $\mathbb{G}_m \to G$ which is a non-trivial group homomorphism. The techniques involved apply also to G-torsors. The second main result is the following.

Theorem 2 [5]**.** Let G be an affine algebraic k–group over a field k. Let X be a G–torsor. If $X\big(k((t))\big) \neq \emptyset$, then $X(k) \neq \emptyset$.

For reductive groups, this statement is due to Bruhat-Tits. The generalization of that statement over a ring is known for GL_n and for tori according to recent results by Bouthier-Česnavičius [2, 2.1.17, 3.1.7]; we generalize it as well for wound closed subgroup schemes of SL_N and for G commutative under further assumptions [5, 4.2,4.3]. It is an open question beyond those cases.

Already over a field it is an open question whether the statement does generalize to homogeneous spaces; this is the case in characteristic 0 according to results by M. Florence [4].

Finally, if G is split reductive, the coset $G(k((t)))/G(k[[t]])$ is described by the k–points of the affine grassmannian $\mathcal{Q}_G$ [8]. This permits to show that an element $g \in G(k((t))) \setminus G(k[[t]])$ is of rank zero iff g is of the shape $g = g_1\mu(t)g_2$ for $g_1, g_2 \in G(k[[t]])$ and $\mu : \mathbb{G}_m \to G$ a homomorphism.

References

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