

INTRODUCTION TO REDUCTIVE GROUP SCHEMES OVER RINGS

In construction

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1. INTRODUCTION

The theory of reductive group schemes is due to Demazure and Grothendieck and was achieved fifty years ago in the seminar SGA 3, see also Demazure's thesis [De]. Roughly speaking it is the theory of reductive groups in family focusing to subgroups and classification issues. It occurs in several areas: representation theory, model theory, automorphic forms, arithmetic groups and buildings, infinite dimensional Lie theory, ...

The story started as follows. Demazure asked Serre whether there is a good reason for the map $\mathrm{SL}_n(\mathbb{Z}) \rightarrow \mathrm{SL}_n(\mathbb{Z}/d\mathbb{Z})$ to be surjective for all $d > 0$. Serre answered it is a question for Grothendieck ... Grothendieck answered it is not the right question !

The right question was the development of a theory of reductive groups over schemes and especially the classification of the “split” ones. The general underlying statement is now that the specialization map $G(\mathbb{Z}) \rightarrow G(\mathbb{Z}/d\mathbb{Z})$ is onto for each semisimple group split (or Chevalley) simply connected scheme G/\mathbb{Z} . It is a special case of strong approximation.

Demazure-Grothendieck's theory assume known the theory of reductive groups over an algebraically closed field due mainly to C. Chevalley ([Ch], see also [Bo], [Sp]) and we will do the same. In the meantime, Borel-Tits achieved the theory of reductive groups over an arbitrary field [BT65] and Tits classified the semisimple groups [Til]. In the general setting, Borel-Tits theory extends to the case of a local base.

Let us warn the reader by pointing out that we do not plan to prove all hard theorems of the theory, for example the unicity and existence theorem of split reductive groups. Our purpose is more to take the user viewpoint by explaining how such results permit to analyse and classify algebraic structures.

It is not possible to enter into that theory without some background on affine group schemes and strong technical tools of algebraic geometry (descent, Grothendieck topologies,...). Up to improve afterwards certain results, half of the lectures avoid descent theory and general schemes.

The aim of the notes is to try to help people attending the lectures. It is very far to be self-contained and quotes a lot in several references starting with [SGA3], Demazure-Gabriel's book [DG], and also the material of the Luminy's summer school provided by Brochard [Br], Conrad [C] and Oesterlé [O].

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Affine group schemes I

We shall work over a base ring R (commutative and unital).

2. SORITES

2.1. R-Functors. We denote by $\mathcal{A}ff_R$ the category of affine R -schemes. We are interested in R -functors, i.e. covariant functors from $\mathcal{A}ff_R$ to the category of sets. If \mathfrak{X} an R -scheme, it defines a covariant R -functor

$$h_{\mathfrak{X}} : \mathcal{A}ff_R \rightarrow \mathbf{Sets}, \quad S \mapsto \mathfrak{X}(S).$$

Given a map $f : \mathfrak{Y} \rightarrow \mathfrak{X}$ of R -schemes, there is a natural morphism of functors $f_* : h_{\mathfrak{Y}} \rightarrow h_{\mathfrak{X}}$ of R -functors.

We recall now Yoneda's lemma in our setting. Let F be an R -functor. If $\mathfrak{X} = \mathrm{Spec}(R[\mathfrak{X}])$ is an affine R -scheme and $\zeta \in F(R[\mathfrak{X}])$, we define a morphism of R -functors

$$\phi(\zeta) : h_{\mathfrak{X}} \rightarrow F$$

by $\phi(\zeta)(S) : h_{\mathfrak{X}}(S) = \mathrm{Hom}_R(R[\mathfrak{X}], S) \rightarrow F(S)$, $x \mapsto F(f_x)(\zeta)$ for each R -ring S where $f_x \in \mathrm{Hom}_R(R[\mathfrak{X}], S)$ is the evaluation function at x .

2.1.1. Lemma. (*Yoneda lemma*)

(1) *The assignment $\zeta \rightarrow \phi(\zeta)$ induces a bijection*

$$F(R[\mathfrak{X}]) \xrightarrow{\sim} \mathrm{Hom}_{R\text{-}func}(h_{\mathfrak{X}}, F).$$

(2) *Let \mathfrak{Y} be an R -scheme. Then we have*

$$\mathrm{Hom}_{R\text{-}sch}(\mathfrak{X}, \mathfrak{Y}) = h_{\mathfrak{Y}}(R[\mathfrak{X}]) \xrightarrow{\sim} \mathrm{Hom}_{R\text{-}func}(h_{\mathfrak{X}}, h_{\mathfrak{Y}}).$$

Proof. (1) The strategy is to construct the inverse map. We are given $\alpha \in \mathrm{Hom}_{R\text{-}func}(h_{\mathfrak{X}}, F)$, it gives rise to a map $\alpha_{R[\mathfrak{X}]} : h_{\mathfrak{X}}(R[\mathfrak{X}]) \rightarrow F(R[\mathfrak{X}])$ so that the universal point $x^{univ} \in h_{\mathfrak{X}}(R[\mathfrak{X}]) = \mathrm{Hom}_R(R[\mathfrak{X}], R[\mathfrak{X}])$ defines an element $\psi(\alpha) = \alpha_{R[\mathfrak{X}]}(id_{R[\mathfrak{X}]}) \in F(R[\mathfrak{X}])$ or for short $\alpha(id_{R[\mathfrak{X}]})$.

Step 1: $\psi \circ \phi = id_{F(R[\mathfrak{X}])}$. Let $\zeta \in F(R[\mathfrak{X}])$. We apply $\phi(\zeta)_{R[\mathfrak{X}]} : h_{\mathfrak{X}}(R[\mathfrak{X}]) \rightarrow F(R[\mathfrak{X}])$ to $R[\mathfrak{X}]$ and obtain $\psi(\phi(\zeta)) = F(id_{R[\mathfrak{X}]}) (\zeta) = \zeta$.

Step 2: $\phi \circ \psi = id_{\mathrm{Hom}_{R\text{-}func}(h_{\mathfrak{X}}, F)}$. Let $\alpha \in \mathrm{Hom}_{R\text{-}func}(h_{\mathfrak{X}}, F)$. Then $\psi(\alpha) = \alpha_{R[\mathfrak{X}]}(id_{R[\mathfrak{X}]}) \in F(R[\mathfrak{X}])$ and we consider the element $\eta = \phi(\psi(\alpha)) \in \mathrm{Hom}_R(h_{\mathfrak{X}}, F)$ defined as follows. For each $f_x \in \mathrm{Hom}_R(R[\mathfrak{X}], S)$, $\eta(S) : h_{\mathfrak{X}}(S) \rightarrow F(S)$ applies f_x to

$$F(f_x)(\psi(\alpha)) = F(f_x)(\alpha_{R[\mathfrak{X}]}(id_{R[\mathfrak{X}]}) = \alpha(f_x \circ id_{R[\mathfrak{X}]}) = \alpha(f_x)$$

where we used the functorial property in the second equality. Thus $\phi \circ \psi = id_{\mathrm{Hom}_{R\text{-}func}(h_{\mathfrak{X}}, F)}$.

(2) We apply (1) to $F = h_{\mathfrak{Y}}$. □

2.1.2. Remarks. (a) The formula $F(f_x)(\psi(\alpha)) = \alpha(f_x)$ arising in the proof expresses the fact that an R -functor $h_X \rightarrow F$ is determined by its value on the universal point of X .

(b) For more on the Yoneda lemma, see [Wa, §1.2], [GW, §4.2] or [Vi, §2.1]. Part (2) holds then for general R -schemes.

An R -functor F is representable by an R scheme (resp. an affine R -scheme) if there exists an R -scheme \mathfrak{X} (resp. an affine R -scheme \mathfrak{X}) together with an isomorphism of functors $h_X \rightarrow F$. We say that \mathfrak{X} represents F .

If \mathfrak{X} is affine, the isomorphism $h_X \rightarrow F$ comes from an element $\zeta \in F(R[\mathfrak{X}])$ which is called the universal element of $F(R[\mathfrak{X}])$. The pair (\mathfrak{X}, ζ) satisfies the following universal property:

For each affine R -scheme \mathfrak{T} and for each $\eta \in F(R[\mathfrak{T}])$, there exists a unique morphism $u : \mathfrak{T} \rightarrow \mathfrak{X}$ such that $F(u^*)(\zeta) = \eta$.

Given a morphism of rings $j : R \rightarrow R'$, an R -functor F defines by restriction an R' -functor denoted by j_*F or $F_{R'}$. If $F = h_{\mathfrak{X}}$ for an affine R -scheme \mathfrak{X} , we have $F_{R'} = h_{\mathfrak{X} \times_R R'}$.

2.1.3. Examples. We will see later more non representable R -functors.

(a) The empty R -functor is not representable by an affine R -scheme (and not actually by any R -scheme). Denote by F the empty functor and assume that $h_{\mathfrak{X}} \cong F$ for an R -scheme \mathfrak{X} . Then $id_{\mathfrak{X}} \in h_{\mathfrak{X}}(R[\mathfrak{X}])$ contradicting the fact that F is the empty R -functor.

(b) We consider the R -functor $F(S) = S^{(\mathbb{N})}$ and claim that it is not representable by an affine R -scheme. Assume that $h_{\mathfrak{X}} \cong F$ so that $\text{Hom}_R(R[\mathfrak{X}], R[\mathfrak{X}]) \cong R[\mathfrak{X}]^{(\mathbb{N})}$. Then the image of $id_{R[\mathfrak{X}]}$ has bounded support d so that $F(S) \subset S^d \subset S^{(\mathbb{N})}$ for each R -ring S . This is a contradiction.

2.1.4. Remark. We denote by $F_0(S) = \{\bullet\}$ for each R -ring S . Let F be an R -functor. Then there is a canonical map $F \rightarrow F_0$; in other words F_0 is a terminal object of the category of R -functors.

2.2. Monomorphisms. The fibered product of R -functors is defined as follows. For $\alpha_1 : F_1 \rightarrow E$ and $\alpha_2 : F_2 \rightarrow E$ two morphisms of R -functors, we set $(F_1 \times_E F_2)(S) = F_1(S) \times_{E(S)} F_2(S)$ for each R -ring S .

2.2.1. Lemma. *Let $\alpha : F \rightarrow E$ be a morphism of R -functors. The following conditions are equivalent:*

- (i) α is a monomorphism;
- (ii) the diagonal $\Delta : F \rightarrow F \times_E F$ is an isomorphism;
- (iii) $F(S) \rightarrow E(S)$ is injective for each R -ring S .

Proof. (i) \implies (ii). We consider the projections $p_i : F \times_E F \rightarrow F$ for $i = 1, 2$. Since $\alpha \circ p_1 = \alpha \circ p_2$, we obtain that $p_1 = p_2$. Thus p_1 is an isomorphism and so is Δ .

(ii) \implies (i). We are given $\beta_1, \beta_2 : G \rightarrow F$ be morphisms of R -functors such that $\alpha \circ \beta_1 = \alpha \circ \beta_2$. This defines a map $\beta : G \rightarrow F \times_E F \xleftarrow{\sim} F$, so that $\beta_1 = \beta_2$.

(iii) \implies (ii). For each R -ring S , we have $F(S) \xrightarrow{\sim} F(S) \times_{E(S)} F(S)$ so that Δ is an isomorphism of R -functors.

(ii) \implies (iii). Obvious. □

We consider now the case of schemes.

2.2.2. Lemma. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of R -schemes. The following conditions are equivalent:*

- (i) *f is a monomorphism;*
- (i') *The R -functor $h_f : h_{\mathfrak{X}} \rightarrow h_{\mathfrak{Y}}$ is a monomorphism;*
- (ii) *the diagonal $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is an isomorphism;*
- (iii) *$F(S) \rightarrow E(S)$ is injective for each R -ring S .*

Proof. The proof of the implications (i) \iff (ii) \implies (iii) is similar with the previous lemma. The implication (iii) \implies (ii) Lemma 2.2.1, (iii) \implies (i) yields the implication (iii) \implies (i').

It remains to establish the implication (i') \implies (ii). Lemma 2.2.1, (i) \implies (ii) shows that the diagonal $h_{\mathfrak{X}} \rightarrow h_{\mathfrak{X}} \times_{h_{\mathfrak{Y}}} h_{\mathfrak{X}}$ is an isomorphism of R -functors. Let \mathfrak{Z} be an R -scheme, we need to establish that the diagonal map $\mathfrak{X}(\mathfrak{Z}) \rightarrow \mathfrak{X}(\mathfrak{Z}) \times_{\mathfrak{Y}(\mathfrak{Z})} \mathfrak{X}(\mathfrak{Z})$ is an isomorphism. If \mathfrak{Z} is affine over R it is true. Let $g, h \in \mathfrak{X}(\mathfrak{Z})$ mapping to the same element of $\mathfrak{Y}(\mathfrak{Z})$.

We consider then an affine cover $(\mathfrak{U}_i)_{i \in I}$ of \mathfrak{Z} so that the restrictions $g_i : \mathfrak{U}_i \subset \mathfrak{Z} \rightarrow \mathfrak{X}$ $h_i : \mathfrak{U}_i \subset \mathfrak{Z} \rightarrow \mathfrak{X}$ define a unique element $f_i \in \mathfrak{X}(\mathfrak{U}_i)$. Since the diagonal is split by the first projection, f_i and f_j agree on $\mathfrak{U}_i \cap \mathfrak{U}_j$ so that define $f : \mathfrak{Z} \rightarrow \mathfrak{X}$. Then $f = g = h$ and we are done. □

2.2.3. Remark. The equivalence (i) \iff (ii) in (1) holds in any category with fiber products, see [St, Tag 01L3].

We consider now the epimorphisms of R -functors. If $\alpha : F \rightarrow E$ satisfies that $F(S) \rightarrow E(S)$ is surjective for each R -ring S , we claim that α is an epimorphism.

Let $\gamma_1, \gamma_2 : E \rightarrow D$ be morphisms of R -functors such that $\gamma_1 \circ \alpha = \gamma_2 \circ \alpha$. Then $\gamma_1 : E(S) \rightarrow D(S)$ agrees with $\gamma_2 : E(S) \rightarrow D(S)$ for each R -ring S so that $\beta_1 = \beta_2$. Thus α is an epimorphism.

It can be shown by using coproducts that the epimorphisms are all of that shape, see [KS, §2, Ex. 2.4, 2.23] or [SGA3, §I.1.4]; those references put also the monomorphism case in a much wider setting.

In the category of R -schemes, we have to pay attention that there are epimorphisms whose associated functor is not surjective, see [GW, Ex. 8.2.(d)]

for the construction of a bunch of epimorphisms. A concrete example is with $k = \mathbb{R}$ and the morphism $u : \mathfrak{X} = \text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{R}) = \mathfrak{Y}$.

Let \mathfrak{Z} be an \mathbb{R} -scheme and let $f_1, f_2 : \mathfrak{Y} \rightarrow \mathfrak{Z}$ such that $f_1 \circ u = f_2 \circ u$. In other words we have two points $z_1, z_2 \in \mathfrak{Z}(\mathbb{R})$ which coincide as complex points. Since $\mathfrak{Z}(\mathbb{R})$ injects in $\mathfrak{Z}(\mathbb{C})$, it follows that $z_1 = z_2$ so that u is an epimorphism. The fact that $\mathfrak{Z}(\mathbb{R})$ injects in $\mathfrak{Z}(\mathbb{C})$ reduces to an affine scheme $\text{Spec}(A)$ for which we have $\text{Hom}_{\mathbb{R}}(A, \mathbb{R}) \subset \text{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, \mathbb{C}) = \text{Hom}_{\mathbb{R}}(A, \mathbb{C})$.

2.3. Zariski sheaves. We say that an R -functor F is a Zariski sheaf if it satisfies the following requirements:

(A) for each R -ring S and each decomposition $1 = f_1 + \cdots + f_n$ in S , then

$$F(S) \xrightarrow{\sim} \left\{ (\alpha_i) \in \prod_{i=1, \dots, n} F(S_{f_i}) \mid (\alpha_i)_{S_{f_i f_j}} = (\alpha_j)_{S_{f_i f_j}} \text{ for } i, j = 1, \dots, n \right\}.$$

(B) $F(0) = \{\bullet\}$.

2.3.1. Lemma. *Let F be an R -functor F which a Zariski sheaf. Then F is additive, i.e. the map $F(S_1 \times S_2) \rightarrow F(S_1) \times F(S_2)$ is bijective for each pair (S_1, S_2) of R -algebras.*

Proof. We are given an R -ring $S = S_1 \times S_2$; we write it $S = S_1 \times S_2 = Se_1 + Se_2$ where e_1, e_2 are idempotents satisfying $e_1 + e_2 = 1$, we have $S_1 = Se_1$, $S_2 = Se_2$ and $Se_1 e_2 = 0$ [St, Tag 00ED]. Then

$$F(S) \xrightarrow{\sim} \{(\alpha_1, \alpha_2) \in F(S_1) \times F(S_2) \mid \alpha_{1,0} = \alpha_{2,0} \in F(0)\}.$$

Since $F(0) = \{\bullet\}$, we conclude that $F(S) = F(S_1) \times F(S_2)$. \square

Representable R -functors are clearly Zariski sheaves. In particular, to be a Zariski sheaf is a necessary condition for an R -functor to be representable.

2.3.2. Lemma. *Let $1 = f_1 + \cdots + f_n$. Let F be an R -functor which is a Zariski sheaf and such that $F_{R_{f_i}}$ is representable by an affine R_{f_i} -scheme for $i = 1, \dots, n$. Then F is representable by an affine R -scheme.*

Proof. Let \mathfrak{X}_i be an R_{f_i} -scheme together with an isomorphism $\zeta_i : h_{\mathfrak{X}_i} \xrightarrow{\sim} F_{R_{f_i}}$ of R_{f_i} -functors for $i = 1, \dots, n$. Then for $i \neq j$, $F_{R_{f_i f_j}}$ is represented by $\mathfrak{X}_i \times_{R_{f_i}} R_{f_i f_j}$ and $\mathfrak{X}_j \times_{R_{f_j}} R_{f_i f_j}$. More precisely, the isomorphism $\zeta_{j, R_{f_i f_j}}^{-1} \circ \zeta_{i, R_{f_i f_j}} : h_{\mathfrak{X}_i \times_{R_{f_i}} R_{f_i f_j}} \xrightarrow{\sim} h_{\mathfrak{X}_j \times_{R_{f_j}} R_{f_i f_j}}$ defines an isomorphism $u_{i,j} : \mathfrak{X}_i \times_{R_{f_i}} R_{f_i f_j} \xrightarrow{\sim} \mathfrak{X}_j \times_{R_{f_j}} R_{f_i f_j}$ and we have compatibilities $u_{i,j} \circ u_{j,k} = u_{i,k}$ once restricted to $R_{f_i f_j f_k}$. It follows that the \mathfrak{X}_i 's glue in an affine R -scheme \mathfrak{X} . Also the map ζ_i^{-1} glue in an R -map $F \rightarrow h_{\mathfrak{X}}$. Since F is a Zariski sheaf, we conclude that $F \xrightarrow{\sim} h_{\mathfrak{X}}$. \square

2.4. Functors in groups.

2.5. Definition. An R -group scheme \mathfrak{G} is a group object in the category of R -schemes. It means that \mathfrak{G}/R is an affine scheme equipped with a section $\epsilon : \text{Spec}(R) \rightarrow \mathfrak{G}$, an inverse $\sigma : \mathfrak{G} \rightarrow \mathfrak{G}$ and a multiplication $m : \mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}$ such that the three following diagrams commute:

Associativity:

$$\begin{array}{ccc}
 (\mathfrak{G} \times_R \mathfrak{G}) \times_R \mathfrak{G} & \xrightarrow{m \times id} & \mathfrak{G} \times_R \mathfrak{G} \\
 \downarrow \wr \text{can} & & \searrow m \\
 \mathfrak{G} \times_R (\mathfrak{G} \times_R \mathfrak{G}) & \xrightarrow{id \times m} & \mathfrak{G} \times_R \mathfrak{G} \\
 & & \nearrow m \\
 & & \mathfrak{G}
 \end{array}$$

Unit:

$$\begin{array}{ccccc}
 \mathfrak{G} \times_R \text{Spec}(R) & \xrightarrow{id \times \epsilon} & \mathfrak{G} \times_R \mathfrak{G} & \xleftarrow{\epsilon \times id} & \text{Spec}(R) \times_R \mathfrak{G} \\
 & \searrow \text{can} & & \nearrow \text{can} & \\
 & \cong & \mathfrak{G} & \cong &
 \end{array}$$

Symmetry:

$$\begin{array}{ccc}
 \mathfrak{G} \times_R \mathfrak{G} & \xrightarrow{id \times \sigma} & \mathfrak{G} \times_R \mathfrak{G} \\
 \downarrow & & \downarrow m \\
 \text{Spec}(R) & \xrightarrow{\epsilon} & \mathfrak{G}.
 \end{array}$$

We say that \mathfrak{G} is commutative if furthermore the following diagram commutes

$$\begin{array}{ccc}
 & \mathfrak{G} \times_R \mathfrak{G} & \\
 m \swarrow & \downarrow \text{switch} & \searrow m \\
 \mathfrak{G} & & \mathfrak{G} \times_R \mathfrak{G}
 \end{array}$$

We will mostly work with affine R -group schemes, that is, when \mathfrak{G} is an affine R -group scheme.

Let $R[\mathfrak{G}]$ be the coordinate ring of \mathfrak{G} . We call $\epsilon^* : R[\mathfrak{G}] \rightarrow R$ the counit (augmentation), $\sigma^* : R[\mathfrak{G}] \rightarrow R[\mathfrak{G}]$ the coinverse (antipode), and denote by $\Delta = m^* : R[\mathfrak{G}] \rightarrow R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ the comultiplication. By means of the

dictionary affine schemes/rings, they satisfy the following commutativity rules:

Co-associativity:

$$\begin{array}{ccc}
 & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] & \xrightarrow{id \otimes \Delta} R[\mathfrak{G}] \otimes_R (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]) \\
 R[\mathfrak{G}] & \begin{array}{l} \nearrow \Delta \\ \searrow \Delta \end{array} & \\
 & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] & \xrightarrow{\Delta \otimes id} (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]) \otimes_R R[\mathfrak{G}].
 \end{array}$$

$\downarrow \wr$

Counit: The following composite maps are $id_{R[\mathfrak{G}]}$

$$R[\mathfrak{G}] \xrightarrow{\Delta} R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \xrightarrow{id \otimes \epsilon} R[\mathfrak{G}] \otimes_R R \xrightarrow{\sim} R[\mathfrak{G}]$$

$$R[\mathfrak{G}] \xrightarrow{\Delta} R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \xrightarrow{\epsilon \otimes id} R[\mathfrak{G}] \otimes_R R \xrightarrow{\sim} R[\mathfrak{G}].$$

Cosymmetry:

$$\begin{array}{ccc}
 & R[\mathfrak{G}] \xrightarrow{\Delta} R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] & \\
 & \swarrow \epsilon^* & \downarrow id \times \sigma^* \\
 R & & R[\mathfrak{G}] \xleftarrow{product} R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]
 \end{array}$$

In other words, $(R[\mathfrak{G}], m^*, \sigma^*, \epsilon^*)$ is a commutative Hopf R -algebra¹. Given an affine R -scheme \mathfrak{X} , there is then a one to one correspondence between group structures on \mathfrak{X} and commutative R -algebra structures on $R[\mathfrak{X}]$.

Also \mathfrak{G} is commutative if and only if the following diagram commutes

$$\begin{array}{ccc}
 & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] & \\
 & \nearrow \Delta & \downarrow switch \\
 R[\mathfrak{G}] & & \\
 & \searrow \Delta & \\
 & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] &
 \end{array}$$

¹This is Waterhouse definition [Wa, §I.4], other people talk about cocommutative coassociative Hopf algebra.

If \mathfrak{G}/R is an (affine) R -group scheme, then for each R -algebra S the abstract group $\mathfrak{G}(S)$ is equipped with a natural group structure. The multiplication is $m(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \rightarrow \mathfrak{G}(S)$, the unit element is $1_S = (\epsilon \times_R S) \in \mathfrak{G}(S)$ and the inverse is $\sigma(S) : \mathfrak{G}(S) \rightarrow \mathfrak{G}(S)$. It means that the functor $h_{\mathfrak{G}}$ is actually a group functor.

2.5.1. Lemma. *Let \mathfrak{X}/R be an affine scheme. Then the Yoneda lemma induces a one to one correspondence between group structures on \mathfrak{X} and group structures on $h_{\mathfrak{X}}$.*

In other words, defining a group law on \mathfrak{X} is the same that to define compatible group laws on each $\mathfrak{G}(S)$ for S running over the R -algebras.

Proof. This is an immediate consequence of Yoneda's lemma. We assume that the R -functor h_X is equipped with a group structure. The Yoneda lemma shows that this group structure arises in an unique way of an affine R -group scheme structure. \square

2.5.2. Remark. We shall encounter certain non-affine group R -schemes. A group scheme \mathfrak{G}/R is a group object in the category of R -schemes. More generally the previous lemma holds for a non affine R -group scheme.

3. EXAMPLES

3.1. Constant group schemes. Let I be a set and consider the R -scheme $I_R = \bigsqcup_{\gamma \in I} \text{Spec}(R) = \bigsqcup_{\gamma \in I} U_i$. We claim that its functor of points h_{I_R} identifies with

$$\left\{ \text{locally constant functions } \text{Spec}(S)_{\text{top}} \rightarrow I \right\}.$$

To see this let S be an R -ring and let $f \in h_{I_R}(S) = \text{Hom}_{\text{Spec}(R)}(\text{Spec}(S), I_R)$. By pulling back the open cover (U_i) of I_R , we obtain a decomposition $S = \bigsqcup_{\gamma \in I} S_i$ in open subschemes of R . This defines a locally constant function $\text{Spec}(S)_{\text{top}} \rightarrow I$ having the value i on each S_i (for more details see [GW, Ex. 4.43] or [St, Tag 03YW]).

Next let Γ be an abstract group. We consider the R -scheme $\Gamma_R = \bigsqcup_{\gamma \in \Gamma} \text{Spec}(R)$. Its functor of points h_{Γ_R} identifies with

$$\left\{ \text{locally constant functions } \text{Spec}(S)_{\text{top}} \rightarrow \Gamma \right\}.$$

The group structure on Γ induces an R -group scheme structure on Γ_R . If R is non zero, this group scheme is affine and only if Γ is finite.

3.2. Vector groups. Let N be an R -module. We consider the commutative group functors

$$\begin{aligned} V_N : \mathcal{A}ff_R &\rightarrow Ab, \quad S \mapsto \text{Hom}_S(N \otimes_R S, S) = (N \otimes_R S)^\vee, \\ W_N : \mathcal{A}ff_R &\rightarrow Ab, \quad S \mapsto N \otimes_R S. \end{aligned}$$

3.2.1. Lemma. *The R -group functor V_N is representable by the affine R -scheme $\mathfrak{V}(N) = \operatorname{Spec}(S^*(N))$ which is then a commutative R -group scheme. Furthermore if the R -module N is of finite presentation then the R -scheme $\mathfrak{V}(N)$ is of finite presentation.*

Proof. It follows readily of the universal property of the symmetric algebra $\operatorname{Hom}_{R'\text{-mod}}(N \otimes_R R', R') \xleftarrow{\sim} \operatorname{Hom}_{R\text{-mod}}(N, R') \xrightarrow{\sim} \operatorname{Hom}_{R\text{-alg}}(S^*(N), R')$ for each R -algebra R' .

We assume that the R -module N is finitely presented, that is, there exists an exact sequence $0 \rightarrow M \rightarrow R^n \rightarrow N \rightarrow 0$ where M is a finitely generated R -module. According to [St, Tag 00DO] the kernel I of the surjective map $S^*(R^n) \rightarrow S^*(N)$ is generated by M (seen in degree one) so is a finitely generated $S^*(R^n)$ -module. Since $S^*(R^n) = R[t_1, \dots, t_n]$, we conclude that the R -algebra $S^*(N)$ is of finite presentation. \square

3.2.2. Remark. The converse of the last assertion holds as well by using the limit characterizations of the finite presentation property, see [St, Tags 0G8P, 0Q0Q].

The commutative group scheme $\mathfrak{V}(N)$ is called the vector group-scheme associated to N . We note that $N = \mathfrak{V}(N)(R)$. In the special case $N = R^d$, this is nothing but the affine space \mathbf{A}_R^d of relative dimension d .

Its group law on the R -group scheme $\mathfrak{V}(N)$ is given by $m^* : S^*(N) \rightarrow S^*(N) \otimes_R S^*(N)$, applying each $X \in N$ to $X \otimes 1 + 1 \otimes X$. The cosymmetry is $\sigma^* : S^*(N) \rightarrow S^*(N)$, $X \mapsto -X$ and the counit is the augmentation map $S^*(N) \rightarrow R$.

If $N = R$, we get the affine line over R . Given a map $f : N \rightarrow N'$ of R -modules, there is a natural map $f^* : \mathfrak{V}(N') \rightarrow \mathfrak{V}(N)$ of R -group schemes.

3.2.3. Lemma. *The assignment $N \rightarrow \mathfrak{V}(N)$ is a faithful contravariant (essentially surjective) functor from the category of R -modules and that of vector group R -schemes.*

Proof. Since this functor is essentially surjective, it is enough to show that it is faithful. Given two R -modules N, N' we want to show that the morphism

$$\operatorname{Hom}_R(N, N') \rightarrow \operatorname{Hom}_{R\text{-gp}}(\mathfrak{V}(N'), \mathfrak{V}(N)), \quad f \mapsto f^*$$

is injective. This is clear since $f_* : S^*(N) \rightarrow S^*(N')$ is a graded morphism and applies N to N' by f . \square

3.2.4. Remark. Let k be a field of characteristic $p > 0$ and consider the Frobenius morphism $\mathbb{G}_{a,k} \rightarrow \mathbb{G}_{a,k}$, $x \mapsto x^p$. It is a k -group homomorphism but is linear. This shows that the functor above is not fully faithful and then not an anti-equivalence of categories. For obtaining an anti-equivalence of categories, we need to restrict the morphisms to linear morphisms, see [SGA3, I.4.6.2].

We consider also the R -functor $W(N)$ defined by $W(N)(S) = N \otimes_R S$. The assignment $N \rightarrow W(N)$ is an equivalence of categories from the category of R -modules and that of functors W with linear maps. Together with Lemma 3.2.3, it follows that there is an anti-equivalence of categories between the category of functors W with linear maps and the category of vector R -group schemes.

If N is projective and finitely generated, we have $W(N) = V(N^\vee)$ so that the R -functor $W(N)$ is representable by an affine group scheme. In this case we denote by $\mathfrak{W}(N)$ the associated R -group scheme.

3.2.5. Theorem. *The R -functor $W(N)$ is representable if and only if N is projective and finitely generated.*

If R is noetherian, this is due to [Ni04]. The general case has been handled by Romagny [Ro, Thm. 5.4.5]. Note that it is coherent with the example 2.1.3.(b).

3.3. Group of invertible elements, linear groups. Let A/R be an algebra (unital, associative). We consider the R -functor

$$S \mapsto \mathrm{GL}_1(A)(S) = (A \otimes_R S)^\times.$$

3.3.1. Lemma. *If A/R is finitely generated projective, then $\mathrm{GL}_1(A)$ is representable by an affine group scheme. Furthermore, $\mathrm{GL}_1(A)$ is of finite presentation.*

Proof. Up to localize for the Zariski topology (Lemma 2.3.2), we can assume that A is a free R -module of rank d .

We shall use the norm map $N : A \rightarrow R$ defined by $a \mapsto \det(L_a)$ where $L_a : A \rightarrow A$ is the R -endomorphism of A defined by the left translation by a . We have $A^\times = N^{-1}(R^\times)$ since the inverse of L_a can be written L_b by using the characteristic polynomial of L_a . More precisely, let $P_a(X) = X^d - \mathrm{Tr}(L_a)X^{d-1} + \cdots + (-1)^{d-1}c_{d-1}(L_a)X + (-1)^d \det(L_a) \in R[X]$ be the characteristic polynomial of L_a ; according to the Cayley-Hamilton theorem we have $P_a(L_a) = 0$ [Bbk1, III, §11] so that $L_{P_a(a)} = 0$ and $P_a(a) = 0$. If $\det(L_a) \in R^\times$, it follows that

$$a \left(a^{d-1} - \mathrm{Tr}(L_a)a^{d-2} + \cdots + (-1)^{d-1}c_{d-1}(L_a)a \right) = (-1)^{d+1} \det(A)$$

so that $ab = ba = 1$ with $b = (-1)^{d+1} \det(A)^{-1} \left(a^{d-1} - \mathrm{Tr}(L_a)a^{d-2} + (-1)^{d-1}c_{d-1}(L_a)a \right)$.

The same is true after tensoring by S , so that

$$\mathrm{GL}_1(A)(S) = \left\{ a \in (A \otimes_R S) = \mathfrak{W}(A)(S) \mid N(a) \in S^\times \right\}.$$

We conclude that $\mathrm{GL}_1(A)$ is representable by the fibered product

$$\begin{array}{ccc} \mathfrak{G} & \longrightarrow & \mathfrak{W}(A) \\ \downarrow & & \downarrow N \\ \mathbb{G}_{m,R} & \longrightarrow & \mathfrak{W}(R). \end{array}$$

□

Given an R -module N , we consider the R -group functor

$$S \mapsto \mathrm{GL}(N)(S) = \mathrm{Aut}_{S\text{-mod}}(N \otimes_R S) = \mathrm{End}_S(N \otimes_R S)^\times.$$

So if N is finitely generated projective, then $\mathrm{GL}(N)$ is representable by an affine R -group scheme. Furthermore $\mathrm{GL}(N)$ is of finite presentation.

3.3.2. Remark. If R is noetherian, Nitsure has proven that $\mathrm{GL}_1(N)$ is representable if and only if N is projective [Ni04].

3.4. Diagonalizable group schemes. Let A be a commutative abelian (abstract) group. We denote by $R[A]$ the group R -algebra of A . As R -module, we have

$$R[A] = \bigoplus_{a \in A} R e_a$$

and the multiplication is given by $e_a e_b = e_{a+b}$ for all $a, b \in A$.

For $A = \mathbb{Z}$, $R[\mathbb{Z}] = R[T, T^{-1}]$ is the Laurent polynomial ring over R . We have an isomorphism $R[A] \otimes_R R[B] \xrightarrow{\sim} R[A \times B]$. The R -algebra $R[A]$ carries the following Hopf algebra structure:

Comultiplication: $\Delta : R[A] \rightarrow R[A] \otimes R[A]$, $\Delta(e_a) = e_a \otimes e_a$,

Antipode: $\sigma^* : R[A] \rightarrow R[A]$, $\sigma^*(e_a) = e_{-a}$;

Augmentation: $\epsilon^* : R[A] \rightarrow R$, $\epsilon^*(\sum_{a \in A} r_a e_a) = r_0$.

We can check easily that it satisfies the axioms of affine commutative group schemes. One important example is that of $A = \mathbb{Z}$. In this case, we find $\mathbb{G}_{m,R} = \mathrm{Spec}(R[T, T^{-1}])$, it is called the multiplicative group scheme. Another one is $A = \mathbb{Z}/n\mathbb{Z}$ for $n \geq 1$ for which we have $\mu_{n,R} = \mathrm{Spec}(R[T]/(T^n - 1))$ called the R -scheme of n -roots of unity.

3.4.1. Definition. We denote by $\mathfrak{D}(A)/R$ (or \widehat{A}) the affine commutative group scheme $\mathrm{Spec}(R[A])$. It is called the diagonalizable R -group scheme of base A . An affine R -group scheme is diagonalizable if it is isomorphic to some $\mathfrak{D}(B)$.

We note also that there is a natural group scheme isomorphism $\mathfrak{D}(A \oplus B) \xrightarrow{\sim} \mathfrak{D}(A) \times_R \mathfrak{D}(B)$.

If $f : B \rightarrow A$ is a morphism of abelian groups, it induces a group homomorphism $f^* : \mathfrak{D}(A) \rightarrow \mathfrak{D}(B)$. In particular, when taking $B = \mathbb{Z}$, we have a natural mapping

$$\eta_A : A \rightarrow \mathrm{Hom}_{R\text{-gp}}(\mathfrak{D}(A), \mathbb{G}_m).$$

3.4.2. Remark. For $a \in A$, put $\chi_a = \eta_A(a) : \mathfrak{D}(A) \rightarrow \mathbb{G}_m$. The map $\chi_a^* : R[t, t^{-1}] \rightarrow R[A]$ applies t to e_a . Using the commutative diagram

$$\begin{array}{ccc} \mathfrak{D}(A)(R[A]) & \xrightarrow{\chi_a} & \mathbb{G}_m(R[A]) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Hom}_R(R[A], R[A]) & \longrightarrow & \mathrm{Hom}_R(R[t, t^{-1}], R[A]) = R[A]^\times, \end{array}$$

we see that the universal element of $\mathfrak{D}(A)$ maps to χ_a^* which corresponds to e_a .

3.4.3. Lemma. *If R is connected, η_A is bijective.*

Proof. We establish first the injectivity. If $\eta_A(a) = 0$, it means that the map $R[T, T^{-1}] \rightarrow R[A]$, $T \mapsto e_a$ factorises by the augmentation $R[T, T^{-1}] \rightarrow R$ hence $a = 0$.

For the surjectivity, let $f : \mathfrak{D}(A) \rightarrow \mathbb{G}_m$ be a morphism of R -group schemes. Equivalently it is given by the map $f^* : R[T, T^{-1}] \rightarrow R[A]$ of Hopf algebra which satisfies in particular the following compatibility

$$\begin{array}{ccc} R[T, T^{-1}] & \xrightarrow{f^*} & R[A] \\ \downarrow \Delta & & \downarrow \Delta_A \\ R[T, T^{-1}] \otimes_R R[T, T^{-1}] & \xrightarrow{f^* \otimes f^*} & R[A] \otimes_R R[A]. \end{array}$$

In other words, it is determined by the function $X = f^*(T) \in R[A]^\times$ satisfying $\Delta(X) = X \otimes X$. Writing $X = \sum_{a \in A} r_a e_a$, we have

$$\sum_{a \in A} r_a e_a \otimes e_a = \sum_{a, a' \in A} r_a r_{a'} e_a \otimes e_{a'}.$$

It follows that $r_a r_b = 0$ if $a \neq b$ and $r_a r_a = r_a$. Since the ring is connected, 0 and 1 are the only idempotents so that $r_a = 0$ or $r_a = 1$. Then there exists a unique a such that $r_a = 1$ and $r_b = 0$ for $b \neq a$. This shows that the map η_A is surjective. We conclude that η_A is bijective. \square

3.4.4. Proposition. *(Cartier duality) Assume that R is connected. The above construction induces an anti-equivalence of categories between the category of abelian groups and that of diagonalizable R -group schemes.*

Proof. It is enough to construct the inverse map $\mathrm{Hom}_{R\text{-gp}}(\mathfrak{D}(A), \mathfrak{D}(B)) \rightarrow \mathrm{Hom}(A, B)$ for abelian groups A, B . We are given a group homomorphism $f : \mathfrak{D}(A) \rightarrow \mathfrak{D}(B)$. It induces a map

$$f^* : \mathrm{Hom}_{R\text{-gp}}(\mathfrak{D}(B), \mathbb{G}_m) \rightarrow \mathrm{Hom}_{R\text{-gp}}(\mathfrak{D}(A), \mathbb{G}_m),$$

hence a map $B \rightarrow A$. It is routine to check that the two functors are inverse of each other. \square

3.4.5. Lemma. *Assume that R is connected. The following are equivalent:*

- (i) *A is finitely generated;*
- (ii) *$\mathfrak{D}(A)/R$ is of finite presentation;*
- (iii) *$\mathfrak{D}(A)/R$ is of finite type.*

Proof. (i) \implies (ii). We use the structure theorem of abelian groups $A \cong \mathbb{Z}^r \times \mathbb{Z}/n_1\mathbb{Z} \cdots \times \mathbb{Z}/n_c\mathbb{Z}$. Using the compatibility with products we are reduced to the case of \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$ which correspond to $\mathbb{G}_{m,R}$ and $\mu_{n,R}$. Both are finitely presented over R .

(ii) \implies (iii). Obvious.

(iii) \implies (i). We assume that $R[A]$ is a finitely generated R -ring. We write $A = \varinjlim_i A_i$ as the inductive limit of finitely generated subgroups. We have $R[A] = \varinjlim_i R[A_i]$. Since the ring $R[A]$ is finitely generated over R , the identity $\mathbb{Z}[A] \rightarrow \mathbb{Z}[A]$ factorizes through $\mathbb{Z}[A_i]$ for some i . It implies that $\mathbb{Z}[A_i] \xrightarrow{\sim} \mathbb{Z}[A]$. Cartier duality shows that $A_i \xrightarrow{\sim} A$. Thus A is finitely generated. \square

There are other notable properties of Cartier duality, see [SGA3, VIII.2.1]. In practice we will work with finiteness assumptions, however it is remarkable that the theory holds for arbitrary abelian groups.

3.5. Monomorphisms of group schemes. We recall that a morphism of R -functors $f : F \rightarrow F'$ is a monomorphism if $f(S) : F(S) \rightarrow F'(S)$ is injective for each R -algebra S/R (§2.2). If F and F' are functors in groups and f respects the group structure, the kernel of f is the R -group functor defined by $\ker(f)(S) = \ker(F(S) \rightarrow F'(S))$ for each R -algebra S .

We recall that a morphism $f : \mathfrak{G} \rightarrow \mathfrak{H}$ of affine R -group schemes is a monomorphism if h_f is a monomorphism (Lemma 2.2.2).

3.5.1. Lemma. *Let $f : \mathfrak{G} \rightarrow \mathfrak{H}$ be a morphism of R -group schemes. Then the R -functor $\ker(f)$ is representable by a closed subgroup scheme of \mathfrak{G} .*

Proof. Indeed the cartesian product

$$\begin{array}{ccc} \mathfrak{N} & \longrightarrow & \mathfrak{G} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(R) & \xrightarrow{\epsilon'} & \mathfrak{H} \end{array}$$

does the job. \square

Summarizing $f : \mathfrak{G} \rightarrow \mathfrak{H}$ is a monomorphism if and only if the kernel R -group scheme $\ker(f)$ is the trivial group scheme.

Over a field F , we know that a monomorphism of algebraic groups is a closed immersion [SGA3, VI_B.1.4.2].

Over a DVR, it is not true in general that an open immersion (and a fortiori a monomorphism as seen in the exercise session) of group schemes of finite type is a closed immersion. We consider the following example

[SGA3, VIII.7]. Assume that R is a DVR and consider the constant group scheme $\mathfrak{H} = (\mathbb{Z}/2\mathbb{Z})_R$. Now let \mathfrak{G} be the open subgroup scheme of \mathfrak{H} which is the complement of the closed point 1 in the closed fiber. By construction \mathfrak{G} is dense in \mathfrak{H} , so that the immersion $\mathfrak{G} \rightarrow \mathfrak{H}$ is not closed. Raynaud constructed a more elaborated example where \mathfrak{H} and \mathfrak{G} are both affine over $\mathbb{F}_2[[t]]$ and a monomorphism which is not an immersion [SGA3, XVI.1.1.c].

However diagonalizable groups have a wonderful behaviour with that respect by using Cartier duality (Proposition 3.4.4).

3.5.2. Proposition. *Assume that R is connected. Let $f : \mathfrak{D}(B) \rightarrow \mathfrak{D}(A)$ be a group homomorphism of diagonalizable R -group schemes. Then the following are equivalent:*

- (i) $f^* : A \rightarrow B$ is onto;
- (ii) f is a closed immersion;
- (iii) f is a monomorphism.

Proof. (i) \implies (ii): Then $R[B]$ is a quotient of $R[A]$ so that $f : \mathfrak{D}(B) \rightarrow \mathfrak{D}(A)$ is a closed immersion.

(ii) \implies (iii): obvious.

(iii) \implies (i): We denote by $B_0 \subset B$ the image of $f^* : A \rightarrow B$. We consider the compositum

$$\begin{array}{ccccccc} & & & f & & & \\ & & & \curvearrowright & & & \\ \mathfrak{D}(B/B_0) & \longrightarrow & \mathfrak{D}(B) & \xrightarrow{v} & \mathfrak{D}(B_0) & \xrightarrow{w} & \mathfrak{D}(A). \\ & & & \curvearrowleft & & & \\ & & & v & & & \end{array}$$

We observe that it is the trivial morphism (v is trivial) and is a monomorphism as compositum of the monomorphisms u and f . It follows that $\mathfrak{D}(B/B_0) = \text{Spec}(R)$ and we conclude that $B_0 = B$ by Cartier duality. \square

Of the same flavour, the kernel of a map $f : \mathfrak{D}(B) \rightarrow \mathfrak{D}(A)$ is isomorphic to $\mathfrak{D}(f(A))$. The case of vector groups is more subtle.

3.5.3. Proposition. *Let $f : N_1 \rightarrow N_2$ be a morphism of finitely generated projective R -modules. Then the morphism of functors $f_* : W(N_1) \rightarrow W(N_2)$ is a monomorphism if and only if f identifies N_1 as a direct summand of N_2 . If it the case, $f_* : \mathfrak{W}(N_1) \rightarrow \mathfrak{W}(N_2)$ is a closed immersion.*

Proof. If N_1 is a direct summand of N_2 , the morphism $f_* : W(N_1) = V(N_1^\vee) \rightarrow W(N_2^\vee)$ is a closed immersion and a fortiori a monomorphism. Conversely we assume that $f_* : \mathfrak{W}(N_1) \rightarrow \mathfrak{W}(N_2)$ is a monomorphism.

Conversely suppose that f_* is a monomorphism. Since $W(N_1)(R)$ injects in $W(N_2)(R)$, we have that $f : N_1 \rightarrow N_2$ is injective. We put $N_3 = N_2/f(N_1)$. To show that N_1 is a direct summand of N_2 it is enough to show that N_3 is (finitely generated projective). This is our plan. Since

N_2 and N_3 are f.g. projective R -modules, the R -module N_3 is of finite presentation. In view of the characterization of f.g. projective modules [Bbk2, II.5.2], it is enough to show that $N_3 \otimes_R R_{\mathfrak{m}}$ is free for each maximal ideal \mathfrak{m} of R . Let \mathfrak{m} be a maximal ideal of R .

Applying the criterion of Lemma 2.2.1 to the residue field $S = R/\mathfrak{m}$ we have that the map

$$f_*(R/\mathfrak{m}) : N_1 \otimes_R R/\mathfrak{m} \rightarrow N_2 \otimes_R R/\mathfrak{m}$$

is injective. It follows that there exists an R/\mathfrak{m} -base $(\bar{w}_1, \dots, \bar{w}_r, \bar{w}_{r+1}, \dots, \bar{w}_n)$ of $N_2 \otimes_R R/\mathfrak{m}$ such that $(\bar{w}_1, \dots, \bar{w}_r)$ is a base of $f(N_1 \otimes_R R/\mathfrak{m})$. We have $\bar{w}_i = f(\bar{v}_i)$ for $i = 1, \dots, r$. We lift the \bar{v}_i 's in an arbitrary way in $N_1 \otimes_R R_{\mathfrak{m}}$ and the $\bar{w}_{r+1}, \dots, \bar{w}_n$ in $N_2 \otimes_R R_{\mathfrak{m}}$. Then (v_1, \dots, v_r) is an $R_{\mathfrak{m}}$ -base of $N_1 \otimes_R R_{\mathfrak{m}}$ and $(f(v_1), \dots, f(v_r), w_{r+1}, \dots, w_n)$ is an $R_{\mathfrak{m}}$ -base of $N_2 \otimes_R R_{\mathfrak{m}}$. Thus $N_3 \otimes_R R_{\mathfrak{m}}$ is free.

We conclude that f identifies N_1 as a direct summand of N_2 . \square

4. SEQUENCES OF GROUP FUNCTORS

4.1. Exactness. We say that a sequence of R -group functors

$$1 \rightarrow F_1 \xrightarrow{u} F_2 \xrightarrow{v} F_3 \rightarrow 1$$

is exact if for each R -algebra S , the sequence of abstract groups

$$1 \rightarrow F_1(S) \xrightarrow{u(S)} F_2(S) \xrightarrow{v(S)} F_3(S) \rightarrow 1$$

is exact. Similarly we can define the exactness of a sequence $1 \rightarrow F_1 \rightarrow \dots \rightarrow F_n \rightarrow 1$. If $w : F \rightarrow F'$ is a map of R -group functors, recall the definition of the R -group functor $\ker(w)$ by $\ker(w)(S) = \ker(F(S) \rightarrow F'(S))$ for each R -algebra S . Also the cokernel $\operatorname{coker}(w)(S) = \operatorname{coker}(F(S) \rightarrow F'(S))$ is an R -functor (but not necessarily an R -functor in groups).

4.1.1. Example. We consider an exact sequence $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$ of finitely generated modules with N_1, N_2 projective. We claim that it induces an exact sequence of R -functors in groups

$$0 \rightarrow W(N_1) \rightarrow W(N_2) \rightarrow W(N_3) \rightarrow 0$$

if and only if the starting sequence is split (equivalently N_3 is projective). The converse implication is obvious. If the sequence above of R -functors in groups is exact, then $W(N_1) \rightarrow W(N_2)$ is a monomorphism so that Proposition 3.5.3 shows that N_1 is a direct summand of N_2 .

We can define also the cokernel of a morphism R -group schemes. But it is very rarely representable. The simplest example is the Kummer morphism $f_n : \mathbb{G}_{m,R} \rightarrow \mathbb{G}_{m,R}$, $x \mapsto x^n$ for $n \geq 2$ for $R = \mathbb{C}$, the field of complex numbers. Assume that there exists an affine \mathbb{C} -group scheme \mathfrak{G} such that there is a four terms exact sequence of \mathbb{C} -functors

$$1 \rightarrow h_{\mu_n} \rightarrow h_{\mathbb{G}_m} \xrightarrow{h_{f_n}} h_{\mathfrak{G}} \rightarrow 1$$

We denote by T' the parameter for the first \mathbb{G}_m and by $T = (T')^n$ the parameter of the second one. Then $T \in \mathbb{G}_m(R[T, T^{-1}])$ defines a non trivial element of $\mathfrak{G}(R[T, T^{-1}])$ which is trivial in $\mathfrak{G}(R[T', T'^{-1}])$. It is a contradiction.

We provide a criterion.

4.1.2. Lemma. *Let*

$$1 \rightarrow \mathfrak{G}_1 \xrightarrow{u} \mathfrak{G}_2 \xrightarrow{v} \mathfrak{G}_3 \rightarrow 1$$

be a sequence of affine R -group schemes. Then the sequence of R -functors

$$1 \rightarrow h_{\mathfrak{G}_1} \rightarrow h_{\mathfrak{G}_2} \rightarrow h_{\mathfrak{G}_3} \rightarrow 1$$

is exact if and only if the following conditions are satisfied:

- (i) $u : \mathfrak{G}_1 \rightarrow \ker(v)$ is an isomorphism;
- (ii) $v : \mathfrak{G}_2 \rightarrow \mathfrak{G}_3$ admits a splitting $f : \mathfrak{G}_3 \rightarrow \mathfrak{G}_2$ as R -schemes.

4.1.3. Remark. Note that if (ii) holds, we have $\mathfrak{G}_2(S) = u(\mathfrak{G}_1(S))f(\mathfrak{G}_3(S))$ for each R -algebra S . Let S be an R -algebra and let $g_2 \in \mathfrak{G}_2(S)$. Since $\mathfrak{G}_1(S) \rightarrow \mathfrak{G}_2(S) \rightarrow \mathfrak{G}_3(S)$ is exact, $g_2 f(v(g_2))^{-1} \in \mathfrak{G}_1(S)$. We conclude that $\mathfrak{G}_2(S) = u(\mathfrak{G}_1(S))f(\mathfrak{G}_3(S))$.

We proceed to the proof of Lemma 4.1.2.

Proof. We assume that the sequence of R -functors $1 \rightarrow h_{\mathfrak{G}_1} \rightarrow h_{\mathfrak{G}_2} \rightarrow h_{\mathfrak{G}_3} \rightarrow 1$ is exact. We have seen that \mathfrak{G}_1 is the kernel of v . This shows (i). The assertion (ii) is an avatar of Yoneda's lemma. We consider the surjective map $\mathfrak{G}_2(R[\mathfrak{G}_3]) \rightarrow \mathfrak{G}_3(R[\mathfrak{G}_3])$ and lift the identity of \mathfrak{G}_3 to a map $t : \mathfrak{G}_2(R[\mathfrak{G}_3]) = \text{Hom}_{R\text{-sch}}(\mathfrak{G}_3, \mathfrak{G}_2)$. Then t is an R -scheme splitting of $v : \mathfrak{G}_2 \rightarrow \mathfrak{G}_3$.

Conversely we assume (i) and (ii). Clearly $h_{\mathfrak{G}_1} \rightarrow h_{\mathfrak{G}_2}$ is a monomorphism and $h_{\mathfrak{G}_2} \rightarrow h_{\mathfrak{G}_3}$ is a epimorphism (see §2.2). We only have to check the exactness of $\mathfrak{G}_1(S) \rightarrow \mathfrak{G}_2(S) \rightarrow \mathfrak{G}_3(S)$ for each S/R but it follows from (ii). \square

4.1.4. Examples. (a) It is not obvious to construct examples of exact sequences of group functors which are not split as R -group functors. An example is the exact sequence of Witt vectors groups over \mathbb{F}_p $0 \rightarrow W_1 \rightarrow W_2 \rightarrow W_1 \rightarrow 0$. It provides a non split exact sequence of commutative affine \mathbb{F}_p -group schemes $0 \rightarrow \mathbb{G}_a \rightarrow W_2 \rightarrow \mathbb{G}_a \rightarrow 0$. For other examples see [DG, III.6]. (b) Also it is natural question to ask whether the existence of sections of the map $\mathfrak{G}_2 \rightarrow \mathfrak{G}_3$ locally over \mathfrak{G}_3 is enough. It is not the case and an example of this phenomenon is by using the \mathbb{R} -group scheme G_2 defined as the unit group scheme of the \mathbb{R} -algebra \mathbb{C} ; recall that its functor of points is $G_2(S) = (S \otimes_{\mathbb{R}} \mathbb{C})^\times$ (§3.3). It comes with a norm morphism $N : G_2(S) \rightarrow \mathbb{G}_{m, \mathbb{R}}$ and we consider the kernel $G_3 = \ker(N)$. Note that G_2

comes with an involution σ given by the complex conjugation. We consider the sequence of \mathbb{R} -group schemes

$$1 \rightarrow \mathbb{G}_m \rightarrow G_2 \xrightarrow{\sigma-id} G_3 \rightarrow 1.$$

The associated sequence for real points is $1 \rightarrow \mathbb{R}^\times \rightarrow \mathbb{C}^\times \rightarrow S^1 \rightarrow 1$, where the last map is $z \mapsto \bar{z}/z$. For topological reasons², there is no continuous section of the map $\mathbb{C}^\times \rightarrow S^1$. A fortiori, there is no algebraic section of the map $G_2 \xrightarrow{\sigma-id} G_3$. On the other hand this map admits local splittings, let us explain how it works for example on $\mathfrak{G}_3 \setminus \{(-1, 0)\}$. We map $t \mapsto (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}) = (\sigma - 1) \cdot (1 + ti)$ induces an isomorphism $\mathbb{R}[\mathfrak{G}_3]_{(-1,0)} \xrightarrow{\sim} \mathbb{R}[t, \frac{1}{t^2+1}]$ and defines a section of $\sigma - id$ on $\mathfrak{G}_3 \setminus \{(-1, 0)\}$. The sequence above is not exact in the category of \mathbb{R} -functors.

4.2. Semi-direct product. Let \mathfrak{G}/R be an affine group scheme acting on another affine group scheme \mathfrak{H}/R , that is we are given a morphism of R -functors

$$\theta : h_{\mathfrak{G}} \rightarrow \text{Aut}(h_{\mathfrak{H}}).$$

The semi-direct product $h_{\mathfrak{H}} \rtimes^\theta h_{\mathfrak{G}}$ is well defined as R -functor.

4.2.1. Lemma. *$h_{\mathfrak{H}} \rtimes^\theta h_{\mathfrak{G}}$ is representable by an affine R -scheme denote by $\mathfrak{H} \rtimes^\theta \mathfrak{G}$. Furthermore we have an exact sequence of affine R -group schemes*

$$1 \rightarrow \mathfrak{H} \rightarrow \mathfrak{H} \rtimes^\theta \mathfrak{G} \rightarrow \mathfrak{G} \rightarrow 1.$$

Proof. We consider the affine R -scheme $\mathfrak{X} = \mathfrak{H} \times_R \mathfrak{G}$. Then $h_{\mathfrak{X}} = h_{\mathfrak{H}} \rtimes^\theta h_{\mathfrak{G}}$ has a group structure so defines a group scheme structure on \mathfrak{X} . The sequence holds in view of the criterion provided by Lemma 4.1.2. \square

A nice example of this construction is the “affine group” of a finitely generated, projective R -module N . The R -group scheme $\text{GL}(N)$ acts on the vector R -group \mathfrak{W}_N so that we can form the R -group scheme $\mathfrak{W}_N \rtimes \text{GL}(N)$ of affine transformations of N .

²The induced map $\mathbb{Z} = \pi(\mathbb{C}^\times, 1) \rightarrow \mathbb{Z} = \pi_1(S^1, 1)$ is the multiplication by 2.

Affine group schemes II

5. FLATNESS

We will explain in this section why flatness is a somehow a minimal reasonable assumption when studying affine group schemes. This includes a nice behaviour of the dimension of geometric fibers, see Thm. 5.3.1 below.

5.1. Examples of flat affine group schemes.

5.1.1. Lemma. *Let \mathfrak{G} be an affine R -group scheme. Then \mathfrak{G} is flat if and only if \mathfrak{G} is faithfully flat.*

Proof. Faithfully flat means that the structural morphism $\mathfrak{G} \rightarrow \operatorname{Spec}(R)$ is flat and surjective. Since $\mathfrak{G} \rightarrow \operatorname{Spec}(R)$ admits the unit section, the structural morphism is surjective. This explains the equivalence between flatness and faithfully flatness in our setting. \square

All examples we have seen so far were flat. Constant group schemes are obviously flat. If A is an abelian group, the diagonalizable R -group scheme $\mathfrak{D}(A)$ is R -flat since $R[A]$ is a free R -module.

If N is a finitely generated projective R -module, the affine group schemes $\mathfrak{V}(N)$ and $\mathfrak{W}(N)$ are flat. Indeed, flatness is a local property for the Zariski topology on $\operatorname{Spec}(R)$ [St, Tag 00HJ] so that it reduces to the case of the affine space \mathbb{A}_R^d which is clear since the R -module $R[t_1, \dots, t_d]$ is free. A more complicated fact is the following.

5.1.2. Lemma. *Let M be an R -module. Then M is flat if and only if $\mathfrak{V}(M)$ is a flat R -scheme.*

Proof. By definition the R -scheme $\mathfrak{V}(M)$ is flat if and only if the symmetric algebra $S^*(M)$ is a flat R -module. Since M is a direct summand of $S^*(M)$ as R -module, the flatness of $S^*(M)$ implies that M is flat.

For the converse we use Lazard's theorem stating that M is isomorphic to a direct limit $\varinjlim_{i \in I} M_i$ of f.g. free R -modules [St, Tag 058G]. Since $S^*(M) = \varinjlim_{i \in I} S^*(M_i)$ and each $S^*(M_i)$ is a free R -module (so a fortiori flat), it follows that $S^*(M)$ is a flat R -algebra in view of [St, Tag 05UU] (use the case $R_i = R$ for all i). \square

Finally the group scheme of invertible elements $U(A)$ of an algebra A/R f.g. projective is flat. We have seen that $U(A)$ is principal open in $\mathfrak{W}(A)$ so that $R[U(A)]$ is flat over $R[\mathfrak{W}(A)]$ [St, Tag 00HT]. Since flatness behaves well for composition [GW, prop. 14.3], we conclude that the affine R -scheme $U(A)$ is flat.

5.2. The DVR case. Assume that R is a *DVR* with uniformizing parameter π and denote by K its field of fractions. We recall the following well-known fact.

5.2.1. Lemma. *Let M be an R -module. Then the following are equivalent:*

- (i) M is flat;
- (ii) M is torsion free, that is $\times \pi : M \rightarrow M$ is injective;
- (iii) $M \rightarrow M \otimes_R K$ is injective.

Furthermore, if M is finitely generated, this is equivalent to $M \cong R^n$.

Proof. (i) \implies (ii). It means that the functor $\otimes_R M$ is exact. Since $\pi : R \rightarrow R$ is injective, it follows that $\times \pi : M \rightarrow M$.

(ii) \implies (i). The R module M is the filtered inductive limit of its finitely generated submodules. Also, submodules of torsionfree modules are torsionfree, and inductive limits of flat modules are flat [St, Tag 05UU]. This is why it suffices to prove that finitely generated torsionfree R -modules are flat, or even free. We assume then that M is a finitely generated R -module. Choose $m_1, \dots, m_n \in M$ such that $\overline{m_1}, \dots, \overline{m_n}$ is a k -basis of the k -vector space $M \otimes_R k$. By Nakayama's Lemma, m_1, \dots, m_n is a generating set of M ; in other words we have a surjective R -map $f : R^n \rightarrow M$. Consider a non zero relation $f(r_1, \dots, r_n) = \sum_{i=1}^n r_i m_i = 0$. Since M is torsionfree, dividing the r_i by the largest possible power π^c occuring so that we get a non-trivial relation $\sum_{i=1}^n \overline{r_i} \overline{m_i} = 0$. This is a contradiction.

(ii) \implies (iii). Once again this reduces to the finitely generated case which is free. Since $R^n \rightarrow K^n$ is injective, we are done.

(iii) \implies (i). Obvious. □

Note that there are generalization to Dedekind domains and valuation rings [St, Tags 0AUW, 0539]. From the lemma, we know that an affine scheme \mathfrak{X}/R is flat, that is, $R[\mathfrak{X}]$ is torsionfree or equivalently that $R[\mathfrak{X}]$ embeds in $K[\mathfrak{X}]$.

5.2.2. Proposition. [EGA4, 2.8.1] (*see also* [GW, §14.3])

Let \mathfrak{X}/R be a flat affine R -group scheme. There is a one to one correspondence between the flat closed R -subschemes of \mathfrak{X} and the closed K -subschemes of the generic fiber \mathfrak{X}_K .

Furthermore this correspondence commutes with fibered products over R and is functorial with respect to R -morphisms $\mathfrak{X} \rightarrow \mathfrak{X}'$ of flat R -schemes.

The correspondence goes as follows. In one way we take the generic fiber and in the way around we take the schematic closure (in the sense of the scheme theoretic image of the immersion map $Y \subset \mathfrak{X}_K \hookrightarrow \mathfrak{X}$ [St, Tag 01R7]). The schematic closure \mathfrak{Y} of Y in \mathfrak{X} is the smallest closed subscheme \mathfrak{X} such that $Y \subset \mathfrak{X}_K \hookrightarrow \mathfrak{X}$ factorizes through \mathfrak{Y} . Let us explain its construction in

terms of rings. If Y/K is a closed K -subscheme of X/K , it is defined by the ideal $I(Y) = \text{Ker}(K[\mathfrak{X}] \rightarrow K[Y])$ of $K[\mathfrak{X}]$. Similarly we deal with the ideal $I(\mathfrak{Y}) = \text{Ker}(R[\mathfrak{X}] \rightarrow R[\mathfrak{Y}])$ of $R[\mathfrak{X}]$. This fits in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I(\mathfrak{Y}) & \longrightarrow & R[\mathfrak{X}] & \longrightarrow & R[\mathfrak{Y}] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I(Y) & \longrightarrow & K[\mathfrak{X}] & \longrightarrow & K[Y] \longrightarrow 0 \end{array}$$

The ideal $I(\mathfrak{Y})$ of $R[\mathfrak{X}]$ is the smallest ideal which maps in $I(Y)$, it follows that $I(\mathfrak{Y}) = I \cap R[\mathfrak{X}]$. Since $I(\mathfrak{Y}) \otimes_R K = I(Y)$, we have $R[\mathfrak{Y}] \otimes_R K = K[Y]$, that is, $\mathfrak{Y} \times_R K = Y_K$. Also the map $R[\mathfrak{Y}] \rightarrow K[Y]$ is injective, i.e. \mathfrak{Y} is a flat affine R -scheme. It remains to show that the other composite is the identity and also the functorial properties. We proceed then to the end of the proof of Proposition 5.2.2.

Proof. Given $\mathfrak{Y} \subset \mathfrak{X}$ a flat closed R -subscheme, we consider the ideal $I(\mathfrak{Y}) = \text{Ker}(R[\mathfrak{X}] \rightarrow R[\mathfrak{Y}])$. We denote by $\mathfrak{Y}' \subset \mathfrak{X}$ the schematic closure of $\mathfrak{Y}_K \subset \mathfrak{X}$. We have $I(\mathfrak{Y}') = I(\mathfrak{Y}_K) \cap R[\mathfrak{X}]$. We consider the commutative diagram of exact sequences of R -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & I(\mathfrak{Y}) & \longrightarrow & R[\mathfrak{X}] & \longrightarrow & R[\mathfrak{Y}] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I(\mathfrak{Y}_K) & \longrightarrow & K[\mathfrak{X}] & \longrightarrow & K[\mathfrak{Y}] \longrightarrow 0 \end{array}$$

where the two vertical maps on the right express flatness of \mathfrak{X} and \mathfrak{Y} . By diagram chase we have $I(\mathfrak{Y}) = I(\mathfrak{Y}')$.

We examine now the behaviour for fibered products. We are given two affine flat R -schemes $\mathfrak{X}_1, \mathfrak{X}_2$ with closed flat R -subschemes $\mathfrak{Y}_1 \subset \mathfrak{X}_1$ and $\mathfrak{Y}_2 \subset \mathfrak{X}_2$. Then $\mathfrak{Y}_1 \times_R \mathfrak{Y}_2$ is a flat closed R -subscheme (using that flatness behaves well with tensor products, see [Bbk2, §1.7]) of $\mathfrak{X}_1 \times_R \mathfrak{X}_2$ and of generic fiber $\mathfrak{Y}_{1,K} \times_K \mathfrak{Y}_{2,K}$ so is the schematic closure of $\mathfrak{Y}_{1,K} \times_K \mathfrak{Y}_{2,K}$ in $\mathfrak{X}_1 \times_R \mathfrak{X}_2$.

Next we deal with a morphism $f : \mathfrak{X} \rightarrow \mathfrak{X}'$ of affine flat R -schemes. For an affine flat closed R -subscheme $\mathfrak{Y} \subset \mathfrak{X}$ (resp. $\mathfrak{Y}' \subset \mathfrak{X}'$), if f induces a morphism $\mathfrak{Y} \rightarrow \mathfrak{Y}'$ then f_K induces a map $\mathfrak{Y}_K \rightarrow \mathfrak{Y}'_K$. Conversely assume that f_K induces a map $f_K : Y' \rightarrow Y$ where $Y \subset \mathfrak{X}_K$ (resp. $Y' \subset \mathfrak{X}'_K$) and denote by $\mathfrak{Y} \subset \mathfrak{X}$ (resp. $\mathfrak{Y}' \subset \mathfrak{X}'$) the schematic adherence of Y . We need

to check that f induces a map $\mathfrak{Y} \rightarrow \mathfrak{Y}'$. We consider the diagram

$$\begin{array}{ccc} R[\mathfrak{X}'] & \xrightarrow{f^*} & R[\mathfrak{X}] \\ \downarrow & & \downarrow \\ K[\mathfrak{X}'] & \xrightarrow{f_K^*} & K[\mathfrak{X}] \\ \parallel & & \parallel \\ I(Y') & \longrightarrow & I(Y) \end{array}$$

It shows that $f^*(R[\mathfrak{X}'] \cap I(Y')) \subseteq R[\mathfrak{X}] \cap I(Y)$ whence $f^*(R[\mathfrak{Y}']) \subseteq R[\mathfrak{Y}]$ as desired. \square

In particular, if \mathfrak{G}/R is a flat group scheme, it induces a one to one correspondence between flat closed R -subgroup schemes of \mathfrak{G} and closed K -subgroup schemes of \mathfrak{G}_K ³.

5.2.3. Example. We consider the centralizer closed subgroup scheme of $\mathrm{GL}_{2,R}$

$$\mathfrak{Z} = \left\{ g \in \mathrm{GL}_{2,R} \mid gA = Ag \right\}$$

of the element $A = \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix}$. Then $\mathfrak{Z} \times_R R/\pi R \xrightarrow{\sim} \mathrm{GL}_{2,R}$ and

$$\mathfrak{Z} \times_R K = \mathbb{G}_{m,K} \times_K \mathbb{G}_{a,K} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \right\}$$

Then the closure of \mathfrak{Z}_K in $\mathrm{GL}_{2,R}$ is $\mathbb{G}_{m,R} \times_R \mathbb{G}_{a,R}$, so does not contain the special fiber of \mathfrak{Z} . We conclude that \mathfrak{Z} is not flat.

5.3. A necessary condition. In the above example, the geometrical fibers were of dimension 4 and 2 respectively. It illustrates then the following general result.

5.3.1. Theorem. [SGA3, VI_B.4.3] *Let R be a ring and let \mathfrak{G}/R be a flat group scheme of finite presentation. Then the dimension of the geometrical fibers is locally constant.*

It means that the dimension of the fibers cannot jump by specialization.

6. REPRESENTATIONS

Let \mathfrak{G}/R be an affine group scheme.

6.0.1. Definition. A (left) R - \mathfrak{G} -module (or \mathfrak{G} -module for short) is an R -module M equipped with a morphism of group functors

$$\rho : h_{\mathfrak{G}} \rightarrow \mathrm{Aut}_{\mathrm{lin}}(W(M)).$$

We say that the \mathfrak{G} -module M is faithful if ρ is a monomorphism.

³Warning: the fact that the schematic closure of a group scheme is a group scheme is specific to Dedekind rings.

Here $\text{Aut}_{lin}(W(M))$ stands for linear automorphisms of the functor $W(M)$, that is, $\text{Aut}_{lin}(W(M))(S) = \text{End}_S(M \otimes_R S)^\times$ for each R -algebra S . We denote by $\text{GL}(M)$ and we bear in mind that is not necessarily representable.

If $M = R^n$, then $\text{GL}(M)$ is representable by $\text{GL}_{n,R}$ so that it corresponds to an R -group homomorphism $\mathfrak{G} \rightarrow \text{GL}_{n,R}$ and faithfulness corresponds to the triviality of the kernel.

Coming back to the general setting, it means that for each algebra S/R , we are given an action of $\mathfrak{G}(S)$ on $W(M)(S) = M \otimes_R S$. We use again Yoneda lemma. The mapping ρ is defined by the image of the universal point $\zeta \in \mathfrak{G}(R[\mathfrak{G}])$ provides an element called the coaction

$$c_\rho \in \text{Hom}_R(M, M \otimes_R R[\mathfrak{G}]) \xrightarrow{\sim} \text{Hom}_{R[\mathfrak{G}]}(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]).$$

Yoneda lemma implies that c_ρ determines ρ . We denote \tilde{c}_ρ its image in $\text{Hom}_{R[\mathfrak{G}]}(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}])$. For $g \in \mathfrak{G}(R)$, we use the evaluation $\epsilon_g : R[\mathfrak{G}] \rightarrow \mathfrak{R}$ and have by functoriality the bottom of the following commutative diagram

$$(6.0.2) \quad \begin{array}{ccccc} & M & & & \\ & \downarrow id & \searrow c_\rho & & \\ id \swarrow & M \otimes_R R[\mathfrak{G}] & \xrightarrow{\tilde{c}_\rho} & M \otimes_R R[G] & \\ & \downarrow id \otimes \epsilon_g & & \downarrow id \otimes \epsilon_g & \\ & M & \xrightarrow{\rho(g)} & M & \end{array}$$

In other words we have

$$(6.0.3) \quad \rho(g).m = \epsilon_g(c_\rho.m) \quad (g \in \mathfrak{G}(R), m \in M).$$

6.0.4. Remark. For the trivial representation, we have that $\tilde{c}_{triv} = id_{M \otimes_R R[\mathfrak{G}]}$ so that $c_{triv}(m) = m \otimes 1$.

6.0.5. Proposition. (1) Both diagrams

$$(6.0.6) \quad \begin{array}{ccc} M & \xrightarrow{c_\rho} & M \otimes_R R[\mathfrak{G}] \\ c_\rho \downarrow & & id \otimes \Delta_{\mathfrak{G}} \downarrow \\ M \otimes_R R[\mathfrak{G}] & \xrightarrow{c_\rho \otimes id} & M \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}], \end{array}$$

$$(6.0.7) \quad \begin{array}{ccc} M & \xrightarrow{c_\rho} & M \otimes_R R[\mathfrak{G}] \\ id \downarrow & \swarrow id \times \epsilon^* & \\ M & & \end{array}$$

commute.

(2) Conversely, if an R -map $c : M \rightarrow M \otimes_R R[\mathfrak{G}]$ satisfying the two rules above, there is a unique representation $\rho_c : h_{\mathfrak{G}} \rightarrow \mathrm{GL}(W(M))$ such that $c_{\rho_c} = c$.

A module M equipped with an R -map $c : M \rightarrow M \otimes_R R[\mathfrak{G}]$ satisfying the two rules above is called a \mathfrak{G} -module (and also a comodule over the Hopf algebra $R[\mathfrak{G}]$). The proposition shows that it is the same to talk about representations of \mathfrak{G} or about \mathfrak{G} -modules (or also $R - \mathfrak{G}$ -modules).

6.0.8. Remark. There is of course a compatibility with the inverse map but it follows from the other rules.

In particular, the comultiplication $R[\mathfrak{G}] \rightarrow R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ defines a \mathfrak{G} -structure on the R -module $R[\mathfrak{G}]$. It is called the regular representation and is studied more closely in Example 6.0.9. We proceed to the proof of Proposition 6.0.5.

Proof. (1) We double the notation by putting $\mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}$. We consider the following commutative diagram

$$\begin{array}{ccc}
 \mathfrak{G}(R[\mathfrak{G}_1]) \times \mathfrak{G}(R[\mathfrak{G}_2]) & \xrightarrow{\rho \times \rho} & \mathrm{GL}(M)(R[\mathfrak{G}_1]) \times \mathrm{GL}(M)(R[\mathfrak{G}_2]) \\
 \downarrow & & \downarrow \\
 \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \times \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) & \xrightarrow{\rho \times \rho} & \mathrm{GL}(M)(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \times \mathrm{GL}(M)(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \\
 m \downarrow & & m \downarrow \\
 \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) & \xrightarrow{\rho} & \mathrm{GL}(M)(R[\mathfrak{G}_1 \times \mathfrak{G}_2])
 \end{array}$$

and consider the image $\eta \in \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2])$ of the couple (ζ_1, ζ_2) of universal elements by the left vertical map. Then η is defined by the ring homomorphism $\eta^* : R[\mathfrak{G}] \xrightarrow{\Delta_{\mathfrak{G}}} R[\mathfrak{G} \times \mathfrak{G}] \xrightarrow{\sim} R[\mathfrak{G}_1 \times \mathfrak{G}_2]$ so that $\rho(\eta)$ is defined by the following commutative diagram (in view of the compatibility (6.0.2))

$$\begin{array}{ccc}
 M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] & \xrightarrow{\rho(\eta)} & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\
 \uparrow id_M \otimes \Delta & & \uparrow id_M \otimes \Delta \\
 M \otimes_R R[\mathfrak{G}] & \xrightarrow{\tilde{c}_\rho} & M \otimes_R R[\mathfrak{G}]
 \end{array}$$

On the other hand we have that $\rho(\eta) = \tilde{c}_{\rho,2} \circ \tilde{c}_{\rho,1}$ where we did not write the extensions to $R[\mathfrak{G}_1 \times \mathfrak{G}_2]$. Reporting that fact in the diagram above provides the commutative diagram

$$\begin{array}{ccccc}
 M & \longrightarrow & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] & \xrightarrow{\tilde{c}_{\rho,2} \circ \tilde{c}_{\rho,1}} & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\
 \uparrow id & & \uparrow id_M \otimes \Delta & & \uparrow id_M \otimes \Delta \\
 M & \longrightarrow & M \otimes_R R[\mathfrak{G}] & \xrightarrow{\tilde{c}_\rho} & M \otimes_R R[\mathfrak{G}].
 \end{array}$$

By restricting to M , we get the commutative square

$$\begin{array}{ccc} M \otimes_R R[\mathfrak{G}_1] & \xrightarrow{c_{\rho_2} \otimes id_{R[\mathfrak{G}_1]}} & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\ c_{\rho,1} \uparrow & & \uparrow \Delta \\ M & \xrightarrow{c_\rho} & M \otimes_R R[\mathfrak{G}] \end{array}$$

as desired. The other rule comes from the fact that $1 \in G(R)$ acts trivially on M and is a special case of the diagram (6.0.2).

(2) We are given $c : M \rightarrow M \otimes_R R[\mathfrak{G}]$ satisfying the two rules. We define first a morphism of R -functors $h_{\mathfrak{G}} \rightarrow W(End_R(M))$. According to Yoneda lemma 2.1.1, we have

$$Hom_{R-func}(h_{\mathfrak{G}}, W(End_R(M))) = W(End_R(M))(R[\mathfrak{G}])$$

$$= Hom_{R[\mathfrak{G}]}(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]) \xleftarrow{\sim} Hom_R(M, M \otimes_R R[\mathfrak{G}]).$$

It follows that c defines a (unique) morphism of R -functors $\rho_c : h_{\mathfrak{G}} \rightarrow W(End_R(M))$ such that the universal element of \mathfrak{G} is applied to \tilde{c} . The first rule insures the multiplicativity (check it) and the second rule says that the unit element $1 \in \mathfrak{G}(R)$ is applied to id_M . It follows that ρ_c factorizes through the subfunctor $GL(M)$ of $W(End_R(M))$ and induces a homomorphism of R -group functors $h_{\mathfrak{G}} \rightarrow GL(M)$. \square

6.0.9. Example. We claim that the regular representation is nothing but the right translation on $R[\mathfrak{G}]$ and that it is faithful. We consider the \mathfrak{G} -module $A = R[\mathfrak{G}]$ defined by the comultiplication $\Delta : A \rightarrow A \otimes_R R[\mathfrak{G}]$. It defines the regular representation $\rho : \mathfrak{G} \rightarrow GL(A)$. Given $g \in \mathfrak{G}(R)$, we consider the following diagram (special case of the diagram (6.0.2))

$$(6.0.10) \quad \begin{array}{ccc} A & & \\ \downarrow id & \searrow \Delta & \\ A \otimes_R R[\mathfrak{G}] & \xrightarrow{\tilde{\Delta}} & A \otimes_R R[\mathfrak{G}] \\ \downarrow id \otimes \epsilon_g & & \downarrow id \otimes \epsilon_g \\ A & \xrightarrow{\rho(g)} & A. \end{array}$$

where ϵ_g is the evaluation at g and where the bottom is the compatibility (6.0.2). In terms of schemes, the map below is $\mathfrak{G} = \mathfrak{G} \times_R \text{Spec}(R) \xrightarrow{id_{\mathfrak{G}} \times g} \mathfrak{G} \times_R \mathfrak{G} \xrightarrow{product} \mathfrak{G}$. It follows that $\rho(g).a = a \circ R_g = R_g^*(a)$ for each $a \in A = R[\mathfrak{G}]$ where $R_g : \mathfrak{G} \rightarrow \mathfrak{G}$ is the right translation by g , $x \mapsto xg$.

Let us show that the regular representation $R[\mathfrak{G}]$ is faithful. Let S be an R -ring and let $g \in \mathfrak{G}(S)$ acting trivially on $S[\mathfrak{G}]$. It means that $f \circ R_g = f$ for each $f \in S[\mathfrak{G}]$ hence $f(g) = f(1)$ for each $f \in S[\mathfrak{G}]$. But

$\mathfrak{G}(S) = \text{Hom}_S(S[\mathfrak{G}], S)$, so that $g = 1$. This shows that the regular representation is faithful.

A morphism of \mathfrak{G} -modules is an R -morphism $f : M \rightarrow M'$ such that $f(S) \circ \rho(g) = \rho'(g) \circ f(S) \in \text{Hom}_S(M \otimes_R S, M' \otimes_R S)$ for each S/R and for $g \in \mathfrak{G}(S)$. Equivalently, this is to require the commutativity of the following diagram

$$(6.0.11) \quad \begin{array}{ccc} M & \xrightarrow{f} & M' \\ \downarrow c_\rho & & \downarrow c_{\rho'} \\ M \otimes_R R[\mathfrak{G}] & \xrightarrow{f \otimes id} & M' \otimes_R R[\mathfrak{G}]. \end{array}$$

It is clear that the R -module $\text{coker}(f)$ is equipped then with a natural structure of \mathfrak{G} -module. For the kernel $\ker(f)$, we cannot proceed similarly because the mapping $\ker(f) \otimes_R S \rightarrow \ker(M \otimes_R S \xrightarrow{f(S)} M' \otimes_R S)$ is not necessarily injective. One tries to use the module viewpoint by considering the following commutative exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(f) & \longrightarrow & M & \xrightarrow{f} & M' \\ & & & & \downarrow c_\rho & & \downarrow c_{\rho'} \\ & & \ker(f) \otimes_R R[\mathfrak{G}] & \longrightarrow & M \otimes_R R[\mathfrak{G}] & \xrightarrow{f \otimes id} & M' \otimes_R R[\mathfrak{G}]. \end{array}$$

If \mathfrak{G} is flat, then the left bottom map is injective, and the diagram defines a map $c : \ker(f) \rightarrow \ker(f) \otimes_R R[\mathfrak{G}]$. This map c satisfies the two compatibilities and define then a \mathfrak{G} -module structure on $\ker(f)$. We have proven the important fact.

6.0.12. Proposition. *Assume that \mathfrak{G}/R is flat. Then the category of \mathfrak{G} -modules is an abelian category.*

6.0.13. Remark. It is actually more than an abelian category since it carries tensor products, see below.

6.1. Tensor products. Given two homomorphisms $\rho_1 : h_{\mathfrak{G}} \rightarrow \text{GL}(M_1)$, $\rho_2 : h_{\mathfrak{G}} \rightarrow \text{GL}(M_2)$ we can form the tensor product

$$\rho_1 \otimes \rho_2 : h_{\mathfrak{G}} \rightarrow \text{GL}(M_1 \otimes_R M_2)$$

by means on the homomorphism

$$h_{\mathfrak{G}} \xrightarrow{\rho_1 \times \rho_2} \text{GL}(M_1) \times \text{GL}(M_2) \xrightarrow{\text{tensor representation}} \text{GL}(M_1 \otimes_R M_2)$$

6.1.1. Lemma. *Let $c_i : M_i \rightarrow M_i \otimes_R R[\mathfrak{G}]$ be the coaction for $i = 1, 2$ and let $c : M_1 \otimes_R M_2 \rightarrow (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}]$ be the coaction of the tensor*

representation. Then the following diagram commutes

$$\begin{array}{ccc}
 M_1 \otimes_R M_2 & \xrightarrow{c_1 \otimes c_2} & (M_1 \otimes_R R[\mathfrak{G}]) \otimes_R (M_2 \otimes_R R[\mathfrak{G}]) \xrightarrow{\sim} (M_1 \otimes_R M_2) \otimes_R (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]) \\
 & \searrow c & \downarrow id \otimes mult \\
 & & (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}].
 \end{array}$$

Proof. We need to identify the coaction c of $M_1 \otimes_R M_2$ starting from

$$\tilde{c} = \tilde{c}_1 \otimes \tilde{c}_2 \in \text{End}_{R[\mathfrak{G}]} \left((M_1 \otimes_R R[\mathfrak{G}]) \otimes_{R[\mathfrak{G}]} (M_2 \otimes_R R[\mathfrak{G}]) \right) \xrightarrow{\sim} \text{End}_{R[\mathfrak{G}]} \left((M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}] \right)$$

where the isomorphism arises from the identification

$$(M_1 \otimes_R R[\mathfrak{G}]) \otimes_{R[\mathfrak{G}]} (M_2 \otimes_R R[\mathfrak{G}]) \xrightarrow[\sim]{\alpha} (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}]$$

$$(m_1 \otimes a_1) \otimes (m_2 \otimes a_2) \mapsto (m_1 \otimes m_2) \otimes (a_1 a_2).$$

We consider then the following commutative diagram

$$\begin{array}{ccccc}
 (M_1 \otimes_R R[\mathfrak{G}]) \otimes_{R[\mathfrak{G}]} (M_2 \otimes_R R[\mathfrak{G}]) & \xrightarrow[\sim]{\alpha} & (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}] & \longleftarrow & M_1 \otimes_R M_2 \\
 \downarrow \tilde{c}_1 \otimes \tilde{c}_2 & & \downarrow \tilde{c} & \swarrow c & \\
 (M_1 \otimes_R R[\mathfrak{G}]) \otimes_{R[\mathfrak{G}]} (M_2 \otimes_R R[\mathfrak{G}]) & \xrightarrow[\sim]{\alpha} & (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}] & &
 \end{array}$$

It follows that $c(m_1 \otimes m_2) = \alpha(c_1(m_1) \otimes c_2(m_2))$ whence the desired statement. \square

In particular, if M_1 is a trivial \mathfrak{G} -module, we have $c(m_1 \otimes m_2) = m_1 \otimes c_2(m_2)$ so that $c = id_{M_1} \otimes c_2$. The coaction has then an interpretation with tensor products.

6.1.2. Lemma. *Let M be a \mathfrak{G} -module with coaction c and denote by $\tilde{c} \in \text{End}_{R[\mathfrak{G}]}(M \otimes_R R[\mathfrak{G}])$ the action of the universal element of \mathfrak{G} . Let M_{tr} be the underlying trivial \mathfrak{G} -module and consider the tensor structure on the R -modules $M_{tr} \otimes_R R[\mathfrak{G}]$ and $M \otimes_R R[\mathfrak{G}]$.*

(1) $c : M \rightarrow M_{tr} \otimes_R R[\mathfrak{G}]$ is a \mathfrak{G} -morphism.

(2) $\tilde{c} : M \otimes_R R[\mathfrak{G}] \rightarrow M_{tr} \otimes_R R[\mathfrak{G}]$ is a \mathfrak{G} -isomorphism.

Proof. (1) The coaction of $M_{tr} \otimes_R R[\mathfrak{G}]$ is $id_M \otimes \Delta$ so that the first rule

$$\begin{array}{ccc}
 M & \xrightarrow{c} & M \otimes_R R[\mathfrak{G}] \\
 c \downarrow & & id \otimes \Delta_{\mathfrak{G}} \downarrow \\
 M \otimes_R R[\mathfrak{G}] & \xrightarrow{c \otimes id} & M \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}],
 \end{array}
 \tag{6.1.3}$$

so that the top horizontal map provides a \mathfrak{G} -morphism $M \rightarrow M_{tr} \otimes_R R[\mathfrak{G}]$.

(2) By using the viewpoint of representations $\tilde{c} : M \otimes_R R[\mathfrak{G}] \rightarrow M \otimes_R R[\mathfrak{G}]$ (which is defined by $\tilde{c}(m \otimes a) = c(m)a$) is a \mathfrak{G} -morphism. It is invertible as we have seen in the beginning of §6. \square

6.2. Representations of diagonalizable group schemes. Let $\mathfrak{G} = \mathfrak{D}(A)/R$ be a diagonalizable group scheme. For each $a \in A$, we can attach a character $\chi_a = \eta_A(a) : \mathfrak{D}(A) \rightarrow \mathbb{G}_m = \mathrm{GL}_1(R)$. It defines then a \mathfrak{G} -structure on the R -module R .

To identify the relevant coaction, we use again Yoneda's technique by considering the homomorphism $\chi_{a,*} = \mathfrak{D}(A)(R[A]) \rightarrow \mathbb{G}_m(R[A]) = R[A]^\times$ and the image of the universal element which is e_a in view of Remark 3.4.2. It follows that the coaction is defined by $\tilde{c}_a : R[A] \xrightarrow{\sim} R[A]$, $u \mapsto e_a u$ so that we have $c_a(r) = r \otimes e_a \in R \otimes_R R[A] = R[A]$.

If $M = \bigoplus_{a \in A} M_a$ is an A -graded R -module, the group scheme $\mathfrak{D}(A)$ acts diagonally on it by χ_a on each piece M_a .

We have constructed a covariant functor from the category of graded A -modules to the category of representations of $\mathfrak{D}(A)$.

6.2.1. Proposition. *The functor above is an equivalence of abelian categories from the category of A -graded R -modules to the category of R - $\mathfrak{D}(A)$ -modules.*

Proof. Step 1: full faithfulness. Let M_\bullet and N_\bullet be A -graded modules. We have maps

$$\prod_{a \in A} \mathrm{Hom}_R(M_a, N_a) \rightarrow \mathrm{Hom}_{\mathfrak{D}(A)\text{-mod}}(M_\bullet, N_\bullet) \hookrightarrow \prod_{a, b \in A} \mathrm{Hom}_R(M_a, N_b).$$

It is then enough to show that $\mathrm{Hom}_R(M_a, N_b) = 0$ if $a \neq b$. For $a \neq b$, let $f : M_a \rightarrow N_b$ be a morphism of $\mathfrak{D}(A)$ -modules. Then for $l \in M_a$, we have $c_{N_b}(f(m)) = f(c_{M_a}(m))$ so that $f(m) \otimes e_b = f(m \otimes e_a) = f(m) \otimes e_a \in N_b \otimes R[A]$. Since $R[A] = \bigoplus_{a \in A} R e_a$, we conclude that $f(m) = 0$. We conclude that $\mathrm{Hom}_R(M_a, N_b) = 0$ if $a \neq b$.

Step 2: Essential surjectivity. Let M be an R - $\mathfrak{D}(A)$ -module and consider the underlying map $c : M \rightarrow M \otimes_R R[A]$. We write $c(m) = \sum_{a \in A} \varphi_a(m) \otimes e_a$.

We apply the first rule (6.0.6), that is, the commutativity of

$$(6.2.2) \quad \begin{array}{ccc} M & \xrightarrow{c} & M \otimes_R R[A] \\ c \downarrow & & \downarrow id \otimes \Delta \\ M \otimes_R R[A] & \xrightarrow{c \otimes id} & M \otimes_R R[A] \otimes_R R[A]. \end{array}$$

We have then

$$(c \otimes id)(c(m)) = (c \otimes id) \left(\sum_{a \in A} \varphi_a(m) \otimes e_a \right) = \sum_{b \in A} \sum_{a \in A} \varphi_b(\varphi_a(m)) e_b \otimes e_a.$$

On the other hand we have

$$(id \otimes \Delta)(c(m)) = (id \otimes \Delta)\left(\sum_{a \in A} \varphi_a(m) \otimes e_a\right) = \sum_{a \in A} \varphi_a(m) e_a \otimes e_a.$$

It follows that

$$\varphi_b \circ \varphi_a = \delta_{a,b} \varphi_a \quad (a, b \in A)$$

We consider also the other compatibility (6.0.7)

$$(6.2.3) \quad \begin{array}{ccc} M & \xrightarrow{c} & M \otimes_R R[A] \\ id \downarrow & \swarrow id \times \epsilon^* & \\ M & & \end{array}$$

It implies that

$$m = id \times \epsilon^*\left(\sum_{a \in A} \varphi_a(m) e_a\right) = \sum_{a \in A} \varphi_a(m).$$

We obtain that

$$\sum_{a \in A} \varphi_a = id_M.$$

Hence the φ_a 's are pairwise orthogonal projectors whose sum is the identity. Thus $M = \bigoplus_{a \in A} \varphi_a(M)$ which decomposes a direct summand of eigenspaces as desired. \square

6.2.4. Corollary. *Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be an exact sequence of $R - \mathfrak{D}(A)$ -modules.*

(1) *For each $a \in A$, it induces an exact sequence $0 \rightarrow (M_1)_a \rightarrow (M_2)_a \rightarrow (M_3)_a \rightarrow 0$.*

(2) *The sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ splits as sequence of $R - \mathfrak{D}(A)$ -modules if and only if it splits as sequence of R -modules.*

Proof. (1) It readily follows of the equivalence of categories stated in Proposition 6.2.1.

(2) The direct sense is obvious. Conversely, let $s : M_3 \rightarrow M_2$ be a splitting. Then for each $a \in A$, the composite $(M_3)_a \rightarrow M_3 \xrightarrow{s} M_2 \xrightarrow{\varphi_a} (M_2)_a$ provides the splitting of $(M_2)_a \rightarrow (M_3)_a$. \square

We record also the following property.

6.2.5. Corollary. *Let M be an $R - \mathfrak{G}$ -module. Then for each S/R and for each $a \in A$, we have $M_a \otimes_R S = (M_a \otimes_R S)_a$.*

6.2.6. Corollary. *Assume that R is a field. Then the category of representations of $\mathfrak{D}(A)$ is semisimple abelian category, that is, all short exact sequences split [KS, 8.3.16].*

Proof. Since the category of k -vector spaces is semisimple so is the category of A -graded vector spaces. Proposition 6.2.1 shows that the category of representations of $\mathfrak{D}(A)$ is semisimple. \square

It is also of interest to know kernels of representations.

6.2.7. Lemma. *Let $A^\#$ be a finite subset of A and denote by A_0 the subgroup generated by $A^\#$. We consider the representation $M = \bigoplus_{a \in A^\#} R^{n_a}$ of $\mathfrak{G} = \mathfrak{D}(A)$, with $n_a \geq 1$. Then the representation $\rho : \mathfrak{G} \rightarrow \mathrm{GL}(M)$ factorizes as*

$$\mathfrak{G} = \mathfrak{D}(A) \rightarrow \mathfrak{D}(A_0) \xrightarrow{\rho_0} \mathrm{GL}(M)$$

where ρ_0 is a closed immersion. Furthermore $\ker(\rho) = \mathfrak{D}(A/A_0)$ is a closed subgroup scheme.

Proof. First case: $A = A_0$. Then the map $\mathfrak{G} \rightarrow \mathrm{GL}(M)$ factorizes by the closed subgroup scheme $T = \prod_{a \in A^\#} \mathbb{G}_{m,R}^{n_a}$. Since the map $\widehat{T} \rightarrow A_0 = A$ is onto, the map $\mathfrak{G} \rightarrow \prod_{a \in A^\#} \mathbb{G}_{m,R}^{n_a}$ is a closed immersion (Proposition 3.5.2).

A composite of closed immersions being a closed immersion, ρ is a closed immersion.

General case. The representation $\rho : \mathfrak{G} \rightarrow \mathrm{GL}(M)$ factorizes as

$$\mathfrak{G} = \mathfrak{D}(A) \xrightarrow{q} \mathfrak{D}(A_0) \xrightarrow{\rho_0} \mathrm{GL}(M)$$

where ρ_0 is a closed immersion. It follows that $\ker(\rho) = \ker(q)$. This kernel $\ker(q)$ is $\mathfrak{D}(A/A_0)$ and is a closed subgroup scheme of \mathfrak{G} (*ibid*). \square

6.2.8. Remark. If R is a field, all finite dimensional representations of $\mathfrak{D}(A)$ are of this shape, so one knows the kernel of each finite dimensional representation.

6.3. Existence of faithful finite dimensional representations (field case). Let k be a field and let \mathfrak{G} be an affine k -group.

6.3.1. Theorem. *Let V be a k – \mathfrak{G} -representation. Then $V = \varinjlim_{i \in I} V_i$ where V_i runs over the f.d. subrepresentations of V .*

Proof. We write $c : V \rightarrow V \otimes_k k[\mathfrak{G}]$ for the coaction. A sum of f.d. subrepresentations of \mathfrak{G} is again one, so it is enough to show that each $v \in V$ belongs in some finite-dimensional subrepresentation. Let $(a_i)_{i \in I}$ be a basis of the k -vector space A . We write $c(v) = \sum_{i \in I} v_i \otimes a_i$, where all but finitely many v_i 's are zero. Next we have $\Delta(a_i) = \sum_{j,l \in I} r_{i,j,l} a_j \otimes_k a_l$. Using the first rule (6.0.6) of comodules we have

$$\sum_{i \in I} c(v_i) \otimes a_i = (c \otimes id)(c(v)) = (id \otimes \Delta)c(v) = \sum_{i,j,l} r_{i,j,l} v_i \otimes a_j \otimes a_l.$$

Comparing the coefficients, we get $c(v_l) = \sum_{i,j \in I} r_{i,j,l} v_i \otimes a_j$. Hence the subspace W spanned by v and the v_i 's is a subrepresentation. \square

6.3.2. Theorem. *Assume that \mathfrak{G} is algebraic, that is, the k -algebra $k[\mathfrak{G}]$ is of finite type. Then \mathfrak{G} admits a finite dimensional faithful k -representation V .*

Proof. We start with the regular representation V of G which is faithful in view of Example 6.0.9. We write $V = \varinjlim_{i \in I} V_i$ as in the previous theorem where the V_i 's are finite dimensional. We put $\mathfrak{H}_i = \ker(\mathfrak{G} \rightarrow \mathrm{GL}(V_i))$, this is a closed k -subgroup of G . For each k -algebra S , we have

$$\bigcap_i \mathfrak{H}_i(S) = \ker(\mathfrak{G}(S) \rightarrow \mathrm{GL}(V)(S)) = 1.$$

We put $\mathfrak{H} = \bigcap_i \mathfrak{H}_i$, this is a closed k -subgroup of \mathfrak{G} with trivial functor of points so that $H = 1$. We write $k[\mathfrak{H}_i] = k[\mathfrak{G}]/J_i$. Then

$$\ker(R[\mathfrak{G}] \rightarrow R) = \bigoplus_{i \in I} J_i.$$

Since the ring $k[\mathfrak{G}]$ is a noetherian ring, its ideals are finitely generated so that there exists $i_1, \dots, i_c \in I$ such that $\ker(R[\mathfrak{G}] \rightarrow R) = J_{i_1} + \dots + J_{i_c}$. We consider the index $i \in I$ defined by $V_i = V_{i_1} + \dots + V_{i_c}$. We have $\mathfrak{H}_i = \mathfrak{H}_{i_1} \cap \dots \cap \mathfrak{H}_{i_c}$ so that $J_i = J_{i_1} + \dots + J_{i_c} = \ker(R[\mathfrak{G}] \rightarrow R)$. Thus $\mathfrak{H}_i = 1$ and V_i is a faithful representation of \mathfrak{G} . \square

6.3.3. Remark. We will see later that a monomorphism of affine algebraic k -group is a closed immersion, see also [DG, §III.7.2] or [Mi2, thm. 3.34]. An easier thing to do is to upgrade Theorem 6.3.2 by requiring that the homomorphism is a closed immersion, see [Wa, Thm. 3.4].

6.4. Existence of faithful finite rank representations. This question is rather delicate for general groups and general rings, see [SGA3, VI_B.13] and the paper [Th] by Thomason. Over a field or a Dedekind ring, faithful representations occur.

6.4.1. Theorem. *Assume that R is a Dedekind ring (e.g. DVR). Let \mathfrak{G}/R be a flat affine group scheme of finite type. Then there exists a faithful \mathfrak{G} -module M which is f.g. free as R -module.*

The key thing is the following fact due to Serre [Se4, §1.5, prop. 2].

6.4.2. Proposition. *Assume that R is noetherian and let \mathfrak{G}/R be an affine flat group scheme. Let M be a \mathfrak{G} -module. Let N be an R -submodule of M of finite type. Then there exists an R - \mathfrak{G} -submodule \tilde{N} of M which contains N and is f.g. as R -module.*

We can now proceed to the proof of Theorem 6.4.1. We take $M = R[\mathfrak{G}]$ seen as the regular representation, it is faithful (Example 6.0.9). The proposition shows that M is the direct limit of the family of \mathfrak{G} -submodules $(M_i)_{i \in I}$ which are f.g. as R -modules. The M_i 's are torsion-free so are flat. Hence the M_i 's are projective (in view of Lemma 5.2.1).

We look at the kernel \mathfrak{N}_i/R of the representation $\mathfrak{G} \rightarrow \mathrm{GL}(M_i)$. The regular representation is faithful and its kernel is the intersection of the \mathfrak{N}_i . Since \mathfrak{G} is a noetherian scheme, there is an index i such that $\mathfrak{N}_i = 1$ (argument of the proof of Theorem 6.3.2). In other words, the representation $\mathfrak{G} \rightarrow \mathrm{GL}(M_i)$ is faithful. Now M_i is a direct summand of a free module R^n , i.e. $R^n = M_i \oplus M'_i$. It provides a representation $\mathfrak{G} \rightarrow \mathrm{GL}(M_i) \rightarrow \mathrm{GL}(M_i \oplus M'_i)$ which is faithful and such that the underlying module is free.

An alternative proof is §1.4.5 of [BT2] which shows that the provided representation $\mathfrak{G} \rightarrow \mathrm{GL}(M)$ is actually a closed immersion. This occurs as special case of the following result.

6.4.3. Theorem. (*Raynaud-Gabber* [SGA3, VI_B.13.2]) *Assume that R is a regular noetherian ring of dimension ≤ 2 . Let \mathfrak{G}/R be a flat affine group scheme of finite type. Then there exists a \mathfrak{G} -module M isomorphic to R^n as R -module such that $\rho_M : \mathfrak{G} \rightarrow \mathrm{GL}_n(R)$ is a closed immersion.*

Finally there are examples due to Grothendieck of rank two tori over the local ring of a nodal curve which do not admit a faithful representation [SGA3, X.1.6], see also [G2, §3].

6.5. Hochschild cohomology. We assume that \mathfrak{G} is flat. If M is a \mathfrak{G} -module, we consider the R -module of invariants $M^{\mathfrak{G}}$ defined by

$$M^{\mathfrak{G}} = \left\{ m \in M \mid m \otimes 1 = c(m) \in M \otimes_R R[\mathfrak{G}] \right\}.$$

It is the largest trivial \mathfrak{G} -submodule of M and we have also $M^{\mathfrak{G}} = \mathrm{Hom}_{\mathfrak{G}}(R, M)$ and is denoted by $H^0(\mathfrak{G}, M)$.

6.5.1. Example. For an R -module N , we consider the tensor product $N \otimes_R R[\mathfrak{G}]$. We claim that the map $N \rightarrow N \otimes_R R[\mathfrak{G}]$ induces an isomorphism

$$N \xrightarrow{\sim} H^0(\mathfrak{G}, N \otimes_R R[\mathfrak{G}]).$$

Clearly the above map is injective. Conversely let $\sum_i n_i \otimes a_i \in H^0(\mathfrak{G}, N \otimes_R R[\mathfrak{G}])$. Then we have

$$\sum_i n_i \otimes a_i \otimes 1 = c\left(\sum_i n_i \otimes a_i\right) = \sum_i n_i \otimes \Delta(a_i) \in N \otimes_R R[\mathfrak{G}] \otimes R[\mathfrak{G}].$$

By applying $id \otimes \epsilon \otimes id$, we get $\sum_i n_i \otimes \epsilon(a_i) = \sum_i n_i \otimes a_i$ so that $\sum_i n_i \otimes a_i$ belongs to N .

We can then mimic the theory of cohomology of groups.

6.5.2. Lemma. *The category of $R - \mathfrak{G}$ -modules has enough injective.*

We shall use the following extrem case of induction, see [J, §2, 3] for the general theory.

6.5.3. Lemma. (*Frobenius reciprocity*) *Let N be an R -module. Then for each \mathfrak{G} -module M the mapping*

$$\psi : \mathrm{Hom}_{\mathfrak{G}}(M, N \otimes_R R[\mathfrak{G}]) \rightarrow \mathrm{Hom}_R(M, N),$$

given by taking the composition with the augmentation map, is an isomorphism.

Proof. We define first the converse map. We are given an R -map $f_0 : M \rightarrow N$ and consider the following map of \mathfrak{G} -modules

$$M \xrightarrow{c_M} M_{tr} \otimes_R R[\mathfrak{G}] \xrightarrow{f_0 \otimes id} N \otimes_R R[\mathfrak{G}]$$

where we use again Lemma 6.1.2.(1). By construction we have $\psi(f) = f_0$. In the way around we are given a \mathfrak{G} -map $h : M \rightarrow N \otimes_R R[\mathfrak{G}]$ and denote by $h_0 : M \rightarrow N \otimes_R R[\mathfrak{G}] \xrightarrow{id \otimes \epsilon} N \rightarrow 0$. We consider the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & N \otimes_R R[\mathfrak{G}] \\ c_M \downarrow & & id \times \Delta_{\mathfrak{G}} \downarrow \\ M \otimes_R R[\mathfrak{G}] & \xrightarrow{h \otimes id} & N \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \\ id \times \epsilon^* \downarrow & & id \times \epsilon^* \downarrow \\ M & \xrightarrow{h \otimes id} & N \otimes_R R[\mathfrak{G}]. \end{array}$$

The composite $N \otimes_R R[\mathfrak{G}] \xrightarrow{id \times \Delta} N \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \xrightarrow{id_N \times \epsilon^* \times id} N \otimes_R R[\mathfrak{G}]$ is the same than the left vertical map and is equal to $h_0 \otimes id$. Thus $h = h_0 \otimes id$ as desired. \square

6.5.4. Remark. Now if N is a \mathfrak{G} -module, we remind the canonical isomorphism $N \otimes_R R[\mathfrak{G}] \xrightarrow{\sim} N_{tr} \otimes_R R[\mathfrak{G}]$ of \mathfrak{G} -modules where N_{tr} denotes the underlying R -module seen as trivial $R - \mathfrak{G}$ -module (Lemma 6.1.2). Lemma 6.5.3 gives rise then to an isomorphism

$$\mathrm{Hom}_{\mathfrak{G}}(M, N \otimes_R R[\mathfrak{G}]) \xrightarrow{\sim} \mathrm{Hom}_R(M, N),$$

for any \mathfrak{G} -module M .

We can proceed to the proof of Lemma 6.5.2.

Proof. The argument is similar as Godement's one in the case of sheaves. Let M be a \mathfrak{G} -module and let us embed the R -module M_{tr} in some injective R -module I (this exists, see [We, Exercise 2.3.5]). Then we consider the following injective \mathfrak{G} -map

$$M \xrightarrow{c_M} M_{tr} \otimes_R R[\mathfrak{G}] \rightarrow I \otimes_R R[\mathfrak{G}]$$

where we use Lemma 6.1.2. We claim that $I \otimes_R R[\mathfrak{G}]$ is an injective \mathfrak{G} -module. We consider a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{i} & N' \\ & & f \downarrow & & \\ & & I \otimes_R R[\mathfrak{G}]. & & \end{array}$$

From Frobenius reciprocity (i.e. Lemma 6.5.3), we have the following

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{G}}(N', I \otimes_R R[\mathfrak{G}]) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathfrak{G}}(N, I \otimes_R R[\mathfrak{G}]) \\ \cong \downarrow & & \cong \downarrow \\ \mathrm{Hom}_R(N', I) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathfrak{G}}(N, I). \end{array}$$

Since I is an injective R -module, the bottom map is onto. Thus f extends to a \mathfrak{G} -map $N' \rightarrow I \otimes_R R[\mathfrak{G}]$. \square

We can then take the right derived functors of the left exact functor $R - \mathfrak{G} - \mathrm{mod} \rightarrow R - \mathrm{Mod}$, $M \rightarrow M^{\mathfrak{G}} = H_0^0(\mathfrak{G}, M)$, see [We, §2.5]. It defines the Hochschild cohomology groups $H_0^i(\mathfrak{G}, M)$. If $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is an exact sequence of \mathfrak{G} -modules, we have the long exact sequence

$$\cdots \rightarrow H_0^i(\mathfrak{G}, M_1) \rightarrow H_0^i(\mathfrak{G}, M_2) \rightarrow H_0^i(\mathfrak{G}, M_3) \xrightarrow{\delta_i} H_0^{i+1}(\mathfrak{G}, M_1) \rightarrow \cdots$$

6.5.5. Lemma. *Let M be an $R[\mathfrak{G}]$ -module. Then $M \otimes_R R[\mathfrak{G}]$ is acyclic, i.e. satisfies*

$$H_0^i(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0 \quad \forall i \geq 1.$$

Proof. We embed the M in an injective R -module I and put $Q = I/M$. The sequence of \mathbf{G} -modules

$$0 \rightarrow M \otimes_R R[\mathfrak{G}] \rightarrow I \otimes_R R[\mathfrak{G}] \rightarrow Q \otimes_R R[\mathfrak{G}] \rightarrow 0$$

is exact. We have seen that $I \otimes_R R[\mathfrak{G}]$ is injective, so that $H_0^i(\mathfrak{G}, I \otimes_R R[\mathfrak{G}]) = 0$ for each $i > 0$. The long exact sequence induces an exact sequence

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathfrak{G}}(R, I \otimes_R R[\mathfrak{G}]) & \longrightarrow & \mathrm{Hom}_{\mathfrak{G}}(R, Q \otimes_R R[\mathfrak{G}]) & \longrightarrow & H_0^1(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) & \rightarrow & 0 \\ \cong \downarrow & & \cong \downarrow & & & & \\ I & \longrightarrow & Q & \longrightarrow & & & 0. \end{array}$$

Therefore $H_0^1(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0$. The isomorphisms

$$H_0^i(\mathfrak{G}, Q \otimes_R R[\mathfrak{G}]) \xrightarrow{\sim} H^{i+1}(\mathfrak{G}, M \otimes_R R[\mathfrak{G}])$$

permits to use the standard shifting argument to conclude that $H^{i+1}(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0$ for each $i \geq 0$. \square

As in the usual group cohomology, these groups can be computed by means of cocycles. This provides a description of $H_0^i(\mathfrak{G}, M)$ in terms of Hochschild cocycles, see [DG, II.3] or [J, §4.16] for details. A n -cocycle (resp. a boundary) in this setting is the data of a n -cocycle (resp. a boundary) $c(S) \in Z^n(\mathfrak{G}(S), M \otimes_R S)$ in the usual sense and which agree with base changes.

6.5.6. Remark. In particular, there is a natural map $Z^n(\mathfrak{G}, M) \rightarrow Z^n(\mathfrak{G}(R), M)$. If $\mathfrak{G} = \Gamma_R$ is finite constant, this map induces an isomorphism $H_0^*(\Gamma, M) \xrightarrow{\sim} H_0^*(\Gamma, M)$ with the usual group cohomology (see [DG, II.3.4]).

We can state an important vanishing statement.

6.5.7. Theorem. (*Grothendieck*) Let $\mathfrak{G} = \mathfrak{D}(A)$ be a diagonalizable group scheme. Then for each \mathfrak{G} -module M , we have $H^i(\mathfrak{G}, M) = 0$ for each $i \geq 1$.

Proof. Once again we embed M in $M_{tr} \otimes_R R[A]$, it is a direct summand as R -module. According to Corollary 6.2.4, the \mathfrak{G} -module M is a direct summand of the flasque \mathfrak{G} -module $M \otimes_R R[\mathfrak{G}]$ (see Lemma 6.5.5). Hence M is flasque as well and has trivial cohomology (for $i \geq 1$). \square

6.6. First Hochschild cohomology group. We just focus on H^1 and H^2 . Then

$$H_0^1(\mathfrak{G}, M) = Z_0^1(\mathfrak{G}, M) / B_0^1(\mathfrak{G}, M)$$

are given by equivalence of Hochschild 1-cocycles. More precisely, a 1-cocycle (or crossed homomorphism) is an R -functor

$$z : h_{\mathfrak{G}} \rightarrow W(M)$$

which satisfies the following rule for each algebra S/R

$$z(g_1 g_2) = z(g_1) + g_1 \cdot z(g_2) \quad \forall g_1, g_2 \in \mathfrak{G}(S).$$

Note that $z(1_S 1_S) = z(1_S) + z(1_S)$ so that $z(1_S) = 0$. The coboundaries are of the shape $g \cdot m \otimes 1 - m \otimes 1$ for $m \in M$. As in the classical case, we can attach to $z \in Z_0^1(\mathfrak{G}, M)$ an R -map

$$s_z \in \text{Hom}_{R\text{-func}}(h_{\mathfrak{G}}, W(M) \rtimes h_{\mathfrak{G}})$$

defined by

$$s_z(g) = (z(g), g) \in (M \otimes_R S) \rtimes \mathfrak{G}(S)$$

for each R -algebra S and each $g \in \mathfrak{G}(S)$. We have the following dictionary.

6.6.1. Lemma. (1) *The assignment*

$$Z_0^1(\mathfrak{G}, M) \rightarrow \text{Hom}_{R\text{-func}}(h_{\mathfrak{G}}, W(M) \rtimes h_{\mathfrak{G}}), \quad z \mapsto s_z,$$

is a bijection between $Z_0^1(\mathfrak{G}, M)$ and the homomorphic sections of the homomorphism of R -group functors $W(M) \rtimes h_{\mathfrak{G}} \rightarrow h_{\mathfrak{G}}$.

(2) *Furthermore it induces a bijection between $Z_0^1(\mathfrak{G}, M)$ and the set of M -conjugacy classes of those sections.*

Proof. (1) Let us check first that s_z is a homomorphic section of the map $W(M) \rtimes h_{\mathfrak{G}} \rightarrow h_{\mathfrak{G}}$. Let S be an R -algebra and let $g_1, g_2 \in \mathfrak{G}(S)$. We have

$$s_z(g_1) s_z(g_2) = (z(g_1), g_1) (z(g_2), g_2) = (z(g_1) + g_1 \cdot z(g_2), g_1 g_2) = (z(g_1 g_2), g_1 g_2) = s_z(g_1 g_2)$$

by using the cocycle condition. Since $s_z(1_S) = (0, 1_S)$, s_z is an homomorphic section of $(M \otimes_R S) \rtimes \mathfrak{G}(S) \rightarrow \mathfrak{G}(S)$.

Conversely we are given a homomorphic section s of $W(M) \rtimes h_{\mathfrak{G}} \rightarrow h_{\mathfrak{G}}$. For each R -algebra S , it is of the shape $s(g) = (a(g), g)$ for each $g \in \mathfrak{G}(S)$ with $a(g) \in M \otimes_R S$. The above computation shows that $a : \mathfrak{G}(S) \rightarrow M \otimes_R S$ satisfies the cocycle relation. The functoriality in S enables us to conclude that a is an Hochschild 1-cocycle.

(2) For an homomorphic section s and $m \in M$, we consider the homomorphic section ${}^m s$ defined by ${}^m s : \mathfrak{G}(S) \rightarrow (M \otimes_R S) \rtimes \mathfrak{G}(S)$; i.e. by $({}^m s)(g) = m s(g) m^{-1}$. We have

$$({}^m s)(g) = (m, 1_S) (a(g), g) (-m, 1_S) = (m+a(g), g) (-m, 1_S) = (m+a(g)-g.m, g).$$

The dictionary tells us that s_z and $s_{z'}$ are M -conjugated if and only if z and z' are cohomologous. \square

6.7. H^2 and group extensions. A 2-cocycle for \mathfrak{G} and M is the data for each S/R of a 2-cocycle $f(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \rightarrow M \otimes_R S$ in a compatible way. It satisfies the rule

$$g_1 \cdot f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = 0$$

for each S/R and each $g_1, g_2, g_3 \in \mathfrak{G}(S)$. The 2-cocycle c is normalized if it satisfies furthermore the rule

$$f(g, 1) = f(1, g) = 0.$$

each S/R and each $g \in \mathfrak{G}(S)$. Up to add a coboundary, we can always deal with normalized cocycles. The link in the usual theory between normalized classes and group extensions [We, §6.6] extends mechanically. Given a normalized Hochschild cocycle $c \in Z^2(\mathfrak{G}, M)$, we can define the following group law on the R -functor $W(M) \times \mathfrak{G}$ by

$$(m_1, g_1) \cdot (m_2, g_2) = (m_1 + g_1 \cdot m_2 + c(g_1, g_2), g_1 g_2)$$

for each S/R and each $m \in M \otimes_R S$ and $g \in \mathfrak{G}(S)$. In other words, we defined a group extension E_f of R -functors in groups of $h_{\mathfrak{G}}$ by $W(M)$.

In the way around, we are given an extension

$$0 \rightarrow W(M) \rightarrow E \rightarrow h_{\mathfrak{G}} \rightarrow 1$$

of R -functors in groups. Since $E \rightarrow h_{\mathfrak{G}}$ is an epimorphism, the universal point $g^{univ} \in \mathfrak{G}(R[\mathfrak{G}])$ lifts to an element $e \in E(R[\mathfrak{G}])$ (see §2.2). In other words we have a section $s : h_{\mathfrak{G}} \rightarrow E$ and we will associate a 2-cocycle which measures how far s is a homomorphism. As in abstract group case [We, th. 6.6.3], for each R -ring S , we set

$$c_s(g_1, g_2) = s(g_1) s(g_2) s(g_1 g_2)^{-1} \quad (g_1, g_2 \in \mathfrak{G}(S)).$$

We can check that c_s is a normalized 2-cocycle and that two normalized cocycles c, c' are cohomologous if and only if the extensions E_c and $E_{c'}$ are isomorphic. Now we denote by $\text{Ext}_{R\text{-functor}}(\mathfrak{G}, W(M))$ the abelian group of classes of extensions (equipped with the Baer sum as defined in the classical setting in [Bn, IV, exercise 1]) of R -group functors of $h_{\mathfrak{G}}$ by $W(M)$ with the given action $h_{\mathfrak{G}} \rightarrow \text{GL}(M)$.

The 0 is the class of the semi-direct product $W(M) \rtimes h_{\mathfrak{G}}$. As in the classical case, it provides a nice description of the H^2 .

6.7.1. Theorem. [DG, II.3.2] *The construction above induces a group isomorphism $H_0^2(\mathfrak{G}, M) \xrightarrow{\sim} \text{Ext}_{R\text{-functor}}(\mathfrak{G}, W(M))$.*

As consequence of the vanishing theorem 6.5.7, we get the following

6.7.2. Corollary. *Let A be an abelian group and let M be a $\mathfrak{D}(A)$ -module. Let $0 \rightarrow W(M) \rightarrow F \rightarrow \mathfrak{D}(A) \rightarrow 1$ be a group R -functor extension. Then F is the semi-direct product of $\mathfrak{D}(A)$ by $W(M)$ and all sections of $F \rightarrow \mathfrak{D}(A)$ are M -conjugated.*

6.8. Linearly reductive algebraic groups. Let k be a field and let G/k be an affine algebraic group. Recall that a $k - G$ -module V is simple if 0 and V are its only G -submodules. Note that simple $k - G$ -module are finite dimensional according to Proposition 6.4.2. A $k - G$ -module is semisimple if it is a direct sum of its simple submodules.

6.8.1. Definition. *The k -group \mathfrak{G} is linearly reductive if each finite dimensional representation of \mathfrak{G} is semisimple.*

We have seen (in an exercise) that diagonalizable groups are linearly reductive. An important point is that this notion is stable by base change and is geometrical, namely G is linearly reductive and only if $G \times_k \bar{k}$ is linearly reductive (see [Mg, prop. 3.2]). Exactly as in the case of diagonalizable groups, we have the following vanishing statement.

6.8.2. Theorem. *Assume that the affine algebraic group G/k is linearly reductive. Then the category of G -modules is semisimple and for each representation V of G , we have $H_0^i(G, V) = 0$ for each $i > 0$.*

Proof. We have to show that each short exact sequence $0 \rightarrow V' \rightarrow V \xrightarrow{p} V'' \rightarrow 0$ of G -modules split.

Step 1: V is finite dimensional. This is clear by decomposing it in a direct sum of simple representations.

Step 2: V'' is finite dimensional. We write $V = \varinjlim_{i \in I} V_i$ of its f.g. G -submodules (Thm. 6.3.1). Then the above sequence induces sequences of \mathfrak{G} -modules

$$0 \rightarrow V'_i \rightarrow V_i \rightarrow V''_i \rightarrow 0$$

which are split. For i large enough, we have $V''_i = V''$ so the sequence is split.

Step 3: General case. We consider the set \mathcal{E} of the pairs (W, s) where W is a G -submodule of V'' and $s : W \rightarrow V''$ is a G -homomorphism such that $p \circ s : W \rightarrow V''$ is the inclusion map. This set is partially ordered, we say that $(W_1, s_1) \leq (W_2, s_2)$ if $W_1 \subseteq W_2$ and $s_{2|W_1} = s_1$. Clearly \mathcal{E} admits upper bounds for every chain so Zorn's lemma provides a maximal element (W, s) of \mathcal{E} . Assume that $W \subsetneq V''$ and pick $x \in V'' \setminus W$. Then x belongs to a finite dimensional G -submodule V''_x in view of Theorem 6.3.1; at least one of the simple G -submodule V''_0 of V''_x is not included in W . Since V_0 is simple, we have $W \cap V''_0 = 0$ hence a direct sum $W \oplus V''_0 \subseteq W$. By the step 2, there exists a section $s_0 : V''_0 \rightarrow V$ so that $s \oplus s_0$ extends s . This contradicts the maximality of W . Thus $W = V''$ and we are done.

The argument for the vanishing of Hochschild cohomology is then the same than for diagonalizable groups. We embed a representation V in $V \otimes_k k[G]$ so that V is a direct summand of $V \otimes_k k[G]$. But $V \otimes_k k[G]$ is flasque (see Lemma 6.5.5), so that $H^i(G, V) = 0$ for all $i \geq 1$. \square

6.8.3. Corollary. *Under the assumptions of Theorem 6.8.2, each extension of group functors of G by a vector algebraic group $W(M)$ (M finite dimensional representation of G) splits. Furthermore M acts transitively on the sections of $W(M) \rtimes G \rightarrow G$.*

Proof. This follows of the interpretation of $0 = H_0^2(G, V)$ in terms of group extensions (Thm. 6.7.1) and $0 = H_0^1(G, V)$ in terms of sections (Lemma 6.6.1.(2)). \square

The smooth connected linearly reductive groups are the reductive groups in characteristic zero and only the tori in positive characteristic (Nagata, see [DG, IV.3.3.6]).

For example, GL_n (for $n \geq 2$) is reductive in characteristic zero but not over a field of positive characteristic.

6.8.4. Example. Let k be a field. The additive k -group \mathbb{G}_a is not linearly reductive. We consider the representation $\rho : \mathbb{G}_a \rightarrow \mathrm{GL}_2$,

$$x \mapsto \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}$$

Then the second projection $p_2 : k^2 \rightarrow k$ is a \mathbb{G}_a -homomorphism with k the trivial representation. We have $H^0(\mathbb{G}_a, k \oplus k) = k.e_1$ and it does surject by p_2 on k . The exact sequence $0 \rightarrow k \rightarrow k \oplus k \xrightarrow{p_2} k \rightarrow 0$ is then not split. Furthermore it induces a sequence

$$0 \rightarrow k \rightarrow H^0(\mathbb{G}_a, k \oplus k) \xrightarrow{p_{2,*}} k \rightarrow H^1(\mathbb{G}_a, k) \rightarrow \dots$$

so that $H^1(\mathbb{G}_a, k)$ is non zero.

For more on the topic, see [Mg] and [Wn]. By using a similar method (involving sheaves) in the non-abelian setting, Demarche gave a proof of the following classical result [De].

6.8.5. Theorem. (Mostow [Mo]) *Assume that $\mathrm{char}(k) = 0$ and let G/k be a linearly reductive group and let U/k be a split unipotent k -group. Then each extension of algebraic groups of G by U is split and the sections are conjugated under $U(k)$.*

Lie algebras, lifting tori

7. WEIL RESTRICTION

We are given the following equation $z^2 = 1 + 2i$ in \mathbb{C} . A standard way to solve it is to write $z = x + iy$ with $x, y \in \mathbb{R}$. It provides then two real equations $x^2 - y^2 = 1$ and $xy = 1$. We can abstract this method for affine schemes and for functors.

We are given a ring extension S/R or $j : R \rightarrow S$. Since a S -algebra is an R -algebra, an R -functor F defines an S -functor denoted by F_S and called the scalar extension of F to S . For each S -algebra S' , we have $F_S(S') = F(S')$. If X is an R -scheme, we have $(h_X)_S = h_{X \times_R S}$.

Now we consider a S -functor E and define its Weil restriction to S/R denoted by $\prod_{S/R} E$ by

$$\left(\prod_{S/R} E \right) (R') = E(R' \otimes_R S)$$

for each R -algebra R' . We note the two following functorial facts.

(I) For an R -map of rings $u : S \rightarrow T$, we have a natural map

$$(7.0.1) \quad u_* : \prod_{S/R} E \rightarrow \prod_{T/R} E_T.$$

(II) For each R'/R , there is natural isomorphism of R' -functors

$$(7.0.2) \quad \left(\prod_{S/R} E \right)_{R'} \xrightarrow{\sim} \prod_{S \otimes_R R' / R'} E_{S \otimes_R R'}.$$

For other functorial properties, see appendix A.5 of [CGP], for example the construction.

At this stage, it is of interest to discuss the example of vector group functors. Let N be an R -module. We denote by $j_* N$ the scalar restriction of N from S to R [Bbk1, §II.1.13]. The module $j_* N$ is N equipped with the R -module structure induced by the map $j : R \rightarrow S$. It satisfies the adjunction property $\mathrm{Hom}_R(M, j_* N) \xrightarrow{\sim} \mathrm{Hom}_S(M \otimes_R S, N)$ (*ibid*, §III.5.2).

7.0.3. Lemma. (1) *We have a canonical isomorphism $\prod_{S/R} W(N) \xrightarrow{\sim} W(j_* N)$.*

(2) *If N is f.g. projective and S/R is finite and locally free, then the R -module $j_* N$ is f.g. projective and $\prod_{S/R} W(N)$ is representable by the vector group scheme $\mathfrak{W}(j_* N)$.*

For a more general statement, see [SGA3, I.6.6].

Proof. (1) For each R -algebra R' , we have

$$\left(\prod_{S/R} W(N)\right)(R') = W(N)(R' \otimes_R S) = N \otimes_S (R' \otimes_R S) = j_* N \otimes_R R' = W(j_* N)(R').$$

(2) We write $N \oplus N' = S^n$ so that $j_* N \oplus j_* N' = (j_* S)^n$. Since the R -module S is f.g. projective, $(j_* S)^n$ is f.g. projective and so is $j_* N$. Hence $W(j_* N)$ is representable by the vector R -group scheme $\mathfrak{W}(j_* N)$. \square

7.0.4. Example. We have $h_{\text{Spec}(R)} = \prod_{S/R} h_{\text{Spec}(S)}$. This is the case $N = 0$ of Lemma 7.0.3.(1).

If F is an R -functor, we have for each R'/R a natural map

$$\eta_F(R') : F(R') \rightarrow F(R' \otimes_R S) = F_S(R' \otimes_R S) = \left(\prod_{S/R} F_S\right)(R');$$

it defines a natural mapping of R -functor $\eta_F : F \rightarrow \prod_{S/R} F_S$ called often the diagonal map. For each S -functor E , it permits to defines a map

$$\phi : \text{Hom}_{S\text{-functor}}(F_S, E) \rightarrow \text{Hom}_{R\text{-functor}}\left(F, \prod_{S/R} E\right)$$

by applying a S -functor map $g : F_S \rightarrow E$ to the composition

$$F \xrightarrow{\eta_F} \prod_{S/R} F_S \xrightarrow{\prod_{S/R} g} \prod_{S/R} E.$$

7.0.5. Lemma. *The map ϕ is bijective.*

Proof. We apply the compatibility (7.0.2) with $R' = S_2 = S$. The map $S \rightarrow S \otimes_R S_2$ is split by the codiagonal map $\nabla : S \otimes_R S_2 \rightarrow S, s_1 \otimes s_2 \rightarrow s_1 s_2$. Then we can consider the map

$$\theta_E : \left(\prod_{S/R} E\right)_{S_2} \xrightarrow{\sim} \prod_{S \otimes_R S_2 / S_2} E_{S \otimes_R S_2} \xrightarrow{\nabla_*} \prod_{S/S} E = E.$$

This map permits to construct the inverse map ψ of ϕ as follows

$$\psi(h) : F_S \xrightarrow{l_S} \left(\prod_{S/R} E\right)_{S_2} \xrightarrow{\theta_E} E$$

for each $l \in \text{Hom}_{R\text{-functor}}\left(F, \prod_{S/R} E\right)$. By construction, the maps ϕ and ψ are inverse of each other. \square

In conclusion, the Weil restriction from S to R is then right adjoint to the functor of scalar extension from R to S .

7.0.6. Proposition. *Assume that S is finite locally free over R . Let \mathfrak{Y}/S be an affine scheme of finite type (resp. of finite presentation). Then the functor $\prod_{S/R} h_{\mathfrak{Y}}$ is representable by an affine scheme of finite type (resp. of finite presentation).*

Again, it is a special case of a much more general statement, see [BLR, §7.6]. We denote by $\prod_{S/R} \mathfrak{Y}$ the R -scheme representing $\prod_{S/R} h_{\mathfrak{Y}}$.

Proof. The R -functor $\prod_{S/R} h_{\mathfrak{Y}}$ is a Zariski sheaf. According to Lemma 2.3.2, up to localize for the Zariski topology, we can assume that S is free over R , namely $S = \oplus_{i=1, \dots, d} R \omega_i$. We see \mathfrak{Y} as a closed subscheme of some affine space \mathbb{A}_S^n , that is given by a system of equations $(P_\alpha)_{\alpha \in I}$ with $P_\alpha \in S[t_1, \dots, t_n]$. Then $\prod_{S/R} h_{\mathfrak{Y}}$ is a subfunctor of $\prod_{S/R} W(S^n) \xrightarrow{\sim} W(j_*(S^n)) \xrightarrow{\sim}$

$W(R^{nd})$ by Lemma 7.0.3. For each I , we write

$$P_\alpha \left(\sum_{i=1, \dots, d} y_{1,i} \omega_i, \sum_{i=1, \dots, d} y_{2,i} \omega_i, \dots, \sum_{i=1, \dots, d} y_{n,i} \omega_i \right) = Q_{\alpha,1} \omega_1 + \dots + Q_{\alpha,r} \omega_r$$

with $Q_{\alpha,i} \in R[y_{k,i}; i = 1, \dots, d; k = 1, \dots, n]$. Then for each R'/R , $\left(\prod_{S/R} h_{\mathfrak{Y}} \right)(R')$

inside R'^{nd} is the locus of the zeros of the polynomials $Q_{\alpha,j}$. Hence this functor is representable by an affine R -scheme \mathfrak{X} of finite type. Furthermore, if \mathfrak{Y} is of finite presentation, we can take I finite so that \mathfrak{X} is of finite presentation too. \square

In particular, if \mathfrak{H}/S is an affine group scheme of finite type, then the R -group functor $\prod_{S/R} h_{\mathfrak{H}}$ is representable by an R -affine group scheme of finite type. There are nice functoriality issues, for example for open (resp. closed) immersions appendix A.5 of [CGP]. There are two basic examples of Weil restrictions.

(a) The case of a finite separable field extension k'/k (or more generally an étale k -algebra). Given an affine algebraic k' -group G'/k' , we associate the affine algebraic k -group $G = \prod_{k'/k} G'$ which is often denoted by $R_{k'/k}(G)$,

see [Vo, §3. 12]. In that case, $R_{k'/k}(G) \times_k k_s \xrightarrow{\sim} (G'_{k_s})^d$. In particular, the dimension of G is $[k' : k] \dim_{k'}(G')$; the Weil restriction of a finite algebraic group is a finite group.

(b) The case where $S = k[\epsilon]$ is the ring of dual numbers which is of very different nature. For example the quotient k -group $\prod_{k[\epsilon]/k} (\mathbb{G}_m)/\mathbb{G}_m$ is the additive k -group. Also if $p = \text{char}(k) > 0$, $\prod_{k[\epsilon]/k} \mu_{p,k[\epsilon]}$ is of dimension 1.

A side statement is the following.

7.0.7. Lemma. *Assume that S is locally free over R of constant rank $d \geq 1$. Let \mathfrak{X} be an affine R -scheme and consider the diagonal map $\eta_{\mathfrak{X}} : \mathfrak{X} \rightarrow \prod_{S/R} (\mathfrak{X} \times_R S)$. Then $\eta_{\mathfrak{X}}$ is a closed immersion.*

Proof. Without loss of generality we may assume that R is non zero and so is S . Let $i : \mathfrak{X} \rightarrow \mathbb{A}_R^n = \mathfrak{W}(R^n)$ be a closed immersion. We consider the

commutative diagram

$$\begin{array}{ccc}
 \mathfrak{X} & \xrightarrow{\eta} & \prod_{S/R}(\mathfrak{X} \times_R S) \\
 \downarrow i & & \downarrow \prod_{S/R}(i_S) \\
 \mathfrak{W}(R^n) & \longrightarrow & \prod_{S/R} \mathfrak{W}(S^n) = \mathfrak{W}(j_* S)^n.
 \end{array}$$

Since the two vertical maps are closed immersions, we are reduced to the case of $\mathfrak{W}(R^n)$ and even to the case of $\mathfrak{W}(R)$. As R -module, we claim that R is a direct summand of S , this implies that $\mathfrak{W}(R) \rightarrow \mathfrak{W}(j_* S)$ is a (split) monomorphism hence a closed immersion in view of Proposition 3.5.3. To establish the claim we embed S as direct summand in R^l . The vector $j(1) = (r_1, \dots, r_l)$ is unimodular, that is, $\sum r_i R = R$ ⁴. Thus R is a direct summand of the R -module S and the claim is proven. \square

Let us give an application of Weil restriction.

7.0.8. Proposition. *Let \mathfrak{G}/R be an affine group scheme. Assume that there exists a finite locally free extension S/R of degree $d \geq 1$ such that $\mathfrak{G} \times_R S$ admits a faithful representation N f.g. locally free as S -module. Then \mathfrak{G} admits a faithful representation M which is f.g. locally free as R -module.*

Proof. Let $\rho : \mathfrak{G} \times_R S \rightarrow \mathrm{GL}(N)$ be a faithful S -representation and denote by M/R the restriction of N from S to R . We have seen that M is f.g. projective in Lemma 7.0.3.(2).

We have a natural embedding $\mathrm{End}_S(N) \subset \mathrm{End}_R(M)$ of R -algebras. Given an R -algebra R' , we can map

$$\prod_{S/R} (W(\mathrm{End}_S(N)))(R') = \mathrm{End}_S(N) \otimes_S (S \otimes R') = \mathrm{End}_S(N) \otimes_R R'.$$

in $\mathrm{End}_R(M) \otimes_R R' = W(\mathrm{End}_R(M))(R')$. We have then a morphism of R -functors

$$\prod_{S/R} (W(\mathrm{End}_S(N))) \rightarrow W(\mathrm{End}_R(M))$$

and we claim that is a monomorphism. The S -module $\mathrm{End}_S(N)$ is finite locally free so that $\mathrm{End}_S(N) \otimes_S (S \otimes R') = \mathrm{End}_{S \otimes_R R'}(N \otimes_S (S \otimes R'))$ [Bbk1, II.5.3, prop. 7]. This embeds in $(W(\mathrm{End}_R(M)))(R') = \mathrm{End}_R(M) \otimes_R R' = \mathrm{End}_{R'}(M \otimes_R R') = \mathrm{End}_{R'}(N \otimes_S (S \otimes_R R'))$ so that the claim is established.

We have then a monomorphism of R -schemes $\prod_{S/R} (\mathfrak{W}(\mathrm{End}_S(N))) \rightarrow \mathfrak{W}(\mathrm{End}_R(M))$. We obtain then a monomorphism of R -group schemes $\prod_{S/R} \mathrm{GL}(N) \rightarrow \mathrm{GL}(M)$

⁴This is a standard argument. If not r_1, \dots, r_l belong to a maximal proper ideal \mathfrak{m} of R , contradicting the fact that $1_S \otimes R/\mathfrak{m}$ is non zero.

of R -group schemes. We consider then the R -map

$$\mathfrak{G} \xrightarrow{\eta_{\mathfrak{G}}} \prod_{S/R} \mathfrak{G} \times_R S \xrightarrow{\prod_{S/R} \rho} \prod_{S/R} \mathrm{GL}(N) \rightarrow \mathrm{GL}(M)$$

Lemma 7.0.7 states that the left handside map is a closed immersion. The map in the diagram is a composite of monomorphisms, hence a monomorphism. \square

7.0.9. Remark. If ρ is a closed immersion, we claim that so is the constructed map $\mathfrak{G} \rightarrow \mathrm{GL}(M)$. Since $\prod_{S/R} \rho$ is a closed immersion, it is enough

to check that $\prod_{S/R} \mathrm{GL}(N) \rightarrow \mathrm{GL}(M)$ is a closed immersion. We claim that we have a cartesian diagram

$$\begin{array}{ccc} \prod_{S/R} \mathrm{GL}(N) & \longrightarrow & \mathrm{GL}(M) \\ \downarrow & & \downarrow \\ \prod_{S/R} \mathfrak{W}(\mathrm{End}_S(N)) = \mathfrak{W}(j_* \mathrm{End}_S(N)) & \longrightarrow & \mathfrak{W}(\mathrm{End}_R(M)) \end{array}$$

where the bottom horizontal map is a closed immersion in view of Proposition 3.5.3. The cartesianity follows from $\mathrm{End}_S(N)^\times = \mathrm{End}_R(M)^\times \cap \mathrm{End}_S(N)$ and similarly after any change of rings R'/R .

7.0.10. Remark. It is natural to ask whether the functor of scalar extension from R to S admits a left adjoint. It is the case and we denote by $\bigsqcup_{S/R} E$ this

left adjoint functor, see [DG, §I.1.6]. It is called the Grothendieck restriction.

If $\rho : S \rightarrow R$ is an R -ring section of j , it defines a structure R^ρ of S -ring. We have $\bigsqcup_{S/R} E = \bigsqcup_{\rho: S \rightarrow R} E(R^\rho)$. If $E = h_{\mathfrak{Y}}$ for a S -scheme \mathfrak{Y} , $\bigsqcup_{S/R} \mathfrak{Y}$ is representable by the R -scheme \mathfrak{Y} . This is simply the following R -scheme $\mathfrak{Y} \rightarrow \mathrm{Spec}(S) \xrightarrow{j^*} \mathrm{Spec}(R)$.

8. TANGENT SPACES AND LIE ALGEBRAS

8.1. Derivations. Let S be an R -ring and let M be an S -module. An R -derivation on M is an R -module homomorphism $d : S \rightarrow M$ to the S -module M satisfying the Leibniz rule

$$d(fg) = f d(g) + g d(f) \quad (f, g \in S).$$

We have $d(1) = d(1 \cdot 1) = 1 d(1) + 1 d(1)$ so that $d(1) = 0$ and $d(R) = 0$. We denote by $\mathrm{Der}_R(S, M)$ the R -module of R -derivations on S to M .

We define the S -module of Kähler differentials $\Omega_{S/R}^1$ as the quotient of the free S -module $S^{(S)} = \bigoplus_{s \in S} S ds$ by the S -module of relations generated by

- (a) $dr, r \in R$;
- (b) $d(s+t) = ds + dt, s, t \in S$;
- (c) $d(st) = s dt + t ds, s, t \in S$.

The map $d : S \rightarrow \Omega_{S/R}^1, s \rightarrow ds$, is then a derivation (note that R -linearity follows from (a)). Next let $f : \Omega_{S/R}^1 \rightarrow M$ be a morphism of S -modules. We define $d_f(s) = f(ds)$, then $d_f(st) = f(d(st)) = f(s dt + t ds) = s f(dt) + t f(ds)$ so it is a derivation. The derivation d is actually universal in the sense of the following statement.

8.1.1. Theorem. [St, Tag 00RO] *For each S -module M , the map*

$$\mathrm{Hom}_S(\Omega_{S/R}^1, M) \rightarrow \mathrm{Der}_R(S, M), \quad f \mapsto d \circ f$$

is an isomorphism.

8.1.2. Example. (see [St, 00RX]) If $S = R[T_1, \dots, T_n]$, we claim that we have

$$\Omega_{S/R}^1 = S dT_1 \oplus \dots \oplus S dT_n \cong R^n.$$

Since S is generated as R -algebra by T_1, \dots, T_n , the map

$$f : S dT_1 \oplus \dots \oplus S dT_n \rightarrow \Omega_{S/R}^1, (P_1, \dots, P_n) \mapsto P_1 dT_1 + \dots + P_n dT_n,$$

is onto. Next consider the R -derivation $\partial/\partial T_i : S \rightarrow S$. By the universal property this corresponds to an S -module map $l_i : \Omega_{S/R}^1 \rightarrow S$ which maps dT_i to 1 and dT_j to 0 for $j \neq i$. Thus it is clear that there are no S -linear relations among the elements dT_1, \dots, dT_n .

In particular for $M = R$ with S -structure $P(T_1, \dots, T_n).r = P(0, \dots, 0).r$, we have $\mathrm{Der}_R(S, R^{(0)}) = \mathrm{Hom}_S(S^n, R^{(0)}) = R^n$ with generators D_1, \dots, D_n defined by $D_i(P) = (\partial P / \partial T_i)(0)$.

8.2. Tangent spaces. We are given an affine R -scheme $\mathfrak{X} = \mathrm{Spec}(A)$. Given a point $x \in \mathfrak{X}(R)$, it defines an ideal $I(x) = \ker(A \xrightarrow{s_x} R)$ and defines an A -structure on R denoted R^x . We denote by $R[\epsilon] = R[t]/t^2$ the ring of R -dual numbers. We claim that we have a natural exact sequence of pointed set

$$\begin{array}{ccccc} 1 & \longrightarrow & \mathrm{Der}_A(A, R^x) & \xrightarrow{i_x} & \mathfrak{X}(R[\epsilon]) & \longrightarrow & \mathfrak{X}(R) \rightarrow 1 \\ & & & & \parallel & & \\ & & & & \mathrm{Hom}_R(A, R[\epsilon]) & & \end{array}$$

where the base points are $x \in \mathfrak{X}(R) \subset \mathfrak{X}(R[\epsilon])$. The map i_x applies a derivation D to the map $f \mapsto s_x(f) + \epsilon D(f)$. It is a ring homomorphism since for $f, g \in A$ we have

$$\begin{aligned} i_x(fg) &= s_x(fg) + \epsilon D(fg) \\ &= s_x(f) s_x(g) + \epsilon D(f) s_x(g) + \epsilon s_x(f) D(g) \quad [\text{derivation rule}] \\ &= (s_x(f) + \epsilon D(f)) \cdot (s_x(g) + \epsilon D(g)) \quad [\epsilon^2 = 0]. \end{aligned}$$

Conversely, one sees that a map $u : A \rightarrow R[\epsilon]$, $f \mapsto u(f) = s_x(f) + \epsilon v(f)$ is a ring homomorphism and only if $v \in \text{Der}_A(A, R^x)$.

8.2.1. Remark. The geometric interpretation of $\text{Der}_A(A, R^x)$ is the tangent space at x of the scheme \mathfrak{X}/R (see [Sp, 4.1.3]).

We have a natural A -map

$$\text{Hom}_{A\text{-mod}}(I(x)/I^2(x), R^x) \rightarrow \text{Der}_A(A, R^x);$$

it applies an A -map $l : I(x)/I^2(x) \rightarrow R$ to the derivation $D_l : A \rightarrow R$, $f \mapsto D_l(f) = l(f - f(x))$. This map is clearly injective but is split by mapping a derivation $D \in \text{Der}_A(A, R^x)$ to its restriction on $I(x)$. Hence the map above is an isomorphism. Furthermore $I(x)/I^2(x)$ is an R^x -module hence the forgetful map

$$\text{Hom}_{A\text{-mod}}(I(x)/I^2(x), R^x) \xrightarrow{\sim} \text{Hom}_{R\text{-mod}}(I(x)/I^2(x), R)$$

is an isomorphism. We conclude that we have the fundamental exact sequence of pointed sets

$$1 \longrightarrow (I(x)/I^2(x))^\vee \xrightarrow{i_x} \mathfrak{X}(R[\epsilon]) \longrightarrow \mathfrak{X}(R) \rightarrow 1.$$

We record that the R -module structure on $I(x)/I(x)^2$ is also induced by the change of variable $\epsilon \mapsto \lambda \epsilon$. This construction behaves well with fibered products.

8.2.2. Lemma. Let $\mathfrak{Y} = \text{Spec}(B)$ be an affine R -scheme and let $y \in \mathfrak{Y}(R)$. The dual of the R -module map $v : I(x)/I^2(x) \oplus I(y)/I^2(y) \rightarrow I(x, y)/I^2(x, y)$ is an isomorphism and fits in the following commutative diagram

$$\begin{array}{ccccc} 1 & \longrightarrow & (I(x)/I^2(x))^\vee \oplus (I(y)/I^2(y))^\vee & \xrightarrow{i_x \times i_y} & \mathfrak{X}(R[\epsilon]) \times \mathfrak{Y}(R[\epsilon]) \longrightarrow \mathfrak{X}(R) \times \mathfrak{Y}(R) \rightarrow 1 \\ & & \uparrow \cong & & \uparrow \cong \\ 1 & \longrightarrow & (I(x, y)/I^2(x, y))^\vee & \xrightarrow{i_{(x, y)}} & (\mathfrak{X} \times_R \mathfrak{Y})(R[\epsilon]) \longrightarrow (\mathfrak{X} \times_R \mathfrak{Y})(R) \rightarrow 1. \end{array}$$

commutes.

Proof. We write the two sequences and the map between them is provided by the fact that the map $(\mathfrak{X} \times_R \mathfrak{Y})(R[\epsilon]) \xrightarrow{\sim} \mathfrak{X}(R[\epsilon]) \times \mathfrak{Y}(R[\epsilon])$ is bijective. \square

We note that an R -module, $I(x)$ is a direct summand of $R[\mathfrak{X}]$. If we consider an R -ring S , it follows that $I(x) \otimes_R S$ is the kernel of $R[\mathfrak{X}] \xrightarrow{s_x \otimes id} S$. In conclusion, we have then defined a (split) exact sequence of pointed R -functors

$$1 \longrightarrow \mathfrak{Y}(I(x)/I(x)^2) \xrightarrow{i_x} \prod_{R[\epsilon]/R} \mathfrak{X}_{R[\epsilon]} \longrightarrow \mathfrak{X} \longrightarrow 1.$$

8.3. Smoothness. There are several equivalent definitions for expressing that an R -algebra S is smooth. We have chosen to follow a variant of [GW2, §10.18] provided by the Stacks Project [St, Tags 00T6, 00T7].

8.3.1. Definition. (1) An R -algebra S is *standard smooth* if

$$S \cong R[T_1, \dots, T_n]/(f_1, \dots, f_c)$$

with $0 \leq c \leq n$ such that

$$\det\left((\partial f_i / \partial T_j)_{i,j=1,\dots,c}\right) \in S^\times.$$

(2) An R -algebra S is *smooth* if it is of finite presentation and if for each geometric point $y \in \operatorname{Spec}(S)$ of image $x \in \operatorname{Spec}(R)$, there exists $f \in R$ and $g \in S$ such that $x \in \operatorname{Spec}(R_f)$, $y \in \operatorname{Spec}(S_g)$, and $R \rightarrow S$ induces a map $R_f \rightarrow S_g$ which is *standard smooth*.

8.3.2. Remarks. (a) If the R -algebra S is *standard smooth* with the above presentation, it follows that the non-empty geometric fibers are of dimension $n - c$ in view of [GW2, Thm. 18.56.(v)]. In particular if S is non zero, the relative dimension d is well-defined. We will see later another way to understand that, see Remark 8.3.7.

(b) *Étale* means smooth of relative dimension 0. We have to pay attention that the notion of *standard étale* is different [St, Tag 00UB], this is $S \cong R[T]_Q/P(T)$ where P is a monic polynomial such that $P'(T) \in S^\times$.

The two notions are stable by arbitrary base change on R .

8.3.3. Examples. (a) A localization R_f is a smooth R -algebra.

(b) The polynomial R -algebra $R[T_1, \dots, T_d]$ is smooth.

The advantage of this definition is to be close of the intuition coming from differential geometry but it is not intrinsic. However a good point is that it behaves well under composition [St, Tags 00T9, 00TD]. It turns out that smooth R -algebras are flat [GW, thm. 14.24], we refer to [GW2, §18.10] for the equivalence with other definitions. The most important result is that smoothness can be characterized on the functor of points.

8.3.4. Theorem. (see [GW2, Cor. 18.57], [St, Tag 00TN, 00UR]) Let $\mathfrak{X} = \operatorname{Spec}(A)$ be an affine R -scheme which is finitely presented.

(1) The R -scheme \mathfrak{X} is *smooth* (i.e. A is an R -smooth algebra) if and only if \mathfrak{X} is *formally smooth*, that is: for each R -ring B and each ideal $I \subset B$ satisfying $I^2 = 0$, the map $\mathfrak{X}(B) \rightarrow \mathfrak{X}(B/I)$ is onto.

(2) The R -scheme \mathfrak{X} is *étale* (i.e. S is an R -étale algebra) if and only if \mathfrak{X} is *formally étale*, that is: for each R -ring B and each ideal $I \subset B$ satisfying $I^2 = 0$, the map $\mathfrak{X}(B) \rightarrow \mathfrak{X}(B/I)$ is bijective.

We make now the connection with tangent spaces.

8.3.5. Lemma. *Let $S = R[T_1, \dots, T_n]/(f_1, \dots, f_c) = R[T_1, \dots, T_n]/I$ be a standard smooth algebra with $0 \leq c \leq n$ and $\det((\partial f_i/\partial T_j)_{i,j=1,\dots,c}) \in S^\times$.*

(1) *The S -module I/I^2 is free of base f_1, \dots, f_c .*

(2) *The S -module $\Omega_{S/R}^1$ is free of base the images of dT_{c+1}, \dots, dT_n .*

Proof. (1) and (2) We put $B = R[T_1, \dots, T_n]$ and denote by $p : B \rightarrow S = B/I$ the quotient map. According to [St, Tag 00RU], we have an exact sequence of S -modules

$$I/I^2 \xrightarrow{d \otimes id} \Omega_{B/R}^1 \otimes_B S \rightarrow \Omega_{S/R}^1 \rightarrow 0.$$

By taking into account Example 8.1.2, we have $\Omega_{B/R}^1 = B^n$, this sequence becomes

$$I/I^2 \xrightarrow{d \otimes id} S^n \rightarrow \Omega_{S/R}^1 \rightarrow 0.$$

We precompose by the surjective map $S^c \rightarrow I/I^2$, $(s_1, \dots, s_c) \mapsto \sum_{j=1}^c s_j f_j$.

The matrix of $S^c \rightarrow S^n$, $(s_1, \dots, s_c) \mapsto \left(\sum_{j=1}^c s_j \partial f_j / \partial T_i \right)_{i=1,\dots,n}$ is

$$\left(\partial f_j / \partial T_i \right)_{j=1,\dots,c, i=1,\dots,n}$$

which admits an invertible minor. It follows that the S -linear map $S^c \rightarrow S^n$ admits a left inverse and that $S^c \xrightarrow{\sim} I/I^2$. Thus I/I^2 is a free S -module of rank c .

We conclude also that $\Omega_{S/R}^1$ is a free S -module of rank $n - c$. □

8.3.6. Lemma. *Let $\mathfrak{X} = \text{Spec}(A)$ be affine R -scheme \mathfrak{X}/R which is smooth of relative dimension d .*

(1) *The $R[\mathfrak{X}]$ -module $\Omega_{R[\mathfrak{X}]/R}^1$ is locally free of rank d .*

(2) *Let $x \in \mathfrak{X}(R)$ and consider the ideal $I(x) = \text{Ker}(R[\mathfrak{X}] \xrightarrow{ev_x} R)$. Then the R -module $(I(x)/I(x)^2)^\vee$ is locally free of rank d .*

Proof. (1) We are allowed to localize on $R[\mathfrak{X}]$ (using [St, Tag 00RT, (2)]), so that the statement boils down to the composite $R \rightarrow R_f \rightarrow S$ for some $f \in R$ where S is a standard smooth R_f -algebra. In view of [St, Tag 00RT, (1)], we have an isomorphism of S -modules $\Omega_{S/R}^1 \xrightarrow{\sim} \Omega_{S/R_f}^1$ so that we are reduced to the case when S is a standard smooth R -algebra. That case is treated by Lemma 8.3.5.(2) so we are done.

(2) By (1), the $R[\mathfrak{X}]$ -module $\Omega_{R[\mathfrak{X}]/R}^1$ is locally free of rank d . We write $\Omega_{R[\mathfrak{X}]/R}^1 \oplus N = R[\mathfrak{X}]^n$, so that the R -module $\text{Hom}_{R[\mathfrak{X}]}(\Omega_{R[\mathfrak{X}]/R}^1, R^x)$ is a direct summand of $\text{Hom}_{R[\mathfrak{X}]}(R[\mathfrak{X}]^n, R^x) = R^n$ so that is locally free. Thus the R -module $(I(x)/I(x)^2)^\vee$ is locally free. To check this is of rank d , we can localize on \mathfrak{X} . □

8.3.7. Remark. It provides another way to see that the relative dimension d of \mathfrak{X} is well-defined by taking a non-empty geometric fiber of $\mathfrak{X} \rightarrow \text{Spec}(R)$.

8.4. Lie algebras. Now let \mathfrak{G}/R be an affine group scheme. We denote by $\text{Lie}(\mathfrak{G})(R)$ the tangent space at the origin $1 \in \mathfrak{G}(R)$. This is the dual of I/I^2 where $I \subset R[\mathfrak{G}]$ is the kernel of the augmentation ideal. We define the “Lie algebra of \mathfrak{G} ” vector R -group scheme by

$$\text{Lie}(\mathfrak{G}) = \mathfrak{V}(I/I^2)$$

and we shall define later the Lie algebra structure. We recall that it fits in the sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Lie}(\mathfrak{G})(R) & \longrightarrow & \mathfrak{G}(R[\epsilon]) & \longrightarrow & \mathfrak{G}(R) \rightarrow 1 \\ & & X & \mapsto & \exp(\epsilon X) & & \end{array}$$

which is a split exact of abstract groups where $\text{Lie}(\mathfrak{G})(R)$ is equipped with the induced group law.

8.4.1. Lemma. *That induced group law is the additive law on $\text{Lie}(\mathfrak{G})(R)$, namely $\exp(\epsilon X + \epsilon Y) = \exp(\epsilon X) \cdot \exp(\epsilon Y)$ for each $X, Y \in \text{Lie}(\mathfrak{G})(R)$.*

Proof. We apply Lemma 8.2.2 and use the product map $m : \mathfrak{G} \times_R \mathfrak{G} \rightarrow \mathfrak{G}$ to construct the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & (I/I^2)^\vee \oplus (I/I^2)^\vee & \xrightarrow{\exp \times \exp} & \mathfrak{G}(R[\epsilon]) \times \mathfrak{G}(R[\epsilon]) & \longrightarrow & \mathfrak{G}(R) \times \mathfrak{G}(R) \rightarrow 1 \\ & & \uparrow v^\vee \cong & & || & & || \\ 1 & \longrightarrow & (I(\mathfrak{G} \times_R \mathfrak{G})/I(\mathfrak{G} \times_R \mathfrak{G})^2)^\vee & \xrightarrow{\exp_{\mathfrak{G} \times_R \mathfrak{G}}} & (\mathfrak{G} \times_R \mathfrak{G})(R[\epsilon]) & \longrightarrow & (\mathfrak{G} \times_R \mathfrak{G})(R) \rightarrow 1. \\ & & \downarrow m_* & & \downarrow m & & \downarrow m \\ 1 & \longrightarrow & (I/I^2)^\vee & \xrightarrow{\exp} & \mathfrak{G}(R[\epsilon]) & \longrightarrow & \mathfrak{G}(R) \rightarrow 1. \end{array}$$

Since the composite $\mathfrak{G} \xrightarrow{id \times \epsilon} \mathfrak{G} \times_R \mathfrak{G} \xrightarrow{m} \mathfrak{G}$ is the identity, the composite map $(I/I^2)^\vee \xrightarrow{id \times 0} (I/I^2)^\vee \oplus (I/I^2)^\vee \rightarrow (I/I^2)^\vee$ is the identity. It is the same for the second summand, so we conclude that the left vertical composite map is the addition. \square

8.4.2. Remark. If \mathfrak{G} is an R -subgroup of some GL_n , the proof of Lemma 8.4.1 boils down to the case of GL_n . In this case

8.4.3. Example. Let M be an R -module and consider the R -vector group scheme $\mathfrak{V}(M)$. For each S/R , we have

$$\mathfrak{V}(M)(S[\epsilon]) = \text{Hom}_{S[\epsilon]}(M \otimes_R S[\epsilon], S[\epsilon]) = \text{Hom}_R(M, S[\epsilon]) = \mathfrak{V}(M)^2(S),$$

hence an R -isomorphism $\mathfrak{V}(M) \xrightarrow{\sim} \text{Lie}(\mathfrak{V}(M))$.

8.4.4. Remarks. (a) The natural map $\mathrm{Lie}(\mathfrak{G})(R) \otimes_R S \rightarrow \mathrm{Lie}(\mathfrak{G})(S)$ is not bijective in general (for example consider the case of a DVR and $\mathfrak{G} = \mathfrak{V}(R/\pi R)$). We have $\mathrm{Lie}(\mathfrak{G})(R) = \mathrm{Hom}_{R[\mathfrak{G}]}(\Omega_{R[\mathfrak{G}]/R}^1, R^{(1)})$.

(b) If $\Omega_{R[\mathfrak{G}]/R}^1$ is a finite locally free $R[\mathfrak{G}]$ -module, we claim that the formation of the Lie algebra commutes with arbitrary base change. Writing $\Omega_{R[\mathfrak{G}]/R}^1 \oplus N' = S^n$ we have that $\mathrm{Lie}(\mathfrak{G})(R) = \mathrm{Hom}_{R[\mathfrak{G}]}(\Omega_{R[\mathfrak{G}]/R}^1, R^{(1)})$ is a direct summand of $\mathrm{Hom}_{R[\mathfrak{G}]}(R[\mathfrak{G}]^n, R^{(1)}) = R^n$. This behaves well under base change.

(c) The preceding fact applies obviously when R is a field and also when \mathfrak{G} is smooth over R (due to Lemma 8.3.5.(2))

(d) The condition that the $R[\mathfrak{G}]$ -module $\Omega_{R[\mathfrak{G}]/R}^1$ is *f.g.* projective is actually necessary for having this base change property in general, see [DG, §II.4.8].

More generally, we can define the Lie algebra R -functor of a group R -functor F by putting

$$\mathrm{Lie}(F)(S) = \ker \left(F(S[\epsilon]) \rightarrow F(S) \right).$$

It is a subgroup equipped with a map $\mathrm{Lie}(F)(R) \otimes_R S \rightarrow \mathrm{Lie}(F)(S)$ coming from the base change $\epsilon \mapsto \lambda\epsilon$. In that generality, we are actually mainly interested in the following examples.

8.4.5. Lemma. *Let M be an R -module. Then $W(M) \xrightarrow{\sim} \mathrm{Lie}(W(M))$ and $\mathrm{End}_S(M \otimes_R S) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{GL}(M))(S)$ for each S/R .*

Proof. The first thing is similar as example 8.4.3. For each S/R , we have indeed a split exact sequence of abstract groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{End}_S(M \otimes_R S) & \longrightarrow & \mathrm{GL}(M)(S[\epsilon]) & \longrightarrow & \mathrm{GL}(M)(S) \longrightarrow 1. \\ & & f & \mapsto & Id + \epsilon f & & \end{array}$$

□

If $f : \mathfrak{G} \rightarrow \mathfrak{H}$ is a morphism of affine R -group schemes, we have a map $\mathrm{Lie}(f) : \mathrm{Lie}(\mathfrak{G}) \rightarrow \mathrm{Lie}(\mathfrak{H})$ of R -vector groups and the commutativity property $f(\exp(\epsilon X)) = \exp(\epsilon \cdot \mathrm{Lie}(f)(X))$.

The exact sequence defines an action of $\mathfrak{G}(R)$ on $\mathrm{Lie}(\mathfrak{G})(R)$ and actually a homomorphism $\mathfrak{G}(R) \rightarrow \mathrm{Aut}_{R\text{-lin}}(\mathrm{Lie}(\mathfrak{G})(R))$ called the adjoint representation.

8.4.6. Lemma. *Let M be a *f.g.* projective R -module and put $\mathfrak{G} = \mathrm{GL}(M)$. Then $\mathrm{End}_R(M) = \mathrm{Lie}(\mathfrak{G})(R)$ and the adjoint action is*

$$\mathrm{Ad}(g) \cdot X = g X g^{-1}.$$

Proof. The R -group scheme \mathfrak{G} is open in $W(\mathrm{End}_R(M))$ so that the tangent space at 1 in \mathfrak{G} is the same than in $W(\mathrm{End}_R(M))$. By example 8.4.3, we get then an R -isomorphism $\mathrm{End}_R(M) \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{G})(R)$. We perform now the

computation of $\text{Ad}(g) \exp(\epsilon X)$ in $\mathfrak{G}(R[\epsilon]) \subset \text{End}_R(M) \otimes_R R[\epsilon]$. We have $\text{Ad}(g) \exp(\epsilon X) = g (Id + \epsilon X) g^{-1} = Id + \epsilon g X g^{-1} = \exp(\epsilon g X g^{-1})$. \square

We assume for simplicity first that $\text{Lie}(\mathfrak{G}) = W(\text{Lie}(\mathfrak{G})(R))$ with $\text{Lie}(G)(R)$ finite locally free (e.g. \mathfrak{G} is smooth over R). We will refer to this property as (LF) .

It follows that the adjoint representation functor

$$\text{Ad} : \mathfrak{G} \rightarrow \text{GL}(W(\text{Lie}(\mathfrak{G})(R))).$$

is actually a representation of \mathfrak{G} called the adjoint representation. By applying the Lie functor, it induces then a morphism of vector R -group schemes

$$\text{ad} : \text{Lie}(\mathfrak{G}) \rightarrow \text{Lie}\left(\text{GL}(\text{Lie}(\mathfrak{G})(R))\right).$$

For each S/R , we have then an S -map

$$\text{ad}(S) : \text{Lie}(\mathfrak{G})(S) \rightarrow \text{Lie}\left(\text{GL}(\text{Lie}(\mathfrak{G}))\right)(S) = \text{End}_S(\text{Lie}(\mathfrak{G})(R) \otimes_R S)$$

in view of the preceding lemma. For each $X, Y \in \text{Lie}(\mathfrak{G})(S)$, we denote by

$$(8.4.7) \quad [X, Y] = \text{ad}(S)(X) \cdot Y \in \text{Lie}(\mathfrak{G})(S)$$

the Lie bracket of X and Y .

8.4.8. Lemma. (1) Let $f : \mathfrak{G} \rightarrow \mathfrak{H}$ be a morphism of affine R -group schemes satisfying both property (LF) . For each $X, Y \in \text{Lie}(\mathfrak{G})(R)$, we have

$$\text{Lie}(f) \cdot [X, Y] = [\text{Lie}(f) \cdot X, \text{Lie}(f) \cdot Y] \in \text{Lie}(\mathfrak{G})(R).$$

(2) In the case $\mathfrak{G} = \text{GL}(M)$ with M f.g. projective, the Lie bracket $\text{End}_R(M) \times \text{End}_R(M) \rightarrow \text{End}_R(M)$ reads $[X, Y] = XY - YX$.

Proof. (1) Up to replace f by $\text{id} \times f : \mathfrak{G} \rightarrow \mathfrak{G} \times \mathfrak{H}$, we may assume that f is a monomorphism. It follows that the R -functor $\text{Lie}(f) : \text{Lie}(\mathfrak{G}) \rightarrow \text{Lie}(\mathfrak{H})$ is a monomorphism. We consider the following diagram of R -functors in groups

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{\text{Ad}_{\mathfrak{G}}} & \text{GL}(\text{Lie}(\mathfrak{G})) \\ & \searrow f \circ \text{Ad}_{\mathfrak{H}} & \uparrow \\ & \text{GL}(\text{Lie}(\mathfrak{H}), \text{Lie}(\mathfrak{G})) & \\ f \downarrow & & \downarrow \text{Ad}_{\mathfrak{H}} \\ \mathfrak{H} & \xrightarrow{\text{Ad}_{\mathfrak{H}}} & \text{GL}(\text{Lie}(\mathfrak{H})) \end{array}$$

where $\text{GL}(\text{Lie}(\mathfrak{H}), \text{Lie}(\mathfrak{G}))$ stands for the normalizer functor of $\text{Lie}(\mathfrak{G})$ in $\text{GL}(\text{Lie}(\mathfrak{H}))$ (as defined in some exercise or in [DG, §II.1.3]). We derive it

and get

$$\begin{array}{ccc}
 \mathrm{Lie}(\mathfrak{G}) & \xrightarrow{ad_{\mathfrak{G}}} & \mathrm{End}(\mathrm{Lie}(\mathfrak{G})) \\
 \downarrow \mathrm{Lie}(f) & \searrow \mathrm{Lie}(f \circ \mathrm{Ad}_{\mathfrak{H}}) & \uparrow \\
 & \mathrm{End}(\mathrm{Lie}(\mathfrak{H}), \mathrm{Lie}(\mathfrak{G})) & \\
 \mathrm{Lie}(\mathfrak{H}) & \xrightarrow{ad_{\mathfrak{H}}} & \mathrm{End}(\mathrm{Lie}(\mathfrak{H}))
 \end{array}$$

whence the desired compatibility.

(2) We consider the adjoint representation $\mathrm{Ad}(R) : \mathrm{GL}(M)(R) \rightarrow \mathrm{GL}(\mathrm{End}_R(M))(R)$ known to be $\mathrm{Ad}(g).X = gXg^{-1}$. We consider $\mathrm{Ad}(R[\epsilon]) : \mathrm{GL}(M)(R[\epsilon]) \rightarrow \mathrm{GL}(\mathrm{End}_R(M))(R[\epsilon])$; for $X, Y \in \mathrm{End}_R(M)$ we compute inside $(\mathrm{End}_R(M))(R[\epsilon])$ using Lemma 8.4.6

$$\begin{aligned}
 \mathrm{Ad}(R[\epsilon])(\exp(\epsilon X)) \cdot Y &= (1 + \epsilon X)Y(1 + \epsilon X)^{-1} \\
 &= (1 + \epsilon X)Y(1 - \epsilon X) \\
 &= Y + \epsilon(XY - YX).
 \end{aligned}$$

We conclude that $[X, Y] = XY - YX$. \square

8.4.9. Proposition. *Assume that \mathfrak{G} satisfies the property (LF) and that \mathfrak{G} admits a faithful linear representation in some GL_n . The Lie bracket defines a Lie R -algebra structure on the R -module $\mathrm{Lie}(\mathfrak{G})(R)$, that is*

- (i) *the bracket is R -bilinear and alternating;*
- (ii) *(Jacobi identity) For each $X, Y, Z \in \mathrm{Lie}(\mathfrak{G})(R)$, we have*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

We give here a short non orthodox proof specific to affine group schemes; for a more general setting, see [DG, II.4.4.3] and [SGA3, Exp. II].

Proof. Let us start with the case where \mathfrak{G} admits a faithful representation in $\mathrm{GL}(R^n)$. Then the R -map $\mathrm{Lie}(\mathfrak{G}) \rightarrow \mathrm{Lie}(\mathrm{GL}(M))$ is a monomorphism. From Lemma 8.4.8, it is then enough to check it for the linear group GL_n . That case is straightforward, we have $\mathrm{Lie}(\mathrm{GL}_n)(R) = M_n(R)$ and the bracket is $[X, Y] = XY - YX$ (lemma 8.4.8). \square

The theory is actually much wider since there is no need of the (LF) condition and also there is need to assume that \mathfrak{G} admits a faithful embedding in some GL_n . Using §3.2, we have an anti-isomorphism of R -functors $\mathrm{Aut}_{lin}(\mathfrak{V}(I/I^2)) \xrightarrow{\sim} \mathrm{GL}(I/I^2)$. This induces an isomorphism of R -functors in abelian groups

$$\mathrm{Lie}\left(\mathrm{Aut}_{lin}(\mathfrak{V}(I/I^2))\right) \xrightarrow{\sim} \mathrm{Lie}\left(\mathrm{GL}(I/I^2)\right)$$

which is nothing but $W(\text{End}_R(I/I^2))$ in view of Lemma 8.4.5 and also $\text{Hom}_{\text{lin}}(\text{Lie}(\mathfrak{G}), \text{Lie}(\mathfrak{G}))$ [DG, II.4.4.1]. This permits to define the Lie bracket as a morphism of R -functors

$$[\cdot, \cdot] : \text{Lie}(\mathfrak{G}) \times \text{Lie}(\mathfrak{G}) \rightarrow \text{Lie}(\mathfrak{G})$$

with the above formula (8.4.7). It turns out that $\text{Lie}(\mathfrak{G})(R)$ is indeed a Lie algebra [DG, II.4.4.5]. The main idea is to embed $\text{Lie}(\mathfrak{G})(R) = \text{Der}_R(R[\mathfrak{G}], R^{(1)})$ in the algebra $\text{Der}_R(R[\mathfrak{G}], R[\mathfrak{G}])$ which is a Lie R -subalgebra of $\text{End}_R(R[\mathfrak{G}], R[\mathfrak{G}])$ [Bbk1, §III.10.4]. In the field case, there is a short proof of this approach in [KMRT, §21.A].

8.4.10. Remark. If $j : R \rightarrow S$ is a finite locally free morphism and \mathfrak{H}/S a group scheme over S , it is a natural question to determine the Lie algebra of \mathfrak{G} . It is done in [CGP, A.7.6]. and we have $\text{Lie}(\mathfrak{G}) = j_* \text{Lie}(\mathfrak{H})$, that is $\text{Lie}(\mathfrak{G})(R') = \text{Lie}(\mathfrak{H})(S \otimes_R R')$ for each R'/R .

8.4.11. Examples. If k is a field of characteristic $p > 0$, $\text{Lie}(\mu_p)(k) = k$ with trivial Lie structure.

8.5. More infinitesimal properties. Our goal is to generalize the exact sequences of §8.2. Let $\mathfrak{X} = \text{Spec}(A)$ be an affine scheme. Let C be an R -ring equipped with an ideal J satisfying $J^2 = 0$. Let $x \in X(C)$. We denote by \bar{x} the image of x in $\mathfrak{X}(C/J)$. We put $I(x) = \ker(C[\mathfrak{X}] \rightarrow C)$. We claim that we have an exact sequence of pointed sets

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Hom}_{C\text{-mod}}(I(x)/I^2(x), J) & \xrightarrow{i_x} & \mathfrak{X}(C) & \longrightarrow & \mathfrak{X}(C/J) \\ & & l & & \mapsto & & x_l \end{array}$$

pointed at 0, x and \bar{x} . More precisely, the point x_l is defined by the morphism of rings

$$\begin{array}{ccc} C[\mathfrak{X}] & \xrightarrow{s_{x_l}} & C \\ f & \mapsto & f(x) + l(f - f(x)). \end{array}$$

It extends indeed the case of §8.2 when taking $C = R[\epsilon]$ and $J = \epsilon R[\epsilon]$. Let us check that the mapping is well defined. The only thing is the multiplicativity. Given $f, g \in C[\mathfrak{X}]$, we compute

$$\begin{aligned} (fg)(x) + l(fg - (fg)(x)) &= f(x)g(x) + l(fg - f(x)g(x)) \\ &= f(x)g(x) + l(f(g - g(x)) + (f - f(x))g(x)) \\ &= f(x)g(x) + f(x)l(g - g(x)) + g(x)l(f - f(x)) \quad [l \text{ is an } R\text{-map}] \\ &= (f(x) + l(f - f(x))) (g(x) + l(g - g(x))) \quad [J^2 = 0]. \end{aligned}$$

Conversely if $s : C[\mathfrak{X}] \rightarrow C$ is a ring homomorphism which coincide modulo I , we put $l_s(f) = s(f)$ for each $f \in I$. Then l_s is C^x -linear and satisfies $l_s(I^2) = 0$.

If \mathfrak{X}/R is smooth of relative dimension d , I/I^2 is locally free of rank d . Also the map $\mathfrak{X}(C) \rightarrow \mathfrak{X}(C/J)$ is onto (theorem 20.0.2).

If C'/C is a ring extension, putting $J' = J \otimes_R C'$ and⁵ $I'(x) = I(x) \otimes_R R'$, we have then an isomorphism [Bbk1, §II.5.3, prop. 7]

$$\mathrm{Hom}_{C\text{-mod}}(I(x)/I^2(x), J) \otimes_R C' \xrightarrow{\sim} \mathrm{Hom}_{C'\text{-mod}}(I'(x)/I'^2(x), J').$$

In this case we have then an exact sequence of C -functors

$$1 \longrightarrow \mathfrak{W}(M) \xrightarrow{i_x} \mathfrak{X}_C \longrightarrow \prod_{C/J} \mathfrak{X}_C \rightarrow 1$$

where $M = \mathrm{Hom}_{C\text{-mod}}(I(x)/I^2(x), J)$.

9. FIXED POINTS OF DIAGONALIZABLE GROUPS

9.1. Representability.

9.1.1. Proposition. *Let \mathfrak{X} be an affine R -scheme equipped with an action of a diagonalizable group scheme $\mathfrak{G}/R = \mathfrak{D}(A)$. Then the R -functor of fixed points F defined by*

$$F(S) = \left\{ x \in \mathfrak{X}(S) \mid G(S') \cdot x_{S'} = x_{S'} \quad \forall S'/S \right\}$$

is representable by a closed subscheme of \mathfrak{X} .

It is denoted by $\mathfrak{X}^{\mathfrak{G}}/R$. The proof below is inspired by [CGP, Lemma 2.1.4].

Proof. The R -module $R[\mathfrak{X}]$ decomposes in eigenspaces $\bigoplus_{a \in A} R[\mathfrak{X}]_a$. We denote by $J \subset R[\mathfrak{X}]$ the ideal generated by the $R[\mathfrak{X}]_a$ for a running over $A \setminus \{0\}$. We denote by \mathfrak{Y} the closed subscheme of \mathfrak{X} defined by J . Since J is a $\mathfrak{D}(A)$ -submodule of $R[\mathfrak{X}]$, $R[\mathfrak{Y}]$ is $\mathfrak{D}(A)$ -module with trivial structure. Hence the R -map $h_{\mathfrak{Y}} \rightarrow h_{\mathfrak{X}}$ factorizes by F , and we have a monomorphism $h_{\mathfrak{Y}} \rightarrow F$. Again by Yoneda, we have

$$F(R) = \left\{ x \in \mathfrak{X}(R) \mid \zeta x_{R[\mathfrak{G}]} = x_{R[\mathfrak{G}]} \right\}$$

where $\zeta \in \mathfrak{G}(R[\mathfrak{G}])$ stands for the universal element of \mathfrak{G} . Let $x \in F(R)$ and denote by $s_x : R[\mathfrak{G}] \rightarrow R$ the underlying map. Then the fact $\zeta x_{R[\mathfrak{G}]} = x_{R[\mathfrak{G}]} \in \mathfrak{X}(R[\mathfrak{G}])$ translates as follows

$$\begin{array}{ccc} R[\mathfrak{X}] & \xrightarrow{c} & R[\mathfrak{X}] \otimes_R R[A] \\ s_x \downarrow & & s_x \otimes id \downarrow \\ R & \longrightarrow & R[A] \\ r & \mapsto & r. \end{array}$$

If $f \in R[\mathfrak{X}]_a$, $a \neq 0$, we have $c(f) = f \otimes e_a$ which maps then to $f(x) \otimes e_a = f(x)$. Therefore $f(x) = 0$. It follows that $J \subset \ker(s_x)$, that is x defines an R -point of $\mathfrak{Y}(R)$. The same holds for any S/R , hence we conclude that $h_{\mathfrak{Y}} = F$. \square

⁵Again we use that I is a direct summand of the R -module $R[\mathfrak{X}]$.

9.2. Smoothness of the fixed point locus.

9.2.1. Theorem. *Assume that R is noetherian. Let \mathfrak{X}/R be an affine smooth R -scheme equipped with an action of the diagonalizable group scheme $\mathfrak{G} = \mathfrak{D}(A)$. Then the scheme of fixed points $\mathfrak{X}^\mathfrak{G}/R$ is smooth.*

For more general statements, see [SGA3, XII.9.6], [CGP, A.8.10] and [De, th. 5.4.4].

9.2.2. Corollary. *Assume that R is noetherian. Let \mathfrak{H}/R be an affine smooth group scheme equipped with an action of the diagonalizable group scheme $\mathfrak{G} = \mathfrak{D}(A)$. Then the centralizer subgroup scheme $\mathfrak{H}^\mathfrak{G}$ is smooth.*

We proceed to the proof of Theorem 9.2.1.

Proof. Since R is noetherian, the closed affine subscheme $\mathfrak{X}^\mathfrak{G}$ of \mathfrak{X} is of finite presentation. According to Theorem 20.0.2, it is enough to show that $\mathfrak{X}^\mathfrak{G}$ is formally smooth. We are given an R -ring C equipped with an ideal J satisfying $J^2 = 0$. We want to show that the map $\mathfrak{X}^\mathfrak{G}(C) \rightarrow \mathfrak{X}^\mathfrak{G}(C/J)$ is surjective. We start then with a point $\bar{x} \in \mathfrak{X}^\mathfrak{G}(C/J)$. Since \mathfrak{X} is smooth, \bar{x} lifts to a point $x \in \mathfrak{X}(C)$. We denote by $I(x) \subset C[\mathfrak{X}]$ the ideal of the regular functions vanishing at x . According to §8.5, we have an exact sequence of pointed C -functors

$$1 \longrightarrow \mathfrak{W}(M) \xrightarrow{i_x} \mathfrak{X}_C \longrightarrow \prod_{C/J} \mathfrak{X}_{C/J} \rightarrow 1.$$

where $M = \text{Hom}_{C\text{-mod}}(I(x)/I^2(x), J)$. Since \mathfrak{X}_C is equipped with an action of \mathfrak{G}_C , it induces an action on $\mathfrak{W}(M)$. In other words, M comes equipped with a \mathfrak{G}_C -module structure. For each $g \in \mathfrak{G}(C)$, we have $\bar{g} \cdot \bar{x} = \bar{g} \cdot \bar{x} = \bar{x}$ since \bar{x} is \mathfrak{G} -invariant. Hence $g \cdot x = i_x(c(g)) = x + c(g)$ for a unique $c(g) \in M$. Now we take $g_1, g_2 \in \mathfrak{G}(C)$ and compute

$$g_1 \cdot (g_2 \cdot x) = g_1 \cdot (x + c(g_2)) = g_1 \cdot x + g_1 \cdot c(g_2) = x + (c(g_1) + g_1 \cdot c(g_2)).$$

By unicity, we have the 1-cocycle formula $c(g_1 g_2) = c(g_1) + g_1 \cdot c(g_2) \in M$ for $g_1, g_2 \in \mathfrak{G}(C)$. We can do the same for each C -ring C' and obtain then a 1-cocycle for the Hochschild cohomology. Since $H_0^1(\mathfrak{G}, M) = 0$ (theorem 6.5.7), there exists $m \in M$ such that $c(g) = g \cdot m - m$ for each C'/C and each $g \in \mathfrak{G}(C')$. It means exactly that the point $i_x(m) \in \mathfrak{X}(R)$ is \mathfrak{G} -invariant. It maps to \bar{x} , so we conclude that $\mathfrak{X}^\mathfrak{G}(C) \rightarrow \mathfrak{X}^\mathfrak{G}(C/J)$ is onto. \square

9.2.3. Remark. If $x \in \mathfrak{X}(R)$, the tangent space at x of $\mathfrak{X}^\mathfrak{G}$ is

$$T_{\mathfrak{X}^\mathfrak{G}, x} = H_0^0(\mathfrak{G}, T_{\mathfrak{X}, x}).$$

10. LIFTING HOMOMORPHISMS

10.1. Rigidity principle. Let $\mathfrak{G} = \mathfrak{D}(A)/R$ be a diagonalizable group scheme. The following fact illustrates the “rigidity” of \mathfrak{G} .

10.1.1. Lemma. *Let I be a nilpotent ideal of R .*

(1) *Let M be an $R - \mathfrak{G}$ -module. Then M is a trivial $R - \mathfrak{G}$ -module if and only if $M \otimes_R R/I$ is a trivial $R/I - \mathfrak{G}_{R/I}$ -module.*

(2) *Assume that \mathfrak{G} acts on an affine R -scheme \mathfrak{X} . Then \mathfrak{G} acts trivially on \mathfrak{X} if and only if $\mathfrak{G} \times_R R/I$ acts trivially on $\mathfrak{X} \times_R R/I$.*

Proof. (1) The direct way is obvious. Conversely, we assume that $M \otimes_R R/I$ is a trivial $R/I - \mathfrak{G}_{R/I}$ -module. We have

$$M_0/IM_0 = (M \otimes_R R/I)_0 = (M \otimes_R R/I)$$

according to Corollary 6.2.5. By the nilpotent Nakayama lemma [St, 18.1.11], the map $M_0 \rightarrow M$ is onto hence an isomorphism.

(2) We apply (1) to the \mathfrak{G} -module $R[\mathfrak{X}]$. □

Here is a variation on the same theme not used in the sequel.

10.1.2. Lemma. *Let I be a nilpotent ideal of R .*

(1) *Let M, M' be two $R - \mathfrak{G}$ -modules which are projective R -modules. Then $M \xrightarrow{\sim} M'$ as \mathfrak{G} -modules if and only if the $\mathfrak{G}_{R/I}$ -modules $M \otimes_R R/I$ and $M' \otimes_R R/I$ are isomorphic.*

(2) *Assume that \mathfrak{G} acts on an affine R -scheme \mathfrak{X} in two ways $u, v : \mathfrak{G} \rightarrow \text{Aut}(\mathfrak{X})$. Assume that $R[\mathfrak{X}]$ is projective. Then $u = v$ if and only if $u \times_R R/I = v \times_R R/I$.*

Proof. (1) The direct way is obvious. In the way around, we fix an isomorphism $f : M \otimes_R R/I \xrightarrow{\sim} M' \otimes_R R/I$ of $R/I - \mathfrak{G}_{R/I}$ -modules. Let $a \in A$. Then M_a and M'_a are projective. We have $M_a \otimes R/I \xrightarrow{\sim} M'_a \otimes R/I$ hence this map lifts in an isomorphism $\tilde{f}_a : M_a \xrightarrow{\sim} M'_a$ by the Nakayama fact below. By summing up the M'_a s, we get an isomorphism of \mathfrak{G} -modules $M \xrightarrow{\sim} M'$.

(2) We apply (1) to $M = R[\mathfrak{X}]$. □

10.1.3. Lemma. *Let I be a nilpotent ideal of R . Let M, M' be projective R -modules. Then M and M' are isomorphic if and only if M/IM and M'/IM' are isomorphic.*

Proof. The direct way is obvious. Conversely, we are given an isomorphism $f : M/IM \xrightarrow{\sim} M'/IM'$. Since M is projective the map $M \rightarrow M \otimes R/I \xrightarrow{f} M' \otimes R/I$ lifts to a map $f^\sharp : M \rightarrow M'$. In the other hand, f^{-1} lifts in a morphism $f^\dagger : M' \rightarrow M$. By construction $f^\dagger \circ f^\sharp = id_M + h$ with

$h \in \text{End}_R(M)$ and $h(M) \subset IM$. Then h is nilpotent so $f^\dagger \circ f^\sharp$ is invertible in $\text{End}_R(M)$. Similarly $f^\sharp \circ f^\dagger$ is invertible in $\text{End}_R(M')$, so we conclude that f is an isomorphism. \square

The next statement also illustrates the rigidity principle.

10.1.4. Theorem. [SGA3, §IX.6] *We assume that A is finitely generated. Let $f : \mathfrak{G} \rightarrow \mathfrak{H}$ be a finitely presented group homomorphism to an affine R -group scheme \mathfrak{H} of finite presentation. Let $x \in \text{Spec}(R)$ be a point such that the homomorphism $f_x : \mathfrak{G}_{\kappa(x)} \rightarrow \mathfrak{H}_{\kappa(x)}$ is a monomorphism. Then there exists a Zariski neighbourhood $\text{Spec}(R')$ of $\text{Spec}(R)$ of x such that $f_{R'}$ is a monomorphism.*

We present an alternative proof.

Proof. We can assume that R is the local ring at x . We denote by $\mathfrak{K} = \ker(f)$. From §3.5, we have to show that $\mathfrak{K}_R = 1$. Our hypothesis reads $\mathfrak{K}_{\kappa(x)} = 1$.

First case: \mathfrak{H}/R admits a faithful linear representation. We have then only to deal with the case of $\mathfrak{H} = \text{GL}_d$, that is with a \mathfrak{G} -module M such that $M \cong R^d$ such that the associated representation ρ_x is a monomorphism. Denote by A^\sharp the (finite set) of weights of $\rho_{\kappa(x)}$. According to Lemma 6.2.7, A^\sharp spans the abelian group A . For $a \in A^\sharp$, M_a is a non-zero module which is projective since it is a direct summand of the free module $M = R^d$. By Nakayama lemma, $M' := \bigoplus_{a \in A^\sharp} M_a$ is isomorphic to M so that $\ker(\rho_M) = 1$.

General case. We shall show that \mathfrak{K} is proper by using the valuative criterion. Let A/R be a valuation ring and denote by F its fraction field. The point is that \mathfrak{H}_A admits a faithful representation (th. 6.4.1). Also the closed point of $\text{Spec}(A)$ maps to the closed point of $\text{Spec}(R)$ so that $\mathfrak{K}_A = 1$ by the first case. Therefore $\mathfrak{K}(A) = \mathfrak{K}(F)$ and \mathfrak{K} is proper. Since \mathfrak{K} is affine, \mathfrak{K} is finite over R [Li, 3.17]. Hence $R[\mathfrak{K}]$ is a finite R -algebra such that $R/\mathfrak{M}_x \xrightarrow{\sim} R[\mathfrak{K}]/\mathfrak{M}_x R[\mathfrak{K}]$. The Nakayama lemma [St, 18.1.11.(6)] shows that the map $R \rightarrow R[\mathfrak{K}]$ is surjective. By using the unit section $1_{\mathfrak{K}}$ we conclude that $R = R[\mathfrak{K}]$. \square

10.1.5. Remark. We shall see later (i.e. Cor. 16.3.1) that a monomorphism $\mathfrak{D}(A)_R \rightarrow \mathfrak{H}$ is a closed immersion.

10.2. Formal smoothness. Let $\mathfrak{G}/R, \mathfrak{H}/R$ be two affine group schemes. We define the following R -functors $\mathbf{Hom}(\mathbf{G}, \mathbf{H}), \overline{\mathbf{Hom}}(\mathbf{G}, \mathbf{H})$ by

$$\frac{\mathbf{Hom}(\mathbf{G}, \mathbf{H})(S)}{\mathbf{Hom}(\mathbf{G}, \mathbf{H})(S)} = \frac{\text{Hom}_{S-\text{gr}}(\mathbf{G}_S, \mathbf{H}_S)}{\text{Hom}_{S-\text{gr}}(\mathbf{G}_S, \mathbf{H}_S)/\mathbf{H}(S)}$$

for each S/R .

10.2.1. Theorem. *Assume that $\mathfrak{G} = \mathfrak{D}(A)$ is diagonalizable and that \mathfrak{H} is smooth.*

- (1) *The R -functor $\mathbf{Hom}(\mathbf{G}, \mathbf{H})$ is formally smooth.*
- (2) *The R -functors $\overline{\mathbf{Hom}}(\mathbf{G}, \mathbf{H})$ and $\mathbf{Homcent}(\mathbf{G}, \mathbf{H})$ are formally étale.*

Proof. (1) Let C be an R -ring equipped with an ideal J satisfying $J^2 = 0$. We are given a C/J -homomorphism $f_0 : \mathfrak{G}_{C/J} \rightarrow \mathfrak{H}_{C/J}$ and want to lift it. We put $I = \ker(C[\mathfrak{H}] \xrightarrow{\epsilon} C)$. We have $\mathrm{Lie}(\mathfrak{H})(C) = (I/I^2)^\vee$. Since \mathfrak{H} is smooth we have an exact sequence of group C -functors

$$1 \longrightarrow \mathfrak{V}(\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J) \xrightarrow{\exp} \mathfrak{H}_C \longrightarrow \prod_{(C/J)/C} \mathfrak{H}_{C/J} \rightarrow 1.$$

Note that the $\prod_{(C/J)/C} \mathfrak{H}_{C/J}$ -structure on $\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{H})(C/J) \otimes_{C/J} J$ arises from the adjoint representation of $\mathfrak{H}_{C/J}$. Now we pull-back this extension by the map of R -functors

$$u : \mathfrak{G}_C \prod_{(C/J)/C} \mathfrak{G}_{C/J} \xrightarrow{\prod_{(C/J)/C} f_0} \prod_{(C/J)/C} \mathfrak{H}_{C/J}.$$

It defines a C -group functor E which fits in the commutative exact diagram of C -group functors

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathfrak{V}(\mathrm{Lie}(\mathfrak{H}) \otimes_C J) & \xrightarrow{\exp} & \mathfrak{H}_C & \longrightarrow & \prod_{(C/J)/C} \mathfrak{H}_{C/J} \longrightarrow 1 \\ & & \parallel & & \uparrow v & & \uparrow u \\ 1 & \longrightarrow & \mathfrak{V}(\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J) & \longrightarrow & E & \longrightarrow & \mathfrak{G} \longrightarrow 1. \end{array}$$

According to Corollary 6.7.2, the bottom extension splits and a splitting defines then an R -group map $\mathfrak{G}_C \rightarrow \mathfrak{H}_C$ which lifts f_0 .

(2) Exactly as in the abstract group setting, the choice of a lifting is the same that the choice of a splitting of the bottom extension. Up to $\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J$ -conjugacy, this choice is encoded by the Hochschild cohomology group $H^1(\mathfrak{G}, \mathrm{Lie}(\mathfrak{H})(C) \otimes_C J)$. But this group vanishes (Th. 6.7.2), hence all liftings are $\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J$ -conjugated. This shows that $\overline{\mathrm{Hom}}(\mathbf{G}, \mathbf{H})(C) \xrightarrow{\sim} \overline{\mathrm{Hom}}(\mathbf{G}, \mathbf{H})(C/J)$ and we conclude that $\overline{\mathrm{Hom}}(\mathbf{G}, \mathbf{H})$ is a formally étale functor.

Now assume that f_0 is central. According to the rigidity principle 10.1.1.(2), any lifting f of f_0 is central as well. If f_1, f_2 lift f_0 , they are $\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J$ -conjugated, hence equal. It yields that $\mathrm{Homcent}(\mathbf{G}, \mathbf{H})$ is a formally étale functor. \square

10.3. Algebraization.

10.3.1. Theorem. [SGA3, IX.7.1] *Assume that R is noetherian and equipped with an ideal I such that A is separated and complete for the I -adic topology. We put $R_n = R/I^{n+1}$ for each $n \geq 0$. Let $\mathfrak{G} = \mathfrak{D}(A)$ be a diagonalizable group scheme and let \mathfrak{H}/R be a smooth affine group scheme. Then the natural map*

$$\mathrm{Hom}(\mathfrak{G}, \mathfrak{H})(R) \rightarrow \varprojlim_n \mathrm{Hom}(\mathfrak{G}, \mathfrak{H})(R_n)$$

is bijective.

10.3.2. Remarks. (1) Injectivity is the easy thing there. Let $u, v : \mathfrak{G} \rightarrow \mathfrak{H}$ be two homomorphisms such that $u_n = u_{R_n}$ and $v_n = v_{R_n}$ agree. We consider then the R -module map $u^* - v^* : R[\mathfrak{H}] \rightarrow R[\mathfrak{G}]$. Our hypothesis implies that $\text{Im}(u^* - v^*) \subset I^{n+1}R[\mathfrak{G}]$ for each $n \geq 0$. Since $\bigcap_n I^n = 0$ and $R[\mathfrak{G}]$ is a free R -module, we conclude that $\text{Im}(u^* - v^*) = 0$ and $u = v$.

(2) The case \mathfrak{G} is finite over R (i.e. A is finite) is easy. Let $u_n : \mathfrak{G} \rightarrow \mathfrak{H}$ be a coherent family of group homomorphisms. Then we have a commutative diagram

$$\begin{array}{ccc} R[\mathfrak{H}] & & R[\mathfrak{G}] \\ \downarrow & & \downarrow \cong \\ \varprojlim_n R_n[\mathfrak{H}] & \xrightarrow{\varprojlim_n u_n} & \varprojlim_n R_n[\mathfrak{H}]. \end{array}$$

The point is that $R[\mathfrak{G}] = R[A]$ is a finite free R -module whence the right vertical map is an isomorphism. The diagram defines then a map $u : \mathfrak{H} \rightarrow \mathfrak{G}$ which is a group homomorphism and lifts the u_n .

Theorem 10.2.1 implies that the transition maps $\text{Hom}(\mathfrak{G}, \mathfrak{H})(R_{n+1}) \rightarrow \text{Hom}(\mathfrak{G}, \mathfrak{H})(R_n)$ are surjective. It yields the first assertion in the next statement.

10.3.3. Corollary. (1) The map $\mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R) \rightarrow \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R/I)$ is surjective.

(2) If $f, f' \in \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R)$ coincide in $\mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R/I)$, then there exists $h \in \ker(\mathfrak{H}(R) \rightarrow \mathfrak{H}(R/I))$ such that $f = g f' h^{-1}$.

(3) $\mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R) \xrightarrow{\sim} \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R/I)$.

(4) If $f \in \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R)$, f is a monomorphism and only if $f_{R/I}$ is a monomorphism.

Proof. (2) We have $\mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R_1) \xrightarrow{\sim} \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R_0)$. More precisely we have seen that f_1 and f'_1 are conjugated under an element of $\ker(\mathfrak{H}(R_1) \rightarrow \mathfrak{H}(R_0))$. Since \mathfrak{H} is smooth, $\mathfrak{H}(R)$ maps onto $\mathfrak{H}(R_1)$, so there exists $h_1 \in \ker(\mathfrak{H}(R) \rightarrow \mathfrak{H}(R_0))$ such that $f'_1 = {}^{h_1}f_1$. We continue and construct by induction a sequence of elements $h_n \in \mathfrak{H}(R)$ such that $h_n \in \ker(\mathfrak{H}(R) \rightarrow \mathfrak{H}(R_n))$ and $f'_n = {}^{h_n h_{n-1} \dots h_1} f_n$. The sequence $h_n h_{n-1} \dots h_1$ converges to an element $h \in \mathfrak{H}(R)$ such that h and h_n agree in $\mathfrak{H}(R_n)$ for each $n \geq 1$. It follows that f and ${}^h f'$ agree in $\text{Hom}(\mathfrak{G}, \mathfrak{H})(R_n)$ for each $n \geq 0$, so are equal.

(3) Using that $\mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R_{n+1}) \xrightarrow{\sim} \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R_n)$, we see that the map $\mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R) \rightarrow \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R/I)$ is injective. For the surjectivity a central homomorphism $u_0 : \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R/I)$ gives rise to coherent system of central homomorphisms $u_n \in \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R_n)$. This system lifts uniquely in $u \in \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R)$ and we have to show that u is central. We consider then the adjoint action of \mathfrak{G} on \mathfrak{H} . By Theorem 9.2.1, $\mathfrak{H}^{\mathfrak{G}}$ is a closed group subscheme \mathfrak{H} which is then of finite presentation. The closed immersion i satisfies $\mathfrak{H}^{\mathfrak{G}} \times_R R/I \xrightarrow{\sim} \mathfrak{H} \times_R R/I$ and $I = \text{rad}(R)$

[Ma, th. 8.2]. Corollary 20.0.6 yields that i is étale, hence $\mathfrak{H}^\mathfrak{G}$ is open in \mathfrak{H} . Since it contains $\mathfrak{H} \times_R R/I$, we have $\mathfrak{H}^\mathfrak{G} = \mathfrak{H}$. Thus u is a central homomorphism.

(4) This is a special case of Theorem 10.1.4. \square

10.4. Rank in family.

10.4.1. Definition. Let H/k be an affine algebraic group defined over a field k . Denote by \bar{k} an algebraic closure of k . We denote by $\text{rank}_{\text{red}}(H)$ the (absolute) reductive rank of H , namely the maximal dimension of a split \bar{k} -torus of $H \times_k \bar{k}$.

Similarly, we denote by $\text{rank}_{\text{red},\text{cent}}(H)$ the (absolute) central reductive rank of H , namely the maximal dimension of a central split \bar{k} -torus of $H \times_k \bar{k}$.

This definition does not depend of the choice of the closure; both ranks remain the same after an arbitrary field extension F/k .

10.4.2. Theorem. Let \mathfrak{H}/R be an affine smooth group scheme. Assume that R is noetherian. Then the map

$$\begin{aligned} \text{Spec}(R) &\longrightarrow \mathbb{Z}_{\geq 0} \\ x &\mapsto \text{rank}_{\text{red}}(H \times_R \kappa(x)) \end{aligned}$$

is lower semi-continuous and idem for $\text{rank}_{\text{red}-\text{cent}}$.

Proof. Firstly, we notice that we are authorised to make an extension R'/R such that $\text{Spec}(R') \rightarrow \text{Spec}(R)$ is surjective (and R' noetherian). Also the statement is of local nature, hence we can suppose that R is local with maximal ideal \mathfrak{M} and residue field k . Let r be the rank of $\mathfrak{H} \times_R k$. Our assumption reads that there exists a finite field extension k'/k such that $\mathfrak{H} \times_R k'$ contains a k' -torus $\mathbb{G}_{m,k'}^r$. There exists a finite flat local morphism of noetherian local rings $R \rightarrow R'$ inducing $k \rightarrow k'$ [EGA3, 10.3.1, 10.3.2]. Then R'/R is finite locally free and faithfully flat. Hence without loss of generality, we can assume that \mathfrak{H}_k contains a k -torus $\mathbb{G}_{m,k}^r$. The completion $\hat{R} = \varprojlim_n R/\mathfrak{M}^n$ is complete and separated for the \mathfrak{M} -adic topology, is noetherian and faithfully flat over R [Li, §1.3.3]. We are then allowed to replace R by \hat{R} .

We fix the closed immersion $f : \mathbb{G}_{m,k}^r \rightarrow \mathfrak{H}_k$. By Corollary 10.3.3.(1), it lifts to an homomorphism $\tilde{f} : \mathbb{G}_{m,\hat{R}}^r \rightarrow \mathfrak{H} \times_R \hat{R}$ which is a monomorphism (Cor. 10.3.3.(4)). For each $y \in \text{Spec}(\hat{R})$, we have then

$$\text{rank}_{\text{red}}(\mathfrak{H} \times_R \kappa(y)) \geq r$$

as desired.

The second statement follows similarly of Corollary 10.3.3.(3). \square

Reductive group schemes

11. REDUCTIVE GROUP SCHEMES

If k is an algebraically closed field, an affine algebraic group G/k is reductive if it is smooth connected and if its unipotent radical is trivial [Sp, §8].

11.0.1. Definition. *An affine R -group scheme \mathfrak{G} is reductive if it satisfies the two following requirements:*

- (1) \mathfrak{G}/R is smooth;
- (2) For each $x \in \operatorname{Spec}(R)$, the geometric fiber $\mathfrak{G} \times_R \bar{\kappa}(x)$ is reductive where $\bar{\kappa}(x)$ stands for an algebraic closure of the residue field $\kappa(x)$.

11.0.2. Remark. A naive approach could be to consider the unipotent radical of \mathfrak{G} but this object does not exist ! The problem occurs already over a non-perfect field, see the introduction of [CGP]. However we shall see later an equivalent definition.

11.0.3. Examples. (1) The diagonalizable group $\mathfrak{D}(\mathbb{Z}^r) = \mathbb{G}_{m,R}^r$ is a reductive group scheme, the linear group GL_n/R is a reductive group scheme and SL_n as well.

(2) A fibered R -product of reductive group schemes is reductive.

Reductivity is stable under base change of the base ring. Also it is an open property among the smooth affine groups with connected fibers [SGA3, XIX.2.6]. We can already prove a useful stability fact.

11.0.4. Proposition. *Let \mathfrak{H}/R be a reductive group scheme and let $f : \mathfrak{T} = \mathbb{G}_m^r \rightarrow \mathfrak{H}$ be a homomorphism. Then the centralizer $\mathfrak{H}^{\mathfrak{T}}/R$ is a reductive group scheme.*

Proof. We know that $\mathfrak{G}^{\mathfrak{T}}/R$ is a smooth group scheme (Th. 9.2.1) so satisfies the first requirement. For the second one, we are reduced to the case of an algebraically closed field. In this case, see [Bo, §13.17]. \square

11.0.5. Definition. *Let \mathfrak{H}/R be an affine group scheme and let $i : \mathfrak{T} = (\mathbb{G}_{m,R})^r \rightarrow \mathfrak{H}$ be a monomorphism. We say that \mathfrak{T} is a maximal (resp. central maximal) R -torus of \mathfrak{H} if for each $x \in \operatorname{Spec}(R)$, $\mathfrak{T} \times_R \bar{\kappa}(x)$ is a maximal (resp. central maximal) $\bar{\kappa}(x)$ -torus of $\mathfrak{H} \times_R \bar{\kappa}(x)$.*

As in the field case, we have the following characterization of maximal tori.

11.0.6. Proposition. *Assume that R is noetherian. Let \mathfrak{H}/R be an affine reductive group scheme and let $i : \mathfrak{T} = (\mathbb{G}_{m,R})^r \rightarrow \mathfrak{H}$ be a monomorphism. Then the following are equivalent:*

- (1) $i : \mathfrak{T} \rightarrow \mathfrak{H}$ is a maximal torus;
- (2) $\mathfrak{T} \xrightarrow{\sim} \mathfrak{H}^{\mathfrak{T}}$.

Note in particular that i is a closed immersion.

Proof. From the field case, the map $f : \mathfrak{T} \rightarrow \mathfrak{H}^{\mathfrak{T}}$ is such that for each $x \in \operatorname{Spec}(R)$, $f_x = f \times_R \kappa(x)$ is an isomorphism. Since \mathfrak{T} is smooth, the fiberwise criterion 20.0.4 yields that f is an isomorphism. \square

We continue with the following local statement.

11.0.7. Proposition. *Let R be a noetherian ring equipped with an ideal I . We assume that R is complete and separated for the I -adic topology. We assume that $\mathfrak{H} \times_R R/I$ admits a maximal R/I -torus $\mathbb{G}_{m,R/I}^r$ (resp. central maximal R/I -torus $\mathbb{G}_{m,R/I}^s$).*

(1) $\mathfrak{H} \times_R R$ admits a maximal (resp. central maximal) torus $\mathbb{G}_{m,R}^r$ (resp. $\mathbb{G}_{m,R}^s$).

(2) For each $x \in \operatorname{Spec}(R)$, we have $\operatorname{rank}_{\text{red}}(\mathfrak{H} \times_R \kappa(x)) = r$ and $\operatorname{rank}_{\text{red}}(\mathfrak{H} \times_R \kappa(x)) = s$.

In other words, $\mathfrak{H} \times_R R/I$ is split and only if \mathfrak{H} is split.

Proof. We do only the case of the reductive rank since the other case is similar.

(1) We are given a monomorphism $f_0 : \mathbb{G}_{m,R/I}^r \rightarrow \mathfrak{H} \times_R R/I$ which is a maximal R/I -torus of \mathfrak{G} . According to Corollary 10.3.3.(1), it lifts to an R -homomorphism $f : \mathfrak{T} = \mathbb{G}_m^r \rightarrow \mathfrak{H}$. We consider the R -subgroup centralizer $\mathbf{Cent}_{\mathfrak{G}}(\mathfrak{T}) = \mathfrak{H}^{\mathfrak{T}}$ which is reductive according to Proposition 11.0.4.

Now the R -map $f : \mathfrak{T} \rightarrow \mathfrak{H}^{\mathfrak{T}}$ is such that $f_{R/I}$ is an isomorphism by Proposition 11.0.6. Both schemes are smooth and again we notice that $I = \operatorname{rad}(R)$. We can apply then the trick 20.0.6, it yields that f is étale. But f is a monomorphism, hence f is an open immersion. Its image contains $\mathfrak{H}^{\mathfrak{T}} \times_R R/I$, so is \mathfrak{H}^T . Thus $\mathfrak{T} \xrightarrow{\sim} \mathfrak{H}^T$ and \mathfrak{T} is then a maximal R -torus.

(2) For each $x \in \operatorname{Spec}(R)$, $\mathfrak{T} \times_R \kappa(x)$ is a maximal $\kappa(x)$ -torus of $\mathfrak{H} \times_R \kappa(x)$ whence the result. \square

This enables us to improve the “lower continuity” theorem (i.e. th. 10.4.2) in the reductive case.

11.0.8. Corollary. *Let \mathfrak{H}/R be an affine smooth group scheme. Assume that R is noetherian. Then the map*

$$\begin{array}{ccc} \operatorname{Spec}(R) & \longrightarrow & \mathbb{Z}_{\geq 0} \\ x & \mapsto & \operatorname{rank}_{\text{red}}(H \times_R \kappa(x)) \end{array}$$

is continuous and idem for $\operatorname{rank}_{\text{red-cent}}$.

The proof goes along the same lines.

12. LIMIT GROUPS

This part is mainly taken from [CGP, §2.1] and [GP3, §15].

12.1. Limit functors. Let \mathfrak{X}/R be an affine scheme equipped with an action $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(\mathfrak{X})$. We define the R -subfunctor \mathfrak{X}_λ of $h_{\mathfrak{X}}$ by

$$\mathfrak{X}_\lambda(S) = \left\{ x \in \mathfrak{X}(S) \mid \lambda(t).x \in \mathfrak{X}(S[t]) \subset \mathfrak{X}(S[t, t^{-1}]) \right\}$$

for each S/R . It is called the limit functor of \mathfrak{X} with respect to λ since $\mathfrak{X}_\lambda(R)$ consists in the elements $x \in \mathfrak{X}(R)$ such that $\lambda(t).x$ has a limit when $t \mapsto 0$.

12.1.1. Lemma. *The R -functor \mathfrak{X}_λ is representable by a closed R -subscheme of \mathfrak{X} .*

Proof. It is similar to that of Proposition 9.1.1. We consider the decomposition in eigenspaces

$$R[\mathfrak{X}] = \bigoplus_{n \in \mathbb{Z}} R[\mathfrak{X}]_n.$$

We denote by I the ideal of $R[\mathfrak{X}]$ generated by $\bigoplus_{n < 0} R[\mathfrak{X}]_n$. We let the reader to check that the closed subscheme $\text{Spec}(R[\mathfrak{X}]/I)$ does the job. \square

We denote by \mathfrak{X}^λ the fixed point locus for the action. Clearly \mathfrak{X}^λ is a \mathbb{G}_m -subscheme of \mathfrak{X} and \mathfrak{X}^λ is an R -subscheme of \mathfrak{X}_λ . The specialization at 0 induces an R -map $q^\dagger : \mathfrak{X}_\lambda \rightarrow \mathfrak{X}$.

12.1.2. Lemma. (1) $\mathfrak{X}^\lambda = \mathfrak{X}_\lambda \times_{\mathfrak{X}} \mathfrak{X}^{-\lambda}$.

(2) *The map q^\dagger factorizes by X^λ . It defines then an R -map $q : \mathfrak{X}_\lambda \rightarrow \mathfrak{X}^\lambda$ and the composite $\mathfrak{X}^\lambda \rightarrow \mathfrak{X}_\lambda \xrightarrow{q} \mathfrak{X}^\lambda$ is the identity.*

Proof. (1) If $x \in \mathfrak{X}_\lambda(R) \cap \mathfrak{X}^{-\lambda}(R)$, we have $\lambda(t).x \in \mathfrak{X}(R[t]) \cap \mathfrak{X}(R[t^{-1}]) = \mathfrak{X}(R)$. Hence $\lambda(t).x = x$ and $x \in \mathfrak{X}^\lambda(R)$.

(2) Let $x \in \mathfrak{X}_\lambda(R)$ and put $x' = q(x) \in \mathfrak{X}(R)$. For each $a \in R^\times$, we have $\lambda(at).x = \lambda(a)(\lambda(t).x)$. By doing $t \mapsto 0$, we get that $x' = \lambda(a).x'$. Hence $R^\times.x' = x'$. The same holds for each R -extension S/R , so we conclude that $x' \in \mathfrak{X}^\lambda(R)$. \square

12.2. The group case. We consider now the case of a group homomorphism $\lambda : \mathbb{G}_m \rightarrow \mathbb{G}$ where \mathbb{G} is an affine R -group scheme. We denote by $\mathfrak{P}_\mathbb{G}(\lambda) = \mathbb{G}_\lambda$. We have then an R -homomorphism $\mathfrak{P}_\mathbb{G}(\lambda) \rightarrow \mathfrak{Z}_\mathbb{G}(\lambda)$ which is split. We denote by $\mathfrak{U}_\mathbb{G}(\lambda) = \ker(q)$ and we have then

$$\mathfrak{P}_\mathbb{G}(\lambda) = \mathfrak{U}_\mathbb{G}(\lambda) \rtimes \mathfrak{Z}_\mathbb{G}(\lambda).$$

For each ring S/R , $\mathfrak{P}_\mathbb{G}(\lambda)(S)$ (resp. $\mathfrak{U}_\mathbb{G}(\lambda)(S)$) consists in the $g \in \mathbb{G}(S)$ such that $\lambda(t)g\lambda(t^{-1})$ admits a limit (resp. converges to 1) when $t \mapsto 0$. The group scheme $\mathfrak{P}_\mathbb{G}(\lambda)/R$ is called the limit group scheme attached to λ .

12.2.1. Example. If we take the diagonal map $\lambda(t) = (t^{a_1}, \dots, t^{a_1}, t^{a_2}, \dots, t^{a_2}, \dots, t^{a_r}, \dots, t^{a_r})$ in $\text{GL}_{m_1 + \dots + m_r}$ with respective multiplicities m_1, \dots, m_r and $a_1 < a_2 < \dots < a_r$,

we find

$$\mathfrak{P}_{\mathrm{GL}_d}(\lambda) = \left(\begin{array}{c|c|c|c|c} A_{1,1} & A_{1,2} & \dots & \dots & A_{1,r} \\ \hline 0 & A_{2,2} & \dots & \dots & A_{2,r} \\ \hline 0 & 0 & A_{3,3} & \dots & \cdot \\ \hline \dots & \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & 0 & A_{r,r} \end{array} \right).$$

From the functor viewpoint, we have

$$\mathrm{Lie}(\mathfrak{P}_\lambda(\mathfrak{G})) = \mathrm{Lie}(\mathfrak{G})(R)_{\geq 0}; \quad \mathrm{Lie}(\mathfrak{U}_\lambda(\mathfrak{G})) = \mathrm{Lie}(\mathfrak{G})(R)_{> 0}.$$

Hence $\mathrm{Lie}(\mathfrak{P}_\lambda(\mathfrak{G})) = \mathrm{Lie}(\mathfrak{Z}_\mathfrak{G}(\lambda))(R)_{\geq 0} \oplus \mathrm{Lie}(\mathfrak{U}_\lambda(\mathfrak{G}))$. Note also that the product R -map

$$i_\lambda : \mathfrak{U}_\mathfrak{G}(-\lambda) \times \mathfrak{P}_\mathfrak{G}(\lambda) \rightarrow \mathfrak{G}$$

is a monomorphism since $\mathfrak{P}_\mathfrak{G}(-\lambda) \times_\mathfrak{G} \mathfrak{P}_\mathfrak{G}(\lambda) = \mathfrak{Z}_\mathfrak{G}(\lambda)$. This map plays an important role in the theory. In the same flavour as Theorem 9.2.1, we have the following fact.

12.2.2. Theorem. [CGP, 2.1.8] *Assume that \mathfrak{G} is smooth.*

- (1) *The R -group schemes $\mathfrak{P}_\lambda(\mathfrak{G})$ and $\mathfrak{U}_\lambda(\mathfrak{G})$ are smooth. Furthermore for each $s \in \mathrm{Spec}(R)$, $\mathfrak{U}_\lambda(\mathfrak{G})_{\kappa(s)}$ is a split unipotent group.*
- (2) *The monomorphism i_λ above is an open immersion.*

We skip the proof which is quite technical.

12.3. Parabolic and Borel subgroup schemes.

12.3.1. Definition. *Let \mathfrak{G}/R be a reductive group scheme. An R -subgroup scheme \mathfrak{P} of \mathfrak{G} is parabolic subgroup (resp. a Borel subgroup) if it satisfies the two requirements:*

- (1) *\mathfrak{P} is smooth;*
- (2) *for each $s \in \mathrm{Spec}(R)$, $\mathfrak{G} \times_R \overline{\kappa(s)}$ is a parabolic (resp. a Borel) subgroup of $\mathfrak{G} \times_R \overline{\kappa(s)}$.*

12.3.2. Corollary. *Let $\lambda : \mathbb{G}_m \rightarrow \mathfrak{G}$ be a homomorphism in a reductive group scheme. Then $\mathfrak{P}_\mathfrak{G}(\lambda)$ is an R -parabolic subgroup.*

This follows from the field case [Sp, §15.1] since $\mathfrak{P}_\mathfrak{G}(\lambda)$ is smooth. By the way, in the field case, each parabolic subgroup is of this shape and this can be extended.

12.3.3. Lemma. *Assume that (R, \mathfrak{M}, k) is noetherian local and let \mathfrak{G}/R be a reductive group scheme and let \mathfrak{P}/R be a parabolic subgroup scheme.*

- (1) *Let $\lambda : \mathbb{G}_m \rightarrow \mathfrak{P}$ be a homomorphism. Then the following are equivalent:*
 - (i) $\mathfrak{P}_\mathfrak{G}(\lambda) = \mathfrak{P}$;
 - (ii) $\mathfrak{P}_\mathfrak{G}(\lambda) \times_R k = \mathfrak{P} \times_R k$.
- (2) *There exists $\lambda \in \mathrm{Hom}_{\widehat{R}}(\mathbb{G}_{m, \widehat{R}}, \mathfrak{P}_{\widehat{R}})$ such that $\mathfrak{P}_{\mathfrak{G}_{\widehat{R}}}(\lambda) = \mathfrak{P}_{\widehat{R}}$.*

Proof. (1) We consider the closed immersion $i : \mathfrak{P}_{\mathfrak{P}}(\lambda) \rightarrow \mathfrak{P}$. The map i_k is an isomorphism and both group schemes are smooth. By Corollary 20.0.6, we get that $i_{\widehat{R}}$ is an isomorphism. By faithfully flat descent, i_R is then an isomorphism. In the same way, one show that $\mathfrak{P}_{\mathfrak{P}}(\lambda) \xrightarrow{\sim} \mathfrak{P}_{\mathfrak{G}}(\lambda)$. We conclude that $\mathfrak{P} = \mathfrak{P}_{\mathfrak{G}}(\lambda)$ as desired.

(2) There exists $\lambda_0 \in \text{Hom}_{k-gp}(\mathbb{G}_{m,k}, \mathfrak{P}_k)$ such that $\mathfrak{P}_k = \mathfrak{P}_{\mathfrak{G}_k}(\lambda_0)$. Since $\text{Hom}_{\widehat{R}}(\mathbb{G}_{m,\widehat{R}}, \mathfrak{P}_{\widehat{R}}) \rightarrow \text{Hom}_{k-gp}(\mathbb{G}_{m,k}, \mathfrak{P}_k)$ is onto (Cor. 10.3.3.(1)), we can pick a lift $\lambda : \mathbb{G}_{m,\widehat{R}} \rightarrow \mathfrak{P}_{\widehat{R}}$ of λ_0 . We apply then (1). \square

12.3.4. Remarks. (1) Assertion (2) is a special case of the same statement without completion, see [GP3, 15.5]. This comes later in the theory.

(2) If $\mathfrak{P}(\lambda)$ is a Borel subgroup, then $\mathfrak{Z}_{\mathfrak{G}}(\lambda)$ is a maximal R -torus. It is true that a Borel subgroup of \mathfrak{G}/R contains a maximal R -torus [SGA3, XXVI.2.3].

(3) The method of the lemma can be used also for lifting parabolic subgroups from the residue field to \widehat{R} .

13. ROOT DATA, TYPE OF REDUCTIVE GROUP SCHEMES

Root systems come from the study of reductive Lie algebras and for studying reductive groups, we need a richer datum which permits to distinguish for example $\text{SL}_{n,\mathbb{C}}$ of $\text{GL}_{n,\mathbb{C}}$ or $\text{PGL}_{n,\mathbb{C}}$. We follow here verbatim [Sp, §7.4], see also [SGA3, XXI].

13.1. Definition. A root datum is a quadruple $\Psi = \Psi(A, \mathcal{R}, A^\vee, \mathcal{R}^\vee)$, where

(a) A and A^\vee are free abelian groups of finite rank, in duality by a pairing $A \times A^\vee \rightarrow \mathbb{Z}$, denote by $\langle \cdot, \cdot \rangle$;

(b) \mathcal{R} (the roots) and \mathcal{R}^\vee (the coroots) are finite subsets of A and A^\vee and we are given a bijection $\alpha \rightarrow \alpha^\vee$ of \mathcal{R} onto \mathcal{R}^\vee .

For each $\alpha \in \mathcal{R}$, we define endomorphisms s_α and s_α^\vee of A and A^\vee by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \cdot \alpha; \quad s_\alpha^\vee(x) = x - \langle \alpha, x \rangle \cdot \alpha^\vee.$$

The following axioms are imposed.

(RD1) For each $\alpha \in \mathcal{R}$, $\langle \alpha, \alpha^\vee \rangle = 2$;

(RD2) For each $\alpha \in \mathcal{R}$, then $s_\alpha \mathcal{R} = \mathcal{R}$ and $s_\alpha^\vee \mathcal{R}^\vee = \mathcal{R}^\vee$.

The first axiom implies that $s_\alpha^2 = 1$ and $s_\alpha(\alpha) = -\alpha$. The Weyl group $W(\Psi)$ is the subgroup of $\text{GL}(A)$ generated by the s_α ($\alpha \in \mathcal{R}$). Let us give here some terminology.

(a) We say that Ψ is *reduced* if for each $\alpha \in \mathcal{R}$, $c \in \mathbb{Q}$ and $c\alpha \in \mathcal{R}$, then $c = \pm 1$.

(b) If $\Psi = \Psi(A, \mathcal{R}, A^\vee, \mathcal{R}^\vee)$ is a root datum, $\Psi^\vee = \Psi(A^\vee, \mathcal{R}^\vee, A, \mathcal{R})$ is a root datum called the dual root datum (or the Langlands dual root datum).

(c) A root datum $\Psi(A, \mathcal{R}, A^\vee, \mathcal{R}^\vee)$ is semisimple if \mathcal{R} generates the vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Furthermore, it is adjoint (resp. simply connected) if \mathcal{R} generates A (resp. \mathcal{R}^\vee generates A^\vee).

(d) Morphisms of root data: to be written.

13.1.1. Remark. Denote by Q the subgroup of A generated by \mathcal{R} . In $\mathcal{R} \neq \emptyset$, then \mathcal{R} is a root system of $Q \otimes_{\mathbb{Z}} \mathbb{R}$ in the sense of [Bbk3, VI.1]. Furthermore W is a subgroup of $\mathrm{GL}(Q \otimes_{\mathbb{Z}} \mathbb{R})$ and is then a finite group.

13.2. Geometric case. Let k be an algebraically closed field. Let G/k be a reductive group. Let $T \subset G$ be a maximal torus. Recall that we can attach to a root datum $\Psi(G, T) = (\hat{T}, \mathcal{R}, (\hat{T})^0, \mathcal{R}^\vee)$ where \hat{T} is the character lattice of T and $(\hat{T})^0$ its dual.

The root datum $\Psi(G, T)$ is reduced. Since the maximal tori of G are conjugated, $\Psi(G, T)$ is independent of the choice of T and we denote it simply by $\Psi(G)$. The main results (showed in Chernousov's lectures) are the following:

(1) (Unicity theorem) Two reductive k -groups G, G' are isomorphic if and only their root data $\Psi(G)$ and $\Psi(G')$ are isomorphic.

(2) (Existence theorem) If Ψ is a reduced root datum, there exists a reductive k -group G such that $\Psi(G) \cong \Psi$.

13.3. Root datum $\Psi(\mathfrak{G}, \mathfrak{T})$. Over a ring, it is technically speaking more delicate to define a root datum with a maximal torus $i : \mathfrak{T} = \mathbb{G}_{m,R}^r \rightarrow \mathfrak{G}$. For simplicity, we assume R connected.

We consider the adjoint action of \mathfrak{G} on the R -module $\mathfrak{g} = \mathrm{Lie}(\mathfrak{G})(R)$ (which is f.g. projective). Its restriction to the torus \mathfrak{T} decomposes as

$$\mathfrak{g} = \bigoplus_{\alpha \in \hat{\mathfrak{T}}} \mathfrak{g}_\alpha.$$

13.3.1. Definition. Assume that $R \neq 0$. A root α for $(\mathfrak{H}, \mathfrak{T})$ is a character $\alpha : \mathfrak{T} \rightarrow \mathbb{G}_m$ such that

- (i) α is everywhere non trivial, that is $\alpha_x \neq 0$ for each $x \in \mathrm{Spec}(R)$.
- (ii) The eigenspace $\mathrm{Lie}(\mathfrak{H})(R)_\alpha$ is an invertible R -module (i.e. projective of rank one).

13.3.2. Lemma. Let α be a root for $(\mathfrak{G}, \mathfrak{T})$. We define $\mathfrak{T}_\alpha = \ker(\alpha)$ and $\mathfrak{Z}_\alpha = \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{T}_\alpha)$. We have

$$\mathrm{Lie}(\mathfrak{Z}_\alpha)(R) = \mathrm{Lie}(\mathfrak{T})(R) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

and $-\alpha$ is a root as well.

Proof. The group scheme \mathfrak{Z}_α is smooth and its Lie algebra is $H_0^0(\mathfrak{T}_\alpha, \mathfrak{g})$ by Remark 9.2.3. From the field case [Bo, 13.18], for each point $x \in \mathrm{Spec}(R)$, we have a decomposition

$$\mathrm{Lie}(\mathfrak{Z}_\alpha)(\kappa(x)) = \mathrm{Lie}(\mathfrak{T})(\kappa(x)) \oplus \mathfrak{g}_\alpha \otimes_R \kappa(x) \oplus \mathfrak{g}_{-\alpha} \otimes_R \kappa(x)$$

and $\mathfrak{g}_{-\alpha} \otimes_R \kappa(x)$ is one dimensional. By the Nakayama lemma, the natural map of f.g. projective R -modules

$$\mathrm{Lie}(\mathfrak{T})(R) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \rightarrow \mathrm{Lie}(\mathfrak{Z}_\alpha)(R)$$

is an R -isomorphism. Furthermore $\mathfrak{g}_{-\alpha}$ is locally free of rank one. \square

The next hard thing is the “integration” of the Lie algebra \mathfrak{g}_α .

13.3.3. Theorem. *Let α be a root for $(\mathfrak{G}, \mathfrak{T})$.*

(1) There exists a unique R -group homomorphism

$$\exp_\alpha : \mathfrak{W}(\mathfrak{g}_\alpha) \rightarrow \mathfrak{G}$$

inducing the canonical inclusion $\mathfrak{g}_\alpha \rightarrow \mathfrak{g}$ and which is \mathfrak{T} -equivariant.

(2) The map \exp_α is a closed immersion, factors through \mathfrak{Z}_α , and its formation commutes with base change.

(3) The multiplication map $\mathfrak{W}(\mathfrak{g}_{-\alpha}) \times_R \mathfrak{T} \times_R \mathfrak{W}(\mathfrak{g}_\alpha) \rightarrow \mathfrak{Z}_\alpha$ is an open immersion.

We postpone in §17.3 the proof of the following characterisation of rank one vector group scheme since it involves descent techniques.

13.3.4. Proposition. *Let \mathfrak{U}/R be an affine smooth group scheme whose geometric fibers are rank one additive groups. We assume that \mathfrak{U}/R is equipped with an action of \mathbb{G}_m such that the \mathbb{G}_m -module $\mathrm{Lie}(\mathfrak{U})(R)$ is non trivial everywhere⁶. Then there exists an invertible R -module L/R such that $\mathfrak{W}(L) \cong \mathfrak{U}$.*

We can sketch the existence part of the proof of Theorem 13.3.3 (see [C, §4.1]). Up to localize, we can assume that $\alpha : \mathfrak{T} \rightarrow \mathbb{G}_m$ is “constant” namely is given by a (non trivial) element of \mathbb{Z}^r . The idea is to choose $\lambda \in (\mathbb{Z}^r)^\vee$ such that $\langle \alpha, \lambda \rangle > 0$ and to consider the homomorphism $\lambda : \mathbb{G}_m \rightarrow \mathfrak{T} \rightarrow \mathfrak{Z}_\alpha$. It gives rise to the limit R -groups $\mathfrak{P}_{\mathfrak{Z}_\alpha}(\pm\lambda)$ and the R -subgroups $\mathfrak{U}_{\mathfrak{Z}_\alpha}(\pm\lambda)$. By taking into account Lemma 13.3.2, we have

$$\mathrm{Lie}(\mathfrak{P}_{\mathfrak{Z}_\alpha}(\pm\lambda))(R) = \mathrm{Lie}(\mathfrak{T})(R) \oplus \mathfrak{g}_{\pm\alpha},$$

$$\mathrm{Lie}(\mathfrak{U}_{\mathfrak{Z}_\alpha}(\pm\lambda))(R) = \mathfrak{g}_{\pm\alpha}.$$

Furthermore $\mathfrak{U}_{\mathfrak{Z}_\alpha}(\pm\lambda)$ is equipped with an action of \mathbb{G}_m within λ hence Proposition 13.3.4 yields that there are both rank one vector group schemes. Note that fact (3) follows from Theorem 12.2.2.(2).

The image \mathfrak{U}_α/R of \exp_α is called the root subgroup relative to α . We come to the definition of coroots.

13.3.5. Theorem. *Let $\alpha \in \hat{T}$ be a root.*

(1) There exists a morphism $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \rightarrow R$, $(X, Y) \mapsto XY$ and a cocharacter $\alpha^\vee : \mathbb{G}_{m,R} \rightarrow \mathfrak{H}$ such that for each S/R , each $X \in \mathfrak{g}_\alpha \otimes_E S$, $Y \in \mathfrak{g}_\alpha \otimes_E S$ we have

$$\exp_\alpha(X) \exp_{-\alpha}(Y) \in \Omega_a(S) \iff 1 - XY \in S^\times$$

⁶that is $\mathrm{Lie}(\mathfrak{U})(\kappa(x))_0 = 0$ for each $x \in \mathrm{Spec}(R)$.

and under this condition we have

$$\exp_{\alpha}(X) \exp_{-\alpha}(Y) = \exp_{-\alpha}\left(\frac{Y}{1-XY}\right) \alpha^{\vee}(1-XY) \exp_{\alpha}\left(\frac{X}{1-XY}\right)$$

(2) The morphism $(X, Y) \rightarrow XY$ and α^{\vee} are uniquely determined by these conditions.

(3) The morphism $(X, Y) \rightarrow XY$ is an R -isomorphism and $\langle \alpha^{\vee}, \alpha \rangle = 2$.

This statement defines the coroot α^{\vee} attached to α . We denote by \mathcal{R} the set of roots and by \mathcal{R}^{\vee} the set of coroots. Both are non necessarily constant morphisms, but are locally constant (we have to be careful with connectness issues).

13.3.6. Definition. We say that the reductive group scheme \mathfrak{G}/R is split if it admits a maximal R -torus $\mathbb{G}_{m,R}^r$ such that the roots and the coroots are constant morphisms and also such that each eigenspace \mathfrak{g}_{α} is a free R -module of rank one.

13.3.7. Remark. (1) If R is connected and $\text{Pic}(R) = 1$, a reductive group scheme \mathfrak{G}/R is split if it admits a maximal R -torus $\mathbb{G}_{m,R}^r$.

(2) In the definition, we say that $\mathbb{G}_{m,R}^r$ is a splitting torus of \mathfrak{G} . For a ring R general enough, GL_n contains maximal split tori which are not splitting it, see Remark 18.4.3.

We assume that \mathfrak{G}/R is split. We see immediately that $\Psi(\mathfrak{G}, \mathfrak{T}) = (\widehat{\mathfrak{T}}, \mathcal{R}, (\widehat{\mathfrak{T}})^0, \mathcal{R}^{\vee})$ is a root datum.

13.3.8. Lemma. Assume that \mathfrak{G}/R is split. Then the isomorphism class of $\Psi(\mathfrak{G}, \mathfrak{T})$ does not depend of the choice of \mathfrak{T} .

Proof. It is true for fields, so we have only to specialise at some maximal ideal of R . \square

Hence we can attach to a split group scheme over R a root datum. The unicity and existence questions analogous with the field case were achieved of Demazure's thesis [D] and need descent techniques to be discussed.

13.4. Center. We record that the center of a split group scheme has the expected shape.

13.4.1. Proposition. Let \mathfrak{G}/R be a reductive split group scheme and let $\mathfrak{T} = \mathbb{G}_m^r$ be a maximal torus of \mathfrak{G} . Then the center of \mathfrak{G} is representable, it is the diagonalizable R -group

$$\ker\left(\mathfrak{T} \longrightarrow \prod_{\alpha \in \mathcal{R}} \mathbb{G}_m\right).$$

In particular, $\mathfrak{Z}(\mathfrak{G}) = 1$ and only if the root datum $\Psi(\mathfrak{G}, \mathfrak{T})$ is adjoint.

Proof. We can assume that R is noetherian. We define $\mathfrak{D}/R = \ker\left(\mathfrak{T} \longrightarrow \prod_{\alpha \in R} \mathbb{G}_m\right)$. We define the “center of \mathfrak{G} ” functor

$$C(S) = \text{Ker}\left(\mathfrak{G}(S) \xrightarrow{\text{int}} \text{Aut}(\mathfrak{G})(S)\right).$$

We have seen that \mathfrak{T} is his own centralizer (prop. 11.0.6), so that C is a subfunctor of \mathfrak{T} . Also $C(S)$ acts trivially on $\text{Lie}(\mathfrak{G})(S)$ for each S/R , so that the action of $C(S) \subset \mathfrak{D}(S)$.

We have proven that C is a subfunctor of \mathfrak{D} . For the converse, we need to prove that $\phi : \mathfrak{D} \rightarrow \mathfrak{G}$ is a central homomorphism. We shall use that the result holds over fields, see [Bo, §14.2].

Let $x \in \text{Spec}(R)$ and denote by \widehat{R}_x the completion. Since $\mathfrak{D} \times_{k(x)} R_x$ is central in $\mathfrak{G}_{k(x)}$, it lifts to a central homomorphism $\psi_x : \mathfrak{D} \times_R \widehat{R}_x \rightarrow \mathfrak{G} \times_R \widehat{R}_x$, according to Theorem 10.3.3.(3). But by assertion (2) of the same statement, ψ_x is $\mathfrak{G}(\widehat{R}_x)$ -conjugated to $\phi_{\widehat{R}_x}$, so that $\mathfrak{D} \times_R \widehat{R}_x$ is central in $\mathfrak{G} \times_x \widehat{R}_x$. Since \widehat{R}_x is faithfully flat over R_x , we conclude that $\mathfrak{D} \times_R R_x$ is central in $\mathfrak{G} \times_x R_x$. Thus \mathfrak{D} is central in \mathfrak{G} . \square

13.4.2. Remark.

Descent techniques

We do a long interlude for developping descent and sheafifications techniques. We use mainly the references [DG, Ro, Wa].

14. FLAT SHEAVES

Our presentation is that of Demazure-Gabriel [DG, III] which involves only rings.

14.1. Covers. A fppf (flat for short) cover of the ring R is a ring S/R which is faithfully flat and of finite presentation⁷ “fppf” stands for “fidèlement plat de présentation finie”.

14.1.1. Remarks. (1) If $1 = f_1 + \dots + f_s$ is a partition of 1_R with $f_1, \dots, f_r \in R$, the ring $R_{f_1} \times \dots \times R_{f_r}$ is a Zariski cover of R and a fortiori a flat cover.

(2) If S_1/R and S_2/R are flat covers of R , then $S_1 \otimes_R S_2$ is a flat cover of R .

(3) If S/R is a flat cover of S and S'/S is a flat cover of S , then S'/R is a flat cover of R .

(4) Finite locally free extensions S/R are flat covers, in particular finite étale surjective maps are flat covers.

14.2. Definition. We consider an R -functor $F : \{R\text{-Alg}\} \rightarrow \text{Sets}$

For each R -ring morphism $S \rightarrow S' = S_1 \times \dots \times S_n$, we can consider the sequence

$$F(S) \longrightarrow \prod_i F(S_i) \xrightarrow[d_{2,*}]{d_{1,*}} \prod_{i,j} F(S_i \otimes_S S_j) \quad .$$

A functor of $F : \{R\text{-Alg}\} \rightarrow \text{Sets}$ is a fppf sheaf (or flat sheaf) for short if if for each R -ring S and each flat cover $S' = S_1 \times S_2 \otimes_R \times S_n/S$, and the sequence

$$F(S) \longrightarrow \prod_i F(S_i) \xrightarrow[d_{2,*}]{d_{1,*}} \prod_{i,j} F(S_i \otimes_S S_j)$$

is exact. It means that the restriction map $F(S) \rightarrow \prod_i F(S_i)$ is injective and its image consists in the sections $(\alpha_i) \in \prod_i F(S_i)$ satisfying $d_{1,*}(\alpha_i) = d_{2,*}(\alpha_j) \in F(S_i \otimes_S S_j)$ for each i, j .

Since Zariski covers are flat covers, Lemma 2.3.1 works as well there and a flat R -sheaf is then an additive R -functor.

Given an R -module M and S'/S as above, the theorem of faithfully flat descent states that we have an exact sequence of S -modules

$$0 \rightarrow M \otimes_R S \rightarrow (M \otimes_R S) \otimes_S S' \xrightarrow{d_{1,*} - d_{2,*}} (M \otimes_R S) \otimes_S S' \otimes_S S' \quad .$$

⁷One may consider also not finitely presented covers, it is called *fpgc*, see [SGA3, IV] and [Vi].

This rephases by saying that the vector group functor $V(M)/R$ (which is additive) is a flat sheaf over $\text{Spec}(R)$. A special case is the exactness of the sequence

$$0 \rightarrow S \rightarrow S' \xrightarrow{d_{1,*} - d_{2,*}} S' \otimes_S S'.$$

If N is an R -module, it follows that the sequence of R -modules

$$0 \rightarrow \text{Hom}_R(N, S) \rightarrow \text{Hom}_R(N, S') \xrightarrow{d_{1,*} - d_{2,*}} \text{Hom}_R(N, S' \otimes_S S')$$

is exact. This shows that the vector R -group scheme $\mathfrak{W}(M)$ is a flat sheaf. More generally we have

14.2.1. Proposition. *Let \mathfrak{X}/R be an affine scheme. Then the R -functor of points $h_{\mathfrak{X}}$ is a flat sheaf.*

Proof. The functor $h_{\mathfrak{X}}$ is additive. We are given an R -ring S and a flat cover S'/S . We write the sequence above with the R -module $R[\mathfrak{X}]$. It reads

$$0 \rightarrow \text{Hom}_{R\text{-mod}}(R[\mathfrak{X}], S) \rightarrow \text{Hom}_{R\text{-mod}}(R[\mathfrak{X}], S') \xrightarrow{d_{1,*} - d_{2,*}} \text{Hom}_{R\text{-mod}}(R[\mathfrak{X}], S' \otimes_S S').$$

It follows that $\mathfrak{X}(S)$ injects in $\mathfrak{X}(S')$ and identifies with $\text{Hom}_{R\text{-rings}}(R[\mathfrak{X}], S') \cap \text{Hom}_{R\text{-mod}}(R[\mathfrak{X}], S)$. Hence the exact sequence

$$\mathfrak{X}(S) \longrightarrow \mathfrak{X}(S') \xrightleftharpoons[d_{2,*}]{d_{1,*}} \mathfrak{X}(S' \otimes_S S') \quad .$$

□

14.2.2. Remark. More generally, the proposition holds with a scheme \mathfrak{X}/R , see [Ro, 2.4.7] or [Vi, 2.5.4].

14.2.3. Examples. (a) If E, F are flat sheaves over R , the R -functor $\text{Hom}(E, F)$ of morphisms from E to F is a flat sheaf. Also the R -functor $\text{Isom}(E, F)$ is a flat sheaf and as special case, the R -functor $\text{Aut}(F)$ is a flat sheaf.

(b) Let $f : E \rightarrow F$ be a morphism of flat sheaves. For each R -algebra B , we consider

$$I(B) = \{x \in F(B) \mid \text{there exists a flat cover } B'/B \text{ such that } x_{B'} \in \text{Im}(E(B') \rightarrow F(B')) \}.$$

Then I is flat S -sheaf, it is called the *image sheaf* of f . By construction, $I \rightarrow F$ is a monomorphism.

(c) (Singleton sheaf). We put $\bullet_R(B) = \{\bullet\}$ for each R -algebra B . Then \bullet_R is a flat R -sheaf and is the final object of that category.

14.3. Monomorphisms and covering morphisms. A morphism $u : F \rightarrow E$ of flat sheaves over R is a monomorphism if $F(S) \rightarrow E(S)$ is injective for each S/R . It is a covering morphism (*couvrant* in French) if for each S/R and each element $e \in E(S)$, there exists a flat cover S'/S and an element $f' \in F(S')$ such that $e|_{S'} = u(f')$.

A morphism of flat sheaves which is a monomorphism and a covering morphism is an isomorphism.

We say that a sequence of flat sheaves in groups over R $1 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 1$ is exact if the map of sheaves $F_2 \rightarrow F_3$ is a covering morphism and if for each S/R the sequence of abstract groups $1 \rightarrow F_1(S) \rightarrow F_2(S) \rightarrow F_3(S)$ is exact.

14.3.1. Examples. (1) For each $n \geq 1$, the Kummer sequence $1 \rightarrow \mu_{n,R} \rightarrow \mathbb{G}_{m,R} \xrightarrow{f_n} \mathbb{G}_{m,R} \rightarrow 1$ is an exact sequence of flat sheaves where f_n is the n -power map. The only thing to check is the epimorphism property. Let S/R be a ring and $a \in \mathbb{G}_m(S) = S^\times$. We put $S' = S[X]/(X^n - a)$, it is finite free over S , hence is faithfully flat of finite presentation. Then $f_n(X) = a|_{S'}$ and we conclude that f_n is a covering morphism of flat sheaves. (2) More generally, let $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ be an exact sequence of f.g. abelian groups. Then the sequence of R -group schemes

$$1 \rightarrow \mathfrak{D}(A_3) \rightarrow \mathfrak{D}(A_2) \rightarrow \mathfrak{D}(A_1) \rightarrow 0$$

is exact.

14.4. Sheafification. Given an R -functor F , there is natural way to sheafify it in a flat functor \tilde{F} . For each S/R , we consider the “set” $\text{Cov}(S)$ of flat covers⁸. Also if $f : S_1 \rightarrow S_2$ is an arbitrary R -ring map, the tensor product defines a natural map $f_* : \text{Cov}(S_1) \rightarrow \text{Cov}(S_2)$. We define then

$$\tilde{F}(S) = \varinjlim_{I \subset \text{Cov}(S)} \ker \left(\prod_{i \in I} F(S_i) \xrightarrow[d_{2,*}]{d_{1,*}} F_{ad}(S_i \otimes_S S_j) \right)$$

where the limit is taken on finite subsets I of $\text{Cov}(S)$. It is an R -functor since each map $f : S_1 \rightarrow S_2$ defines $f_* : \tilde{F}(S_1) \rightarrow \tilde{F}(S_2)$. We have also a natural mapping $u_F : F \rightarrow \tilde{F}$.

14.4.1. Proposition. (1) For each R -functor F , the R -functor \tilde{F} is a flat sheaf.

(2) The functor $F \rightarrow \tilde{F}$ is left adjoint to the forgetful functor applying a flat sheaf to its underlying R -functor. For each R -functor F and each flat sheaf E , the natural map

$$\text{Hom}_{\text{flat sheaves}}(\tilde{F}, E) \xrightarrow{\sim} \text{Hom}_{R\text{-functor}}(F, E)$$

(applying a morphism $u : \tilde{F} \rightarrow E$ to the composite $F \rightarrow \tilde{F} \rightarrow E$) is bijective.

(1) follows essentially by construction [DG, III.1.8]. Note that in this reference, the two steps are gathered in one. For (2) one needs to define the inverse mapping. Observe that the sheafification of E is itself, so that the sheafification of $F \rightarrow E$ yields a natural morphism $\tilde{F} \rightarrow E$.

⁸We do not enter in set-theoretic considerations but the reader can check there is no problem there.

Given a morphism of flat R -sheaves $f : E \rightarrow F$, we can sheafify the functors

$$S \mapsto E(S)/R_f(S), \quad S \mapsto \operatorname{Im}(E(S) \rightarrow F(S)),$$

where $R_f(S)$ is the equivalence relation defined by $f(S)$. We denote by $\operatorname{Coim}(f)$ and $\operatorname{Im}(f)$ their respective sheafifications [the image sheaf has been already constructed in the Example 14.2.3.(b)]. We have an induced mapping

$$f_* : \operatorname{Coim}(f) \rightarrow \operatorname{Im}(f)$$

between the coimage sheaf and the image sheaf. We say that f is strict when f_* is an isomorphism of flat sheaves.

14.4.2. Lemma. *If f is a monomorphism (resp. an covering morphism), then f is strict.*

In the first case, we have $E \xrightarrow{\sim} \operatorname{Coim}(f) \xrightarrow{\sim} \operatorname{Im}(f)$; in the second case, we have $\operatorname{Coim}(f) \xrightarrow{\sim} \operatorname{Im}(f) \xrightarrow{\sim} F$.

14.4.3. Lemma. *Let $f : E \rightarrow F$ be a morphism of flat R -sheaves. Let I be the image sheaf of f . Then the following are equivalent:*

- (i) $I = F$;
- (ii) f is a covering morphism;
- (iii) f is an universal epimorphism, that is f_B is an epimorphism of flat B -sheaves for each R -algebra B .
- (iv) f is an epimorphism.

Proof. (i) \implies (ii). Let B be an R -ring. Let $x \in F(B) = I(B)$. Then there exists a flat cover B' of B such that $x \in f(E(B'))$. Thus f is a covering morphism.

(ii) \implies (iii). Let B_0 be an R -algebra. Let $u_1, u_2 : F_{B_0} \rightarrow G$ be two morphisms of flat B_0 -sheaves such that $u_1 \circ f_{B_0} = u_2 \circ f_{B_0}$. We want to show that $u_1 = u_2$. We are given an B_0 -algebra B and $x \in F(B)$. Since there exists a flat cover B' of B such that $x \in f(E(B'))$, it follows that $u_1(x)_{B'} = u_2(x)_{B'}$. Thus $u_1(x) = u_2(x) \in F(B)$. This establishes that f_{B_0} is an epimorphism.

(iii) \implies (iv). Obvious.

(iv) \implies (i). We assume that $f : E \rightarrow F$ is an epimorphism. Since f factorizes through $i : I \rightarrow F$, it follows that $i : I \rightarrow F$ is an universal epimorphism as well. We consider the R -functor C defined by $C(B) = (F(B) \sqcup \{\bullet\}) / \sim$ where \sim is the following equivalence relation: $x, y \in F(B) \sqcup \{\bullet\}$ are equivalent if $x, y \in I(B) \sqcup \{\bullet\}$ or if $x = y$.

The point is that C is *separated*, that is, $C(B) \rightarrow C(B')$ is injective for each flat cover B' of B . This implies that $\tilde{C} = \tilde{\tilde{C}}$ is the sheafification of C . It comes with a morphism $v : \bullet_R \rightarrow \tilde{C}$. The canonical map $u : F \rightarrow \tilde{C}$ and $F \rightarrow \bullet_R \xrightarrow{v} \tilde{C}$ agree on I so are equal.

Let B be an R -algebra and let $x \in F(B)$. Then there exists a flat cover B' of B such that $u(x)$ is the image of $\{\bullet\}$ in $C(B') = F(B')/\sim$, that is x belongs to $I(B')$. Thus $x \in I(B)$. We conclude that $I = B$. \square

14.4.4. Corollary. *Let $E \rightarrow F$ be an epimorphism of R -functors. Then the map of flat R -sheaves $\tilde{\tilde{E}} \rightarrow \tilde{\tilde{F}}$ is a covering morphism.*

Proof. We need to show that $\tilde{\tilde{E}} \rightarrow \tilde{\tilde{F}}$ is an epimorphism of flat sheaves. Let $u_1, u_2 : \tilde{\tilde{F}} \rightarrow G$ be maps of flat R -sheaves which agree on $\tilde{\tilde{E}}$. We consider the composite R -mappings $v_i : F \rightarrow \tilde{\tilde{E}} \xrightarrow{u_i} G$. Since v_1 and v_2 agree on E , it follows that $v_1 = v_2$. But $\text{Hom}_{R\text{-functors}}(E, G) = \text{Hom}_{R\text{-sheaves}}(\tilde{\tilde{E}}, G)$ (universal property), hence $u_1 = u_2$. \square

14.5. Group actions, quotients sheaves and contracted products.

Let G be an R -group flat sheaf and let F be a flat sheaf equipped with a right action of G . The quotient functor is $Q(S) = F(S)/G(S)$ and its sheafification is denoted by F/G . It is called the quotient sheaf⁹.

When G and F are representable, the natural question is to investigate whether the quotient sheaf Q is representable. It is quite rarely the case. A first evidence to that is the following fact.

14.5.1. Proposition. *We are given an affine R -group scheme \mathfrak{G} and a monomorphism $\mathfrak{G} \rightarrow \mathfrak{H}$ into an affine group scheme. Assume that the quotient sheaf $\mathfrak{H}/\mathfrak{G}$ is representable by an R -scheme \mathfrak{X} . We denote by $p : \mathfrak{H} \rightarrow \mathfrak{X}$ the quotient map and by $\epsilon_X = p(1_{\mathfrak{G}}) \in \mathfrak{X}(R)$.*

(1) *The map $\mathfrak{H} \rightarrow \mathfrak{X}$ is a covering map and R -map $\mathfrak{H} \times_R \mathfrak{G} \rightarrow \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H}$ is an isomorphism.*

(2) *The diagram*

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{i} & \mathfrak{H} \\ \downarrow & & \downarrow \\ \text{Spec}(R) & \xrightarrow{\epsilon_X} & \mathfrak{X} \end{array}$$

is cartesian.

(3) *The map i is an immersion. It is a closed immersion and only if \mathfrak{X}/R is separated.*

(4) *\mathfrak{G}/R is flat and only if p is flat.*

(5) *\mathfrak{G}/R is smooth and only if p is smooth.*

The general statement is [SGA3, VI_B.9.2].

⁹One can work in a larger setting, that of equivalence relations and groupoids, see [DG, §III.2].

Proof. (1) The first assertion follows from Corollary 14.4.4. The map $\mathfrak{H} \times_R \mathfrak{G} \rightarrow \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H}$ is a monomorphism. We are given S/R and $(h_1, h_2) \in \mathfrak{H}(S)^2$ such that $p(h_1) = p(h_2)$. There exists a flat cover S'/S and $g \in \mathfrak{G}(S')$ such that $h_1|_{S'} = h_2|_{S'} g$. Hence $g \in \mathfrak{G}(S') \cap \mathfrak{H}(S)$. Since i is a monomorphism, we conclude by descent that $g \in \mathfrak{G}(S)$ whence (h_1, h_2) comes from (h_1, g) .

(2) It follows that the following diagram

$$\begin{array}{ccccc} \mathfrak{G} & \xrightarrow{1_{\mathfrak{H}} \times id} & \mathfrak{H} \times_R \mathfrak{G} \cong \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H} & \longrightarrow & \mathfrak{H} \\ \downarrow & & \downarrow p_1 & & \downarrow p \\ \mathrm{Spec}(R) & \xrightarrow{1_{\mathfrak{H}}} & \mathfrak{H} & \xrightarrow{p} & \mathfrak{X} \end{array}$$

is cartesian as desired.

(3) If \mathfrak{X} is separated, $\epsilon_{\mathfrak{X}}$ is a closed immersion and so is i .

(4) and (5) If p is flat (resp. smooth), so is i by base change. \square

One very known case of representability result is the following.

14.5.2. Theorem. *Let k be a field. Let H/k be an affine algebraic group and G/k be a closed subgroup. Then the quotient sheaf H/G is representable by a k -scheme of finite type X .*

One needs the following

14.5.3. Proposition. [DG, III.3.5.2] *Let G acts on a quasi-projective k -variety X . Let $x \in X(k)$ and denote by G_x the stabilizer of x .*

(1) *The quotient G/G_x is representable by a quasi-projective k -variety.*

(2) *The orbit map induces an immersion $G/G_x \rightarrow X$.*

It can be suitably generalized over rings, see [SGA3, XVI.2], by means of the theorem of Grothendieck-Murre.

Sketch of proof: We assume for simplicity that G is smooth, that is absolutely reduced. By faithfully flat descent, one can assume that k is algebraically closed.

(1) We know denote by X_0 the reduced subscheme of the schematic image of f_x . Since G is smooth, it is (absolutely) reduced and acts then on X_0 . We know that the $X_0 \setminus G.x$ consists in orbits of smaller dimensions so that $G.x$ is an open subset of X_0 . We denote it by U_x . We claim that the map $h_x : G \rightarrow U_x$ is faithfully flat. The theorem of generic flatness [DG, I.3.3.7] shows that the flat locus of h_x is not empty. By homogeneity, it is U_x , hence h_x is faithfully flat. Let us show now that it implies that U_x represents the orbit of x . The morphism $h_x : G \rightarrow U_x$ gives rise to a morphism of k -sheaf $h_x^\dagger : G/G_x \rightarrow U_x$. Since the map $h_x : G \rightarrow U_x$ is faithfully flat, the morphism h_x is a covering morphism of flat sheaves¹⁰.

¹⁰We are given S/R and a point $u \in U_x(S)$. Then $h_x^{-1}(u) = \mathrm{Spec}(S')$ is a flat cover of S and there is a point $v \in G(S)$ mapping to u .

h_x^\dagger is a monomorphism. Let S be an R -ring and let $y_1, y_2 \in (G/G_x)(S)$ having same image u in $U_x(S)$. There exists a flat cover S'/S such that y_1 (resp. y_2) comes from some $g_1 \in G(S')$ (resp. g_2). Then $g_1 \cdot x = g_2 \cdot x \in U_x(S')$ so that $g_2^{-1} g_1 \in G_x(S')$. Thus $u_1 = u_2 \in (G/G_x)(S)$.

(2) By construction, U_x is locally closed in X . \square

Theorem 14.5.2 follows then of the fact that G admits a representation V such that there exists a point $x \in \mathbf{P}(V)(k)$ such that $G = H_x$ [DG, II.2.3.5].

14.5.4. Remark. One interest of the Chevalley quotient is the fact it is universal. That is for each k -algebra R , $(H/G) \times_k R$ represents the quotient R -sheaf $(H \times_k R)/(G \times_k R)$. It can use as follows (see [CTS2, 6.12]). Assume we are given a closed immersion $\iota : \mathfrak{G} \rightarrow \mathfrak{H}$ of R -group schemes, a flat cover R'/R and a commutative square

$$\begin{array}{ccc} \mathfrak{G} \times_R R' & \xrightarrow{\iota_{R'}} & \mathfrak{H} \times_R R' \\ u \downarrow \cong & & v \downarrow \cong \\ G \times_k R' & \xrightarrow{i \times_k R'} & H \times_k R' \end{array}$$

where u, v are isomorphisms. We claim then that the quotient sheaf $\mathfrak{H}/\mathfrak{G}$ is representable by an R -scheme. According to Theorem 14.5.2, $\mathfrak{H}_{R'}/\mathfrak{G}_{R'}$ is representable by a quasi-projective R' -scheme which indeed descends to R .

14.6. Contracted products. We are given two flat R -sheaves in sets F_1, F_2 and and a flat sheaf G in groups. If F_1 (resp. F_2) is equipped with a right (resp. left) action of G , we have a natural right action of G on the product $F_1 \times F_2$ by $(f_1, f_2) \cdot g = (f_1 g, g^{-1} f_2)$. The sheaf quotient of $F_1 \times F_2$ under this action by G is denoted by $F_1 \wedge^G F_2$ and is called the contracted product of F_1 and F_2 with respect to G .

14.6.1. Remark. This construction occurs for group extensions. Let $1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$ be an exact sequence of flat sheaves in groups with A abelian. Given a map $A \rightarrow B$ of abelian flat sheaves equipped with compatible G -actions, the contracted product $B \wedge^A E$ is a sheave in groups and is an extension of G by B .

14.7. Sheaf torsors. Let G/R be a flat sheaf in groups (e.g. an affine group scheme over R).

14.7.1. Definition. A sheaf G -torsor over R is a flat sheaf E/R equipped with a right action of G submitted to the following requirements:

(T1) The R -map $E \times G \rightarrow E \times E$, $(e, g) \mapsto (e, e.g)$ is an isomorphism of flat sheaves over R .

(T2) There exists a flat cover S/R such that there is a G_S -isomorphism $E_S \xrightarrow{\sim} G_S$.

Condition (T2) says that a G -torsor sheaf is locally trivial with respect to the flat topology.

The basic example of such an object is the trivial G -torsor sheaf G equipped with the right action. For avoiding confusions, we denote it sometimes E_{tr} .

Now if F is a flat sheaf over R equipped with a right G -action and E/R is a G -torsor, we call the contracted product $E \wedge^G F$ the twist of F by E . It is denoted sometimes ${}^E F$ or ${}_E F$. We record the two special cases:

- (1) The action of G on E_{tr} by left translations, we get then $E = {}^E E_{tr}$.
- (2) The action of G on itself by inner automorphisms, the twist ${}^E G$ is called the inner twisted form of G associated to E .

- (3) We can twist the left action (by translation) $G \times E_{tr} \rightarrow E_{tr}$, where G acts on itself by inner automorphisms. It provides a left action ${}^E G \times E \rightarrow E$.

In the case G is representable by an affine group scheme \mathfrak{G}/R , then descent theory shows that sheaf G -torsors are representable as well and we say that the relevant schemes are G -torsors/ Furthermore if \mathfrak{G}/R is flat (resp. smooth), so are the G -torsors. We give some examples of torsors.

14.7.2. Examples. (1) Galois covers $\mathfrak{Y} \rightarrow \mathfrak{X}$ under a finite group Γ , see below 14.8.1.

- (2) The Kummer cover $\mathbb{G}_m \rightarrow \mathbb{G}_{m,n}$.

- (3) The Chevalley quotient 14.5.2 gives rise to the H -torsor $G \rightarrow G/H$.

14.7.3. Lemma. (1) Let S/R be a flat cover which splits E . Then $({}^E F)_S \xrightarrow{\sim} F_S$.

- (2) ${}^E(\text{Aut}(F)) = \text{Aut}({}^E F)$.

(3) If F is representable by an R -affine scheme \mathfrak{X} , so is ${}^E F$. Furthermore if \mathfrak{X} is finitely presented (resp. faithfully flat, smooth), so are F .

(4) If G is representable by an affine R -scheme \mathfrak{G} , so are E and ${}^E G$. Furthermore if \mathfrak{G}/R is finitely presented (resp. faithfully flat, smooth), so are E and ${}^E \mathfrak{G}$.

Note that the R -functor $\text{Aut}(F)$ is a flat sheaf, see 14.2.3.

Proof. (1) The formation of contracted products commute with arbitrary base change, hence $({}^E F)_S = {}^{E_S} F_S \xrightarrow{\sim} {}^{E_{tr,S}} F_S = F_S$.

(2) Twisting the morphism of flat sheaves $\text{Aut}(F) \times F \rightarrow F$ by E yields a morphism ${}^E \text{Aut}(F) \times F \rightarrow F$. It defines then a map ${}^E \text{Aut}(F) \rightarrow \text{Aut}({}^E F)$. It is an isomorphism since it is after making the base change S/R .

(3) It is a special case of faithfully flat descent.

(4) It comes from the permanence properties kept by faithfully flat descent. \square

Statement (1) says that ${}^E F$ is an S/R -form of F , that is a flat sheaf F' such that $F'_S \xrightarrow{\sim} F_S$.

14.8. Quotient by a finite constant group. An important case of torsor and quotients is the following

14.8.1. Theorem. [DG, §III.6] *Let Γ be a finite abstract group. We assume that Γ_R acts freely on the right on an affine R -scheme \mathfrak{X} . It means that the graph map $\mathfrak{X} \times_R \Gamma_R \rightarrow \mathfrak{X} \times_R \mathfrak{X}$ is a monomorphism. We put $\mathfrak{Y} = \text{Spec}(R[\mathfrak{X}]^\Gamma)$.*

- (1) *The map $\mathfrak{X} \rightarrow \mathfrak{Y}$ is a Γ_R -torsor, i.e. a Galois cover of group Γ ;*
- (2) *The scheme \mathfrak{Y}/R represents the quotient sheaf $\mathfrak{X}/\mathfrak{G}$.*

See also [R, X, p. 108] for another proof.

14.9. Quotient by a normalizer. A more advanced result is the following representability theorem used only at the end of the lectures.

14.9.1. Theorem. [SGA3, XVI.2.4] (see also [Br, §3.8]) *Let $i : \mathfrak{H} \rightarrow \mathfrak{G}$ be a monomorphism of affine group schemes. We assume that \mathfrak{G} is finitely presented and that \mathfrak{H} is smooth with connected geometric fibers.*

- (1) *Then the normalizer functor N defined by*

$$N(S) = \left\{ g \in \mathfrak{G}(S) \mid g\mathfrak{H}(S')g^{-1} = \mathfrak{H}(S') \quad \forall S'/S \right\}$$

for each S/R is representable by a closed subscheme of \mathfrak{G}/R of finite presentation.

- (2) *We assume that N is flat. Then the quotient sheaf G/N is representable by a scheme which is of finite presentation over R and quasi-projective.*

15. NON-ABELIAN COHOMOLOGY, I

15.1. Definition. We denote by $H^1(R, G)$ the set of isomorphism classes of G -torsors over R . It is a pointed set pointed by the class of the trivial G -torsor. If S/R is a cover, we denote by $H^1(S/R, G)$ the subset consisting of G -torsors split by S/R . This set $H^1(S/R, G)$ can be computed by means of cocycles modulo coboundaries [P]. More precisely, a 1-cocycle is an element $g \in G(S \otimes_R S)$ satisfying the rule

$$d_{2,*}(g) d_{1,2,*}(g) = d_{1,3,*}(g) \in G(S \otimes_R S \otimes_R S).$$

Two 1-cocycles $g_1, g_2 \in G(S \times_R S)$ are equivalent if there exists $g \in G(S)$ such that

$$g_2 = d_{2,*}(g)^{-1} g_1 d_{1,*}(g) \in G(S \otimes_R S).$$

15.1.1. Remark. If S/R is a Galois covering for an abstract group Γ , then $S \otimes_S R \xrightarrow{\sim} S^\Gamma$ and this leads to non-abelian Galois cohomology, see [P].

15.2. Twisting. If G is not abelian, there is no natural group structure on $H^1(E, G)$. We have however the torsion operation (change of origin)

$$\tau_E : H^1(R, {}^E G) \xrightarrow{\sim} H^1(R, G)$$

for a G -torsor E . Its definition (and also of the converse map) requires some preparation. Firstly the left action of G on itself gives rise to an action of ${}^E G$ to E . We have then a bitorsor structure

$${}^E G \times E \times G \rightarrow E.$$

Given a ${}^E G$ -torsor F , the contracted product $F \wedge {}^E G E$ is equipped with a right G -action and is indeed a G -torsor. We put $\tau_E(F) = [F \wedge {}^E G E]$.

The opposite torsor E^{op} of E is the right ${}^E G$ -torsor obtained by taking the opposite actions above. It comes then with a left action of G . Now, given a G -torsor L , the contracted product $L \wedge^G E^{op}$ is similarly a right ${}^E G$ -torsor. It defines the converse of the torsion bijection map.

Also the contracted product permits to define $H^1(R, G) \rightarrow H^1(R, H)$ for a map $u : G \rightarrow H$.

15.2.1. Proposition. *There is one to one correspondence*

$$\left\{ S/R\text{-forms of } F \right\} \xrightarrow{\sim} H^1(S/R, \text{Aut}(F)).$$

Proof. We explain only the maps. Given an S/R -form F' of F , we observe that $\text{Aut}(F)$ acts on the right on the flat sheaf $\text{Isom}(F, F')$ which is a $\text{Aut}(F)$ -torsor since it is so after extension to S/R . Conversely, given a $\text{Aut}(F)$ -torsor E , the twisted sheaf ${}^E F$ is an S/R -form of F . \square

A special case is the following, see [P].

15.2.2. Theorem. (*Hilbert-Grothendieck 90*) *Let M be an R -module which is locally free of rank d .*

(1) *The set $H^1(R, \text{GL}(M))$ classifies the isomorphism classes of R -modules of rank d .*

(2) *If R is semilocal, $H^1(R, \text{GL}(M)) = 1$.*

Another nice example is that of the even orthogonal group, see [DG, III.5.2].

15.3. Weil restriction II. Let S be an R -ring. Let H/S be a flat sheaf in groups and consider the R -functor $G = \prod_{S/R} H$, that is the Weil restriction of

H from S to R . We note that G is a flat R -sheaf in groups. The adjunction map $\psi : G_S \rightarrow H$ defines a natural map

$$H^1(R, G) \rightarrow H^1(S, G_S) \xrightarrow{\psi_*} H^1(S, H).$$

15.3.1. Proposition. [SGA3, XXIV.8.2]

(1) *The map $H^1(R, G) \rightarrow H^1(S, H)$ is injective and its image consists in H -torsors which are split after a flat cover coming from R .*

(2) *If S/R is a flat cover, we have $H^1(R, G) \xrightarrow{\sim} H^1(S, H)$.*

Proof. (1) We denote by $H_R^1(S, H)$ the subset of $H^1(H, S)$ consisting in classes of H -torsors split after a flat cover coming from R . Let R'/R be a flat cover of R and put $S' = S \otimes_R R'$. Then $G(R') = H(S')$ and $G(R' \otimes_R R') = H(S' \times_S S')$ so that $H^1(R'/R, G) \xrightarrow{\sim} H^1(S'/S, H)$. By passing to the limit we get the desired bijection $H^1(R, G) \xrightarrow{\sim} H_R^1(S, H)$.

Assertion (2) follows. □

16. QUOTIENTS BY DIAGONALIZABLE GROUPS

Let A be a finitely generated abelian group and consider the diagonalizable R -group scheme $\mathfrak{G}/R = \mathfrak{D}(A) = \text{Spec}(R[A])$. We assume it acts on the right on an affine R -scheme \mathfrak{X} . We have then the decomposition in eigenspaces

$$R[\mathfrak{X}] = \bigoplus_{a \in A} R[\mathfrak{X}]_a.$$

16.1. Torsors. We are interested in understanding when $\mathfrak{X} \rightarrow \text{Spec}(R)$ is a \mathfrak{G} -torsor.

16.1.1. Proposition. *Assume that \mathfrak{X} is of finite presentation. Then \mathfrak{X}/R is a \mathfrak{G} -torsor if and only if the two following conditions hold*

- (i) *For each $a \in A$, $R[\mathfrak{X}]_a$ is an invertible R -module;*
- (ii) *For each pair $(a, b) \in A^2$, the multiplication homomorphism $R[\mathfrak{X}]_a \otimes_R R[\mathfrak{X}]_b \rightarrow R[\mathfrak{X}]_{a+b}$ is an isomorphism.*

Furthermore, these two conditions are equivalent to the next conditions

- (iii) $R \xrightarrow{\sim} R[\mathfrak{X}]_0$;
- (iv) $R[\mathfrak{X}]_a R[\mathfrak{X}] = R[\mathfrak{X}]$ for each $a \in A$.

Proof. We observe first that the trivial torsor \mathfrak{G}/R satisfies conditions (i) and (ii). Assume that \mathfrak{X} is a \mathfrak{G} -torsor. There exists a flat cover S/R such that $\mathfrak{X} \times_R S \xrightarrow{\sim} \mathfrak{G}_S$ in an equivariant way. so that \mathfrak{X}_S satisfies (i) and (ii). By faithfully flat descent, \mathfrak{X} satisfies (i) and (ii).

Conversely, we assume that \mathfrak{X} satisfies (i) and (ii). Then $R[\mathfrak{X}]$ is a projective module. The cograph map $h : R[\mathfrak{X}] \otimes_R R[\mathfrak{X}] \rightarrow R[\mathfrak{X}] \otimes_R R[A]$ applies an homogeneous element $f_a \otimes f_b$ to $(f_a f_b) \otimes e_a$. Hence h splits in a direct summand

$$h_a : R[\mathfrak{X}]_a \otimes_R R[\mathfrak{X}] \longrightarrow R[\mathfrak{X}]$$

$$f_a \otimes f \longmapsto f_a f$$

Condition (ii) ensures that h_a is an isomorphism and so is h . This shows that \mathfrak{X} is a pseudo \mathfrak{G} -torsor.

Since $R[\mathfrak{X}]$ is a projective R -module, it is then faithfully flat over R . Then $R \rightarrow R[\mathfrak{X}]$ is a flat cover which splits $\mathfrak{X} \rightarrow \text{Spec}(R)$, therefore \mathfrak{X}/R is a \mathfrak{G} -torsor.

Conditions (i) and (ii) imply (iii) and (iv). Conversely assume (iii) and (iv). Let $a \in A$. There are elements f_1, \dots, f_r of $R[\mathfrak{X}]_{-a}$ and h_1, \dots, h_r of $R[\mathfrak{X}]_a$

such that $1 = f_1 h_1 + \cdots + h_r f_r$. Then the family $R_{f_i h_i}$ is a Zariski cover of R and up to localize, we can assume that there exists $f \in R[\mathfrak{X}]_{-a} \cap R[\mathfrak{X}]^\times$. For each $b \in A$, it follows that the homomorphism $R[\mathfrak{X}]_b \rightarrow R[\mathfrak{X}]_{a+b}$, $u \mapsto fu$ is an isomorphism. In particular, $R = R[\mathfrak{X}]_0 \xrightarrow{\sim} R[\mathfrak{X}]_a$ and the multiplication $R[\mathfrak{X}]_a \otimes_R R[\mathfrak{X}]_b \rightarrow R[\mathfrak{X}]_{a+b}$ is an isomorphism. \square

16.1.2. Example. The case of $A = \mathbb{Z}$, that is of $\mathfrak{G} = \mathbb{G}_{m,R}$. In this case, we know from the yoga of forms that a \mathbb{G}_m -torsor \mathfrak{X}/R is the same thing than an invertible R -module M . Another way to see it is to consider the invertible module $R[\mathfrak{X}]_1$.

16.2. Quotients.

16.2.1. Theorem. *We assume that \mathfrak{G} acts freely on \mathfrak{X} , that is the map $\mathfrak{X} \times_R \mathfrak{G} \rightarrow \mathfrak{X} \times_R \mathfrak{X}$ is a monomorphism. We put $\mathfrak{Y} = \text{Spec}(R[\mathfrak{X}]_0)$.*

(1) *The R -map $p : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a $\mathfrak{G}_{\mathfrak{Y}}$ -torsor;*

(2) *\mathfrak{Y}/R represents the flat quotient sheaf $\mathfrak{X}/\mathfrak{G}$.*

Proof. (1) Without lost of generality, we can assume that $R = R[\mathfrak{X}]_0$. The morphism $\mathfrak{X} \rightarrow \mathfrak{Y} = \text{Spec}(R)$ is \mathfrak{G} -invariant. From Proposition 16.1.1, we need to check that $R[\mathfrak{X}]_a R[\mathfrak{X}] = R[\mathfrak{X}]$ for each $a \in A$.

Let \mathfrak{M} be a maximal ideal of R and consider the subset $A_{\#}$ of A consisting in the elements $a \in A$ such that $R[\mathfrak{X}]_a R[\mathfrak{X}]_{-a} \not\subset \mathfrak{M}$. We note that $A_{\#}$ is a subgroup of A and consider the ideal

$$\mathcal{I} = \sum_{a \notin A_{\#}} R[\mathfrak{X}]_a R[\mathfrak{X}]$$

of $R[\mathfrak{X}]$. We have $\mathcal{I} \cap R \subset \mathfrak{M}$.

16.2.2. Claim. $A_{\#} = A$.

The point is that \mathcal{I} is a graded ideal of $R[\mathfrak{X}]$ so that $\text{Spec}(R[\mathfrak{X}]/\mathcal{I})$ carries an induced \mathfrak{G} -action which is fixed by the closed subgroup R -scheme $\mathfrak{D}(A/A_{\#})$ of \mathfrak{G} . Since the action is free, we conclude that $A_{\#} = A$.

From the claim, we get that for each $a \in A$, the ideal $R[\mathfrak{X}]_a R[\mathfrak{X}]_{-a}$ of R is R .

(2) Denote by Q the quotient sheaf $\mathfrak{X}/\mathfrak{G}$. The map $p : \mathfrak{X} \rightarrow \mathfrak{Y}$ factorizes by Q , that is defines a map of flat sheaves $\tilde{p} : Q \rightarrow \mathfrak{Y}$. Since p is faithfully flat, q is a covering morphism (same argument as at the end of proof of Proposition 14.5.3). Let us show that q is a monomorphism. We are given an R -ring S and two elements $q_1, q_2 \in Q(S)$ such that $\tilde{p}(q_1) = \tilde{p}(q_2) = y \in \mathfrak{Y}(S)$. Let S'/S be a cover such that q_1 and q_2 come from $x_1, x_2 \in \mathfrak{X}(S')$. Since $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{G}_{\mathfrak{Y}} \xrightarrow{\sim} \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$, there exists $g \in \mathfrak{G}(S')$ such that $x_1 g = x_2$. Therefore $q_1 = q_2 \in Q(S)$. \square

16.2.3. Corollary. (1) *The graph morphism $\mathfrak{X} \times_R \mathfrak{G} \rightarrow \mathfrak{X} \times_R \mathfrak{X}$ is a closed immersion.*

(2) *For each $x \in \mathfrak{X}(R)$, the orbit map $\mathfrak{G} \rightarrow \mathfrak{X}$, $g \mapsto x.g$ is a closed immersion.*

16.3. Homomorphisms to a group scheme.

16.3.1. Corollary. *Let $f : \mathfrak{G} = \mathfrak{D}(A) \rightarrow \mathfrak{H}$ be an R -group monomorphism where \mathfrak{H}/R is an affine group scheme. Then f is a closed immersion.*

Proof. The action of \mathfrak{G} on \mathfrak{H} is free. Then \mathfrak{G} arises as the fiber at 1 of the quotient map $\mathfrak{H} \rightarrow \mathfrak{H}/\mathfrak{G}$. \square

More difficult is the following

16.3.2. Theorem. [SGA3, IX.6.4] *Let $f : \mathfrak{G} = \mathfrak{D}(A) \rightarrow \mathfrak{H}$ be a group homomorphism where \mathfrak{H}/R is a smooth affine group scheme. Assume that R is noetherian and connected. Then the kernel of f is a closed subgroup scheme $\mathfrak{D}(A/B)$ of \mathfrak{G} and f factorizes in an unique way as*

$$\mathfrak{G} = \mathfrak{D}(A) \rightarrow \mathfrak{D}(B) \xrightarrow{\tilde{f}} \mathfrak{H}$$

where \tilde{f} is a closed immersion.

Proof. We can assume that R is local with maximal ideal \mathfrak{M} , residue field κ .

We denote by \hat{R} the completion of R with respect to the ideal \mathfrak{M} . Since $\mathfrak{H} \times_R \kappa$ admits a faithful representation, the kernel of f_k is $\mathfrak{D}(A/B)$ for some B . By the rigidity principle 10.1.1, f_{R/\mathfrak{M}^n} is trivial on $\mathfrak{D}(A/B) \times_R R/\mathfrak{M}^n$ for each $n \geq 1$. Therefore $f_{\hat{R}}$ is trivial on $\mathfrak{D}(A/B)$ (injectivity in Theorem 10.3.1) and f is then trivial on $\mathfrak{D}(A/B)$ because \hat{R} is faithfully flat over R . Up to mod out by $\mathfrak{D}(A/B)$, we can then assume then f_k is a monomorphism. Then f is a monomorphism according to Theorem 10.1.4. \square

17. GROUPS OF MULTIPLICATIVE TYPE

17.1. Definitions.

17.1.1. Definition. *A finitely presented affine group scheme \mathfrak{G}/R is of multiplicative type if there exists a flat cover $S_1 \times \dots \times S_l$ of R such that $\mathfrak{G} \times_R S_i$ is a diagonalizable S_i -group scheme.*

If \mathfrak{G}_{S_i} is isomorphic to some $\mathfrak{D}(\mathbb{Z}^{r_i})_{S_i}$ for each i , we say that \mathfrak{G} is a torus.

If R is connected, this is equivalent to ask that $\mathfrak{G} \times_R S$ is diagonalizable for a single flat cover S/R . By descent, the nice properties of diagonalizable groups generalize. More precisely:

- (1) The rigidity properties;
- (2) Existence of quotients for free actions on affine schemes;
- (3) The category of group of multiplicative type admits kernels and cokernels, it is an abelian category. We denote it by \mathcal{M}/R .
- (4) Each R -group of multiplicative type \mathfrak{G} fits in a canonical exact sequence $1 \rightarrow \mathfrak{T} \rightarrow \mathfrak{G} \rightarrow \mathfrak{G}' \rightarrow 1$ where \mathfrak{T}/R is a R -torus and \mathfrak{G}'/R is finite.

17.1.2. Example. If S/R is a finite étale cover and A a f.g. abelian group, the Weil restriction $\mathfrak{G} = \prod_{S/R} \mathfrak{D}(A)_S$ is an R -group of multiplicative type¹¹.

We have a natural map $\mathfrak{D}(A) \rightarrow \mathfrak{G}$.

17.1.3. Definition. An R -group of multiplicative type is called

isotrivial if there exists a finite étale cover S/R such that \mathfrak{G}_S is diagonalizable;

quasi-isotrivial if there exists an étale cover S/R such that \mathfrak{G}_S is diagonalizable.

For example, the Weil restriction example is isotrivial. We consider the case of a connected Galois cover S/R of finite group Γ . By the yoga of forms, for each f.g. abelian group A , we have a correspondence

$$\left\{ S/R\text{-forms of } \mathfrak{D}(A) \right\} \quad \longleftrightarrow \quad H^1(S/R, \mathrm{GL}(A)) = H^1(\Gamma, \mathrm{GL}(A)).$$

The point is that $\mathrm{Aut}(\mathfrak{D}(A))^{op} = (\mathrm{GL}(A))_R$. Also since $H^1(\Gamma, \mathrm{GL}(A)) = \mathrm{Hom}_{gr}(\Gamma, \mathrm{GL}(A))/\mathrm{GL}(A)$, it implies that there is a minimal Galois subextension $S_{\mathfrak{G}}/R$ of S/R which splits \mathfrak{G} . This can be pushed further as follows.

17.1.4. Proposition. *The subcategory of \mathcal{M}/R consisting of R -groups of multiplicative type split by S/R is full and abelian. It is antiequivalent to the category of f.g. Γ -modules over \mathbb{Z} .*

We can pass that to the limit on Galois covers.

17.1.5. Corollary. *Suppose that R is connected and let $f : R \rightarrow F$ be a base point where f is a separably closed field. The subcategory \mathcal{M}_R consisting of isotrivial R -groups of multiplicative type is a full and abelian. It is antiequivalent to the category of discrete $\pi_1(R, f)$ -modules which are finitely generated over \mathbb{Z} .*

17.1.6. Remarks. (1) For an isotrivial R -group of multiplicative type \mathfrak{G} , there is a minimal Galois subextension $R_{\mathfrak{G}}/R$ of R^{sc} which splits \mathfrak{G} .

(2) Since a Γ -module M (f.g. over \mathbb{Z}) is a quotient of a free module $\mathbb{Z}[\Gamma]^r$, it follows that each isotrivial R -group of multiplicative type embeds in a quasi-trivial torus namely of the shape $\prod_{S/R} \mathbb{G}_{m,S}$ for a suitable finite étale cover of R .

To find the best way to present a given R -group of multiplicative type is then a natural question which is linked with representation theory. We can mention here the theory of flasque resolutions by Colliot-Thélène and Sansuc which deals with isotrivial objects [CTS1]. In the general case, not much in known beyond the following fact.

¹¹Up to localize, we can assume that S/R is locally free of rank $d \geq 1$. We prove it by using a finite étale cover T such that $S \otimes_R T \cong T^d$. Then $\mathfrak{G}_T = \prod_{S \otimes_R T/T} \mathfrak{D}(A)_{S \otimes_R T} =$

$\prod_{T^d/T} \mathfrak{D}(A)_{T^d} = \mathfrak{D}(A)_T^d$.

17.1.7. Proposition. (Conrad, [C, B.3.8]) *Let \mathfrak{G} be an R -group of multiplicative type. Then \mathfrak{G} embeds as closed subgroup scheme in an R -torus.*

17.2. Splitting results.

17.2.1. Lemma. *Assume that R is connected. Let \mathfrak{G}/R be a finite group of multiplicative type. Then \mathfrak{G} is isotrivial.*

Proof. There exist a flat cover S/R such that $\mathfrak{G} \times_R S \xrightarrow{\sim} \mathfrak{D}(A)_R$ where A is a finite abelian group. In other words, \mathfrak{G} is an R -form of $\mathfrak{D}(A)$. But those forms are classified by the pointed set $H^1(R, \mathrm{GL}_1(A))$ which classifies also Galois R -covers of group $\mathrm{GL}_1(A)$. Therefore \mathfrak{G} defines a class of Galois covers $[S/R]$ which split \mathfrak{G} . \square

17.2.2. Proposition. *Let k be a field. Then the k -groups of multiplicative type are isotrivial.*

Proof. Since it holds in the finite case, we can deal with a torus T/k of rank d . There exists a finite field extension L/k and an isomorphism $\phi : \mathbb{G}_{m,L}^d \xrightarrow{\sim} T_L$. If L is separable, there is nothing to do. If not, k is of characteristic $p > 0$ and there exists a subextension $F \subsetneq L$ such that $L = F(\sqrt[p]{x})$. We claim that ϕ descend to F . We consider the ring $R = L \otimes_F L = L[t]/(t^p - x) \cong L[u]/u^p$, it is an artinian local ring of residue field L . Theorem 10.2.1.(2) shows that

$$\mathrm{Hom}_{R\text{-gp}}(\mathbb{G}_{m,R}^d, T_R) \xrightarrow{\sim} \mathrm{Hom}_{L\text{-gp}}(\mathbb{G}_{m,L}^d, T_L).$$

It follows that $d_{1,*}(\phi) = d_{2,*}(\phi) : \mathbb{G}_R^d \rightarrow T_R$. By faithfully flat descent, ϕ descends then to F . and this is an R -group isomorphism. We can continue this process which stops when reaching the maximal separable subextension of L/k . \square

17.2.3. Corollary. *Let A be a f.g. abelian group. Then we have*

$$\mathrm{Hom}_{ct}(\mathrm{Gal}(k_s/k), \mathrm{GL}(A))/\mathrm{GL}(A) \xrightarrow{\sim} H^1(k, \mathrm{GL}(A)).$$

17.2.4. Theorem. *Let \mathfrak{G}/R be an R -group of multiplicative type. Then \mathfrak{G}/R is quasi-isotrivial.*

In the present proof, we use Artin's approximation theorem which came six years after the SGA3 seminar.

Proof. By the classical limit argument, we can assume that R is of finite type over \mathbb{Z} and in particular that R is noetherian. Up to localize, we can assume that \mathfrak{G} is an R -form of $\mathfrak{D}(A)$. The statement holds in the case A finite (Lemma 17.2.1) and behaves well under extensions, so we can assume that $A = \mathbb{Z}^d$. We switch then to the torus notation $\mathfrak{T} = \mathfrak{G}$. We consider the R -functor (which is a flat sheaf)

$$F(S) = \mathrm{Hom}_{S\text{-gr}}(\mathbb{G}_{m,S}^d, \mathfrak{T}_S).$$

We have seen that F is formally étale (Theorem 10.2.1.(2)) and we observe that F is locally finitely presented, that is commutes with filtered direct limits of rings.

Let $x \in \operatorname{Spec}(R)$ be a closed point and denote by \mathfrak{M}_x the underlying prime ideal of R and $k(x) = R/\mathfrak{M}_x$. We denote by R_x^{sh} the strict henselization of the local ring R_x , see §21.4. The plan is to construct an element of $F(R_x^{sh})$ which is an isomorphism. We choose firstly a separable field extension $k'/k(x)$ which splits $\mathfrak{T}_{k(x)}$. Up to shrink R , it lifts to a finite étale connected cover R'/R . Without loss of generality, we can assume then that $\mathfrak{T}_{k(x)}$ is split. In other words, there exists a group $k(x)$ -isomorphism

$$\phi_0 : \mathbb{G}_{m,k(x)}^d \xrightarrow{\sim} \mathfrak{T}_{k(x)}.$$

We denote by \widehat{R}_x the \mathfrak{M}_x -adic completion of R . From 10.3.3.(3), we have see that ϕ_0 lifts uniquely to a morphism

$$\widehat{\phi} : \mathbb{G}_{m,\widehat{R}_x}^d \rightarrow \mathfrak{T}_{\widehat{R}_x}$$

and $\widehat{\phi}$ is a monomorphism. The cokernel of $\widehat{\phi}$ is a \widehat{R}_x -group of multiplicative type whose special fiber is trivial. It follows that $\widehat{\phi}$ is an isomorphism. We apply now the Artin's approximation theorem 21.5.1 to the locally finitely presented functor $E(S) = \operatorname{Isom}_{S-gr}(\mathbb{G}_{m,S}^d, \mathfrak{T}_S)$. It implies that

$$\operatorname{Im}\left(E(R_x^h) \rightarrow E(k(x))\right) = \operatorname{Im}\left(E(\widehat{R}) \rightarrow E(k(x))\right).$$

In our case, it provides an isomorphism $\phi : \mathbb{G}_{m,R_x^h}^d \xrightarrow{\sim} \mathfrak{T} \times_R R_x^h$. This isomorphism is defined on some étale neighborhood R'/R of x . \square

17.2.5. Corollary. *Let R be strictly henselian ring. Then the R -groups of multiplicative type are split.*

17.2.6. Theorem. *Assume that R is normal. Let K be the fraction field of R and let $f : R \rightarrow K \rightarrow K_s$ be an embedding in a separable closure of K .*

(1) *The R -groups of multiplicative type are isotrivial.*

(2) *The category of R -groups of multiplicative type is equivalent to the category of discrete $\pi_1(R, f)$ -modules which are f.g. over \mathbb{Z} .*

Assertion (2) is a formal consequence of (1). We present an alternate proof based on the following step.

17.2.7. Lemma. *Let \mathfrak{T}/R be a torus of dimension d . There is a Galois cover R'/R of group $\operatorname{GL}_d(\mathbb{F}_3)$ such that $\mathfrak{T}_{R' \otimes_R K}$ splits.*

Proof. We have an exact sequence of groups

$$1 \rightarrow \Theta \rightarrow \operatorname{GL}_d(\mathbb{Z}) \rightarrow \operatorname{GL}_d(\mathbb{F}_3) \rightarrow 1$$

and Minkowski's lemma states that Θ is torsion free [N, IX.11]. We consider then the exact commutative diagram of pointed sets

$$1 \longrightarrow H^1(R, \Theta) \longrightarrow H^1(R, \operatorname{GL}_d(\mathbb{Z})) \longrightarrow H^1(R, \operatorname{GL}_d(\mathbb{F}_3)).$$

The R -torus \mathfrak{T} defines a class in $H^1(R, \mathrm{GL}_d(\mathbb{Z}))$ whose image in $H^1(R, \mathrm{GL}_d(\mathbb{F}_3))$ is represented by a Galois cover R'/R of group $\mathrm{GL}_d(\mathbb{F}_3)$. As usual we take the right connected component S/R , it a connected Galois cover of group $\Gamma \subset \mathrm{GL}_d(\mathbb{F}_3)$. We put $L = \mathrm{Frac}(S) = S \otimes_R K$ and look at the commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & H^1(R, \Theta) & \longrightarrow & H^1(R, \mathrm{GL}_d(\mathbb{Z})) & \longrightarrow & H^1(R, \mathrm{GL}_d(\mathbb{F}_3)) \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H^1(S, \Theta) & \longrightarrow & H^1(S, \mathrm{GL}_d(\mathbb{Z})) & \longrightarrow & H^1(S, \mathrm{GL}_d(\mathbb{F}_3)) \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H^1(L, \Theta) & \longrightarrow & H^1(L, \mathrm{GL}_d(\mathbb{Z})) & \longrightarrow & H^1(L, \mathrm{GL}_d(\mathbb{F}_3))
\end{array}$$

where the vertical maps are base change maps.

17.2.8. Claim. $H^1(L, \Theta) = 1$.

Indeed a class of $H^1(L, \Theta)$ is represented in $H^1(L, \mathrm{GL}_d(\mathbb{Z}))$ by a continuous map $\phi : \mathrm{Gal}(K_s/L) \rightarrow \Theta$, so is trivial. By diagram chase, it follows that $[\mathfrak{T}] \in \ker\left(H^1(R, \mathrm{GL}_d(\mathbb{Z})) \rightarrow H^1(L, \mathrm{GL}_d(\mathbb{Z}))\right)$. Thus \mathfrak{T}_L is a split torus. \square

We can proceed to the proof of Theorem 17.2.6.

Proof. We can assume that R is noetherian. Since the result holds in the finite case (lemma 17.2.1) and behaves well under exact sequences, it is enough to deal with the torus case. Let \mathfrak{T}/R be an R -torus of dimension d . Granting to Lemma 17.2.7, we can assume that \mathfrak{T}_K is split. We have then an isomorphism map $\alpha : \mathbb{G}_{m,K}^d \xrightarrow{\sim} \mathfrak{G}_K$. We want to show that it extends to R .

First case: R is a *DVR*. We have $R^{sc} = R^{hs}$ (see §21.4), so that $\mathfrak{T} \times_R R^{sc}$ splits according to Corollary 17.2.5. Hence there exists a finite Galois connected cover S/R and an isomorphism $\beta : \mathbb{G}_K^d \xrightarrow{\sim} \mathfrak{T}_K$. We put $L = \mathrm{Frac}(S)$ and observe that $\alpha_L \circ \beta_L \in \mathrm{GL}_d(\mathbb{Z})(L) = \mathrm{GL}_d(\mathbb{Z})(S)$. So we can modify β such that $\beta_L = \alpha_L$. It follows that β is $\mathrm{Gal}(S/R)$ -invariant and descends to an isomorphism $\mathbb{G}_R^d \xrightarrow{\sim} \mathfrak{T}$, which extends α .

General case: Given $x \in \mathrm{Spec}(R)^{(1)}$, we know from the first case that there exists an isomorphism $\beta_x : \mathbb{G}_{m,R_x}^d \xrightarrow{\sim} \mathfrak{T}_{R_x}$ which extends α . Hence there exists an open subset \mathfrak{U} of $\mathrm{Spec}(R)$ containing all points of codimension one such that α extends to an isomorphism $\tilde{\alpha} : \mathfrak{D}(A)_{\mathfrak{U}} \xrightarrow{\sim} \mathfrak{G}_{\mathfrak{U}}$.

The map $\tilde{\alpha} : \mathbb{G}_m^d \times_{\mathbb{Z}} \mathfrak{U} \rightarrow \mathfrak{G}$ is defined everywhere in codimension one on the normal scheme¹² $\mathbb{G}_m^d \times_{\mathbb{Z}} R$ so it extends uniquely to a map $\mathbb{G}_{m,R}^d \rightarrow \mathfrak{G}$ ([EGA4, 20.4.6] or [Li, 4.1.14]). This map is a group isomorphism with the same kind of arguments. \square

¹²Recall that a smooth affine scheme over a normal noetherian ring is normal [Li, 8.2.25].

17.3. Back to the recognition statement. We state once again the important Proposition 13.3.4.

17.3.1. Proposition. [C, 4.2.2] *Let \mathfrak{U}/R be an affine smooth group scheme whose geometric fibers are rank one additive groups. We assume that \mathfrak{U}/R is equipped with an action of \mathbb{G}_m such that the \mathbb{G}_m -module $L = \text{Lie}(\mathfrak{U})(R)$ is non trivial everywhere. Then there exists a natural R -group isomorphism $\mathfrak{W}(L) \cong \mathfrak{U}$.*

Proof. We can assume that R is noetherian. The statement is of local nature for the flat topology so we can assume that $R = (R, \mathfrak{M}, k)$ is local. Also up to make an essentially étale extension of R , we can assume that the map $\mathfrak{U}(R) \rightarrow \mathfrak{U}(k)$ is not trivial.

In particular \mathbb{G}_m acts on L by a character α_n , $n \geq 1$ (up to take the opposite action). Let $\sigma \in \mathfrak{U}(R)$ be a point whose specialization is not trivial. We consider the orbit map

$$q : \mathbb{G}_{m,R} \rightarrow \mathfrak{U}, \quad t \mapsto t \cdot \sigma \quad .$$

It extends to an R -map $\tilde{q} : \mathbb{A}_R^1 \rightarrow \mathfrak{U}$ which is \mathbb{G}_m -equivariant for the scaling action on $\mathbb{G}_{a,R}$. The induced map $\tilde{q}_k : \mathbb{A}_k^1 \rightarrow \mathfrak{U}_k \xrightarrow{\sim} \mathbb{G}_{a,k}$ is a non constant endomorphism f of the affine line \mathbb{A}_k^1 which satisfies $f(t.x) = t^n f(x)$, hence $f(t) = a t^n$ for some $a \in k^\times$. In particular, f is $\mu_{n,k}$ -invariant.

17.3.2. Claim. *q is μ_n -invariant.*

Equivalently we have to show that $\mu_n \rightarrow \mathbb{G}_m \rightarrow \mathfrak{U}$ is a constant map (of value σ) or that $R[\mathfrak{U}] \rightarrow R[\mu_n]/R$ is trivial. Since q_k is trivial, $R_m[\mathfrak{U}] \rightarrow R_m[\mu_n]/R_m = R[\mathbb{Z}/n\mathbb{Z}]/R_m$ is trivial where $R_m = R/\mathfrak{M}^m$. By passing to the limit, we get that $\hat{R}[\mathfrak{U}] \rightarrow \hat{R}[\mu_n]/\hat{R} = \hat{R}[\mathbb{Z}/n\mathbb{Z}]/\hat{R}$ is trivial. Thus $q_{\hat{R}}$ is a constant map so q is trivial as well. The claim is proved.

By moding out by μ_n , we get a factorization $q' : \mathfrak{G}'_{m,R} \rightarrow \mathfrak{U}$ and a map $\tilde{q}' : (\mathbb{A}_{m,R}^1)' \rightarrow \mathfrak{U}$ which is \mathbb{G}_m -equivariant where the action on $(\mathbb{A}_{m,R}^1)'$ is by α_n . It follows that the map \tilde{q}' is an R -group monomorphism. Also \tilde{q}' is étale by the differential criterion so that it is an immersion. But its image contains the closed fiber \mathfrak{U}_k which permits to conclude that \tilde{q}' is an isomorphism. \square

17.3.3. Remark. The key thing in the proof is the \mathbb{G}_m -action. In positive characteristic, the additive group \mathbb{G}_a has a large automorphism group [DG, II.1.2.7] which is under control there.

Reductive group schemes and descent techniques

18. SPLITTING REDUCTIVE GROUP SCHEMES

18.1. Local splitting. The next result generalizes the torus case 17.2.4.

18.1.1. Theorem. *Let \mathfrak{G}/R be a reductive group scheme. Then there exists an étale cover $S_1 \times \cdots \times S_l$ such that $\mathfrak{G} \times_R S_i$ is a split reductive S_i -group scheme for $i = 1, \dots, r$.*

Proof. The proof goes on the same lines than Theorem 17.2.4. By the classical limit argument, we can assume that R is of finite type over \mathbb{Z} and in particular that R is noetherian. Let $x \in \text{Spec}(R)$ be a closed point and denote by \mathfrak{M}_x the underlying prime ideal of R and $k(x) = R/\mathfrak{M}_x$. The $k(x)$ -group $\mathfrak{G} \times_R k(x)$ admits a maximal torus $T/k(x)$. It splits after a separable field extension $k(x)'$. Up to shrink R , the extension $k'(x)/k$ lifts to a finite étale connected cover R'/R . It boils down then to the case when $\mathfrak{G} \times_R k(x)$ admits a maximal torus $\mathbb{G}_m^d/k(x)$. According to Theorem 10.3.3.(1), our given $k(x)$ -embedding

$$\phi_0 : \mathbb{G}_{m,k(x)}^d \rightarrow \mathfrak{G}_{k(x)}.$$

lifts to a \widehat{R}_x -monomorphism

$$\widehat{\phi} : \mathbb{G}_{m,\widehat{R}_x}^d \rightarrow \mathfrak{G}_{\widehat{R}_x}.$$

We apply now the Artin's approximation theorem 21.5.1 to the locally finitely presented functor $F(S) = \text{Hom}_{S\text{-gr}}(\mathbb{G}_{m,S}^d, \mathfrak{G}_S)$. It implies that

$$\text{Im}\left(F(R_x^h) \rightarrow F(k(x))\right) = \text{Im}\left(F(\widehat{R}_x) \rightarrow F(k(x))\right).$$

In our case, it provides an R_x^h -map $\phi : \mathbb{G}_{m,R_x^h}^d \rightarrow \mathfrak{G}_{R_x^h}$. This map ϕ is a monomorphism by 10.1.4 and a closed immersion by Corollary 16.3.1. Since the absolute rank is a locally constant function (Cor. 11.0.8), we conclude that ϕ defines a split maximal R_x^h -torus of $\mathfrak{G}_{R_x^h}$. The $\mathfrak{G}_{R_x^h}$ -group is then split according to Remark 13.3.6. \square

18.1.2. Corollary. *Assume that R is a strictly henselian local ring. Then each reductive group scheme \mathfrak{G}/R is split.*

18.1.3. Definition. *Let \mathfrak{H}/R be a reductive group scheme. For each $x \in \text{Spec}(R)$, we define the type of \mathfrak{H} at x as the isomorphism class of the root datum $\Psi(\mathfrak{H}_{\overline{\kappa(x)}})$. It is denoted $\text{type}_x(\mathfrak{H})$.*

The isomorphism classes of root data form a set, we denote it by *Type*. We refine then the continuity of the rank by the

18.1.4. **Corollary.** *Let \mathfrak{H}/R be an affine smooth group scheme. The map*

$$\begin{aligned} \mathrm{Spec}(R) &\longrightarrow \mathcal{T}ype \\ x &\mapsto \mathrm{type}_x(\mathfrak{H}) \end{aligned}$$

is continuous.

We get also a local characterization of reductive group schemes.

18.1.5. **Corollary.** *Let \mathfrak{G}/R be an affine group scheme. Then the following are equivalent:*

- (i) \mathfrak{G} is a reductive group scheme;
- (ii) there exists an étale cover $S_1 \times \cdots \times S_l$ such that $\mathfrak{G} \times_R S_i$ is a split reductive S_i -group scheme for $i = 1, \dots, l$;
- (iii) there exists an flat cover $S_1 \times \cdots \times S_l$ such that $\mathfrak{G} \times_R S_i$ is a split reductive S_i -group scheme for $i = 1, \dots, l$.

Proof. Theorem 18.1.1 is exactly (i) \implies (ii). The implication (ii) \implies (iii) is obvious. The implication (iii) \implies (i) is the easy one. Assuming (iii), faithfully flat descent theory yields that \mathfrak{G} is smooth. Also the geometric fibers of \mathfrak{G} are reductive, so we conclude that \mathfrak{G} is a reductive group scheme. \square

18.1.6. **Proposition.** *Let \mathfrak{G}/R be a smooth affine group scheme. Maximal tori exist locally for the étale topology and are locally conjugated.*

Proof. Since split reductive group schemes admit maximal tori, Theorem 18.1.1 yields the existence of maximal tori locally for the étale topology. For conjugacy, we can assume that R is finitely generated over \mathbb{Z} . Let $\mathfrak{T}_1, \mathfrak{T}_2$ be two maximal R -tori of \mathfrak{G} . Let $x \in \mathrm{Spec}(R)$. By Theorem 17.2.4, we can localize for the étale topology in order to split \mathfrak{T}_1 and \mathfrak{T}_2 . We can then assume that \mathfrak{T}_1 and \mathfrak{T}_2 are split and see them as the images of $\phi_i : \mathbb{G}_R^d \rightarrow \mathfrak{G}$ for $i = 1, 2$. Up to localize furthermore, we know that $\mathfrak{T}_1 \times_R k(x)$ and $\mathfrak{T}_2 \times_R k(x)$ are conjugated [CGP, A.2.10] by an element $g_x \in G(k(x))$ which lifts in $g \in \mathfrak{G}(R)$. This boils down to the case when $\phi_{1,k(x)} = \phi_{2,k(x)}$. By Corollary 10.3.3.(1), there exist $g \in \mathfrak{G}(\widehat{R}_x)$ such that $\phi_1 =^g \phi_2$. Artin approximation's theorem applied to the transporter functor

$$E(S) = \left\{ h \in \mathfrak{G}(S) \mid \phi_{1,S} = {}^h \phi_{2,S} \right\}$$

shows that $E(R_x^h)$ is not empty. \square

18.2. Weyl groups.

18.2.1. **Proposition.** *Let \mathfrak{G}/R be a reductive group scheme equipped with a maximal R -torus \mathfrak{T} . We denote by $N = N_{\mathfrak{G}}(\mathfrak{T})$ the normalizer functor of \mathfrak{T} defined by*

$$N(S) = \left\{ g \in \mathfrak{G}(S) \mid g\mathfrak{T}(S')g^{-1} = \mathfrak{T}(S') \quad \forall S'/S \right\}.$$

for each S/R .

(1) The functor N is representable by a closed subgroup scheme $\mathfrak{N} = \mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})$ of \mathfrak{G} .

(2) \mathfrak{N} is smooth and the quotient $\mathfrak{N}/\mathfrak{T}$ is locally a twisted finite constant group scheme.

(3) If \mathfrak{B} is a Borel subgroup containing \mathfrak{T} , we have $\mathfrak{T} = \mathfrak{N} \times_{\mathfrak{G}} \mathfrak{B}$.

(4) If \mathfrak{T} is split and \mathfrak{G} splits respectively to \mathfrak{T} , then $\mathfrak{N}/\mathfrak{T}$ is isomorphic to W_R where W is the Weyl group of the root datum $\Psi(G, T)$. Furthermore, the map $\mathfrak{N}(R) \rightarrow W_R(R)$ is surjective.

Proof. To be written. □

It applies to Borel subgroups.

18.2.2. Theorem. *Let \mathfrak{G}/R be a reductive group scheme and let \mathfrak{B}/R be a Borel R -subgroup scheme. Then \mathfrak{B} is its own normalizer.*

Proof. We consider the normalizer functor of \mathfrak{B} defined by

$$N(S) = \left\{ g \in \mathfrak{G}(S) \mid g\mathfrak{B}(S')g^{-1} = \mathfrak{B}(S') \quad \forall S'/S \right\}$$

for each S/R . We note that N is a flat sheaf so that we can localize for flat topology. From Theorems 18.1.1 and 18.5.3, we can assume that \mathfrak{G} is split and that \mathfrak{B} contains a maximal split R -torus \mathfrak{T} . Let us show that $\mathfrak{B}(R) = N(R)$. We are given $g \in N(R)$. Then ${}^g\mathfrak{T}$ is a maximal R -torus of \mathfrak{B} , so that up to localize for the étale topology, there exists $g \in \mathfrak{B}(R)$ such that ${}^g\mathfrak{T} = {}^b\mathfrak{T}$ (Th. 18.1.6). We can then assume that ${}^g\mathfrak{T} = \mathfrak{T}$, that is $g \in \mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$. By Proposition 18.2.1.(3), we get that $g \in \mathfrak{T}(R)$. □

18.2.3. Remark. An alternative way is to use Theorem 14.9.1. It implies that the normalizer functor is representable by a closed subgroup scheme \mathfrak{N} of \mathfrak{G} . Since $\mathfrak{B}_{k(x)} = \mathfrak{N}_{k(x)}$ for each $x \in \text{Spec}(R)$, the closed immersion $\mathfrak{B} \rightarrow \mathfrak{N}$ is surjective, hence an isomorphism.

18.3. Center of reductive groups. We say that a split reductive group is semisimple (resp. adjoint, simply connected) if its root datum is semisimple (resp. adjoint, simply connected). The general definition is then provided by descent.

18.3.1. Corollary. *Let \mathfrak{G}/R be a reductive group scheme. Then the center of \mathfrak{G} is representable by an R -group of multiplicative type. Furthermore the quotient $\mathfrak{G}/\mathfrak{Z}(G)$ is an adjoint reductive R -group.*

18.4. Isotriviality issues. We shall discuss firstly examples. The (normal) ring \mathbb{Z} is simply connected so that all tori are split. Also $\text{Pic}(\mathbb{Z}) = 0$, hence by Remark 13.3.7, we have

A reductive group scheme \mathfrak{G}/\mathbb{Z} is split if and only if \mathfrak{G} carries a maximal \mathbb{Z} -torus.

There are semisimple group schemes over \mathbb{Z} , the simplest one being the special orthogonal group of the \mathbb{Z} -quadratic form Γ_8 , see [CG]. Those groups have no maximal tori and this is somehow the first obstruction for splitting a reductive group scheme. A reasonable question is the following.

18.4.1. Question. Assume that R is normal. Let \mathfrak{G}/R be a reductive group scheme such that $\mathfrak{G} \times_R R^{sc}$ admits a maximal (split) R -torus. Is \mathfrak{G}/R isotrivial namely split by R^{sc} ?

We discuss this question by means of the following example. Let M/R be a locally free module of rank $d \geq 1$. Then the R -group $\mathfrak{G} = \mathrm{GL}(M)/R$ is isomorphic locally (for the Zariski topology) to GL_d . This R -group is reductive.

18.4.2. Lemma. (1) $\mathfrak{G} = \mathrm{GL}(M)$ admits a split R -torus of rank d if and only if $M = L_1 \oplus \cdots \oplus L_d$ where the L_i 's are invertible R -modules.

(2) $\mathfrak{G} = \mathrm{GL}(M)$ is split if and only if there exists an invertible R -module L such that $M \cong L^d$.

Proof. (1) If M decomposes as a sum of invertible modules, \mathfrak{G} contains \mathbb{G}_m^d as closed R -subgroup scheme. Conversely, assume that there is a closed immersion $i : \mathbb{G}^d \rightarrow \mathrm{GL}(M)$. By diagonalization, we get a decomposition $M = M_1 \oplus \cdots \oplus M_d$ and the M_i 's are projective locally free of rank one.

(2) If $M = L^d$, we have $\mathrm{GL}(M) = \mathrm{GL}_d$. Conversely, assume that \mathfrak{G} is split. In particular, there is a closed immersion $i : \mathbb{G}^d \rightarrow \mathrm{GL}(M)$ and we have $M = M_1 \oplus \cdots \oplus M_d$. We consider the adjoint action of $\mathbb{G}_{m,R}^d$ on $\mathrm{End}_R(M) = \mathrm{Lie}(\mathfrak{G})(R)$. We have the decomposition

$$\mathrm{End}_R(M) = R^d \oplus \bigoplus_{i \neq j} \mathrm{Hom}_R(M_i, M_j)$$

where $\mathrm{Hom}_R(M_i, M_j)$ is the eigenspace for the root $\alpha_i^{-1} \alpha_j$. Since \mathfrak{G} is split, the eigenspaces are free modules, so we conclude that $M_1 \xrightarrow{\sim} M_i$ for $i = 2, \dots, d$. \square

18.4.3. Remark. Assume that R is a Dedekind ring and let L be an invertible module. Then $L \oplus L^*$ is free so that $\mathrm{GL}_2 \xrightarrow{\sim} \mathrm{GL}(L \oplus L^*)$ contains a maximal split torus of \mathfrak{T} rank two and the root decomposition is $\mathfrak{gl}_{2,R} = \mathrm{Lie}(\mathfrak{T}) \oplus L^{\otimes 2} \oplus (L^{\otimes 2})^*$. Hence \mathfrak{T} is a splitting torus of GL_2 if and only if $[L] \in {}_2\mathrm{Pic}(R)$

Now let E/\mathbb{C} be a projective elliptic curve and put $\mathrm{Spec}(R) = E \setminus \{0\}$. Then R is a Dedekind ring and we have an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathrm{Pic}(E) \rightarrow \mathrm{Pic}(R) \rightarrow 0.$$

But $\mathrm{Pic}(E) \xrightarrow{\sim} \mathbb{Z} \oplus E(\mathbb{C})$ so that $E(\mathbb{C}) \xrightarrow{\sim} \mathrm{Pic}(R)$. Since $E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/\mathbb{Z}^2$, it follows that $\mathrm{Pic}(R)$ contains a class $[L]$ which is not torsion. Now we consider the R -group $\mathrm{GL}(R \oplus L)$ and claim it is not isotrivial, namely cannot

be split by a finite étale extension of R . We reason by sake of contradiction. Assume there exists a finite étale cover R'/R such that $\mathfrak{G} \times_R R'$ splits. Lemma 18.4.2.(2) implies then that $L \otimes_R R' \xrightarrow{\sim} R'$, i.e.

$$[L] \in \ker(\mathrm{Pic}(R) \rightarrow \mathrm{Pic}(R')).$$

By means of the norm map $\mathrm{Pic}(R') \rightarrow \mathrm{Pic}(R)$ (see [EGA4, 21.5.5]), this kernel is torsion (killed by the degree of $[R' : R]$). This contradicts our assumption over L .

This example shows that the naive question 18.4.1 has a negative answer. In other words, Theorem 17.2.6 is not true for reductive (and semisimple) R -group schemes even for an R -group carrying a split maximal R -torus.

18.4.4. Proposition. *Let \mathfrak{G}/R be a reductive group scheme admitting a maximal R -torus which is locally isotrivial. Then \mathfrak{G} is locally isotrivial, that is there exists a Zariski cover $R_1 \times \cdots \times R_l$ of R such that $\mathfrak{G} \times_R R_i$ is split for $i = 1, \dots, l$.*

Note that it applies in particular when R is normal and \mathfrak{G} contains a maximal R -torus by Theorem 17.2.6.

Proof. Let \mathfrak{T} be a maximal R -torus. Up to localize, we can suppose that \mathfrak{G} has constant type and that there exists a finite étale connected cover such that $\mathfrak{T} \times_R S \cong \mathbb{G}_S^d$. For each root α , the weight space $\mathrm{Lie}(\mathfrak{G})(S)_\alpha$ is an invertible S -module. For each $x \in \mathrm{Spec}(R)$, $S \otimes_R R_x$ is a semi-local ring so that $\mathrm{Pic}(S \otimes_R R_x) = 1$. It follows that $\mathfrak{G} \times_R (S \otimes_R R_x)$ splits. By quasi-compactness, we conclude that \mathfrak{G} is locally isotrivial. \square

18.4.5. Remark. The statement is rather weak and can be strengthened as follows: A semisimple R -group scheme is locally isotrivial, see [SGA3, XXIX.4.1.5].

18.5. Killing pairs.

18.5.1. Definition. *A Killing couple is a pair $(\mathfrak{B}, \mathfrak{T})$ where \mathfrak{B} is a Borel R -subgroup (see 12.3.1) and \mathfrak{T} is a maximal R -torus of \mathfrak{B} .*

18.5.2. Example. Let \mathfrak{G}/R be a split reductive group and \mathfrak{T}/R be a maximal split torus of \mathfrak{G}/R . We denote by $\Psi = \Psi(\mathfrak{G}, \mathfrak{T})$ the associated root datum. Choose $\lambda \in \widehat{\mathfrak{T}}^0$ such that $\langle \alpha, \lambda \rangle \neq 0$ for each root α . We get then a subset of positive roots $\{\alpha \mid \langle \alpha, \lambda \rangle > 0\}$ and a basis Δ of the root system $\Phi(\mathfrak{G}, \mathfrak{T})$ [Bbk3, VI.1.7, cor. 2]. Then we claim that the limit R -group $\mathfrak{B} = \mathfrak{B}_G(\lambda)$ defined in 12.3.1 is a Borel subgroup of \mathfrak{G}/R . It is a closed subgroup scheme which is indeed smooth. Its Lie algebra is

$$\mathrm{Lie}(\mathfrak{B})(R) = \mathrm{Lie}(\mathfrak{T})(R) \oplus \bigoplus_{\alpha > 0} \mathrm{Lie}(\mathfrak{G})_\alpha(R).$$

Also its geometric fibers are parabolic subgroups [Sp, §15.1] whose Lie algebras are Borel subalgebras. Therefore the geometric fibers of \mathfrak{B} are Borel subgroups and we conclude that \mathfrak{B} is a Borel subgroup scheme of \mathfrak{G} .

18.5.3. Theorem. *Let \mathfrak{G}/R be a reductive group scheme.*

(1) *Locally for the étale topology, Killing couples of \mathfrak{G} exist and are conjugated.*

(2) *Locally for the étale topology, Borel subgroups of \mathfrak{G} exist and are conjugated.*

Proof. By Theorem 18.1.1, \mathfrak{G} is locally split for the étale topology. The example above shows that \mathfrak{G} admits a Killing couple locally for the étale topology on $\mathrm{Spec}(R)$. It remains to treat the two conjugacy questions with essentially the same method than for 18.1.6. We can assume that R is finitely generated over \mathbb{Z} .

(1) Let $(\mathfrak{B}_1, \mathfrak{T}_1), (\mathfrak{B}_2, \mathfrak{T}_2)$ be two Killing couples of \mathfrak{G}/R . Let $x \in \mathrm{Spec}(R)$, we want to show the statement étale-locally at x . Since the result holds for separably closed fields, we can localize for the étale topology so that $(\mathfrak{B}_{1,k}, \mathfrak{T}_{1,k}) = (\mathfrak{B}_{2,k}, \mathfrak{T}_{2,k})$. By Theorem 10.3.3.(3), there exist unique $\lambda_i : \mathbb{G}_{m, \widehat{R}_x} \rightarrow \mathfrak{T}_{i, \widehat{R}_x}$ which lifts $\lambda_{i,k}$ for $i = 1, 2$.

Lemma 12.3.3.(1) shows that $\mathfrak{B}_{i, \widehat{R}} = \mathfrak{P}_{\mathfrak{G}}(\lambda_i)$ for $i = 1, 2$. Now we use that λ_1 and λ_2 are $\mathfrak{G}(\widehat{R})$ -conjugated according to Corollary 10.3.3.(2), i.e. $\lambda_{1, \widehat{R}} = {}^g \lambda_{2, \widehat{R}}$ for some $g \in \mathfrak{G}(\widehat{R})$. Since $\mathfrak{T}_{i, \widehat{R}} = \mathfrak{Z}_{\mathfrak{G}(\widehat{R})}(\lambda_i)$ for $i = 1, 2$, it follows that $(\mathfrak{B}_1, \mathfrak{T}_1)_{\widehat{R}} = {}^g (\mathfrak{B}_2, \mathfrak{T}_2)_{\widehat{R}}$. Once again the Artin approximation theorem enables to conclude that $(\mathfrak{B}_1, \mathfrak{T}_1)$ and $(\mathfrak{B}_2, \mathfrak{T}_2)$ are locally conjugated for the étale topology.

(2) It is a simplification of the previous argument. □

19. TOWARDS THE CLASSIFICATION OF SEMISIMPLE GROUP SCHEMES

19.1. Kernel of the adjoint representation. Let \mathfrak{G}/R be a reductive group scheme and denote by $\mathfrak{g} = \mathrm{Lie}(\mathfrak{G})(R)$ its Lie algebra. We consider the adjoint representation $\mathrm{Ad} : \mathfrak{G} \rightarrow \mathrm{GL}(\mathfrak{g}) = \mathfrak{H}$, it factorizes in the sequence of R -group functors

$$\mathfrak{G} \xrightarrow{\mathrm{int}} \mathrm{Aut}(\mathfrak{G}) \xrightarrow{L} \mathrm{GL}(\mathfrak{g}) = \mathfrak{H}$$

where L maps an S -isomorphism $\varphi : \mathfrak{G}_S \rightarrow \mathfrak{G}_S$ to its differential $\mathrm{Lie}(\varphi) : \mathfrak{g} \otimes_R S \rightarrow \mathfrak{g} \otimes_R S$.

19.1.1. Proposition. (1) [SGA3, XXII.5.14] *The adjoint representation $\mathfrak{G} \rightarrow \mathrm{GL}(\mathfrak{g})$ induces a monomorphism $\mathfrak{G}_{\mathrm{ad}} \rightarrow \mathrm{GL}(\mathfrak{g})$.*

(2) *If \mathfrak{G} is adjoint, $L : \mathrm{Aut}(\mathfrak{G}) \rightarrow \mathrm{GL}(\mathfrak{G}) = \mathfrak{H}$ is a monomorphism.*

(3) *If \mathfrak{G} is adjoint, the morphism of R -functors $\mathrm{Aut}(\mathfrak{G}) \rightarrow N_{\mathfrak{H}}(\mathfrak{G})$ is an isomorphism and $\mathrm{Aut}(\mathfrak{G})$ is representable by a closed subgroup scheme of $\mathrm{GL}(\mathfrak{g})$.*

Proof. (1) The field case is due to Rosenlicht [Rt, Lemma 1 p. 39]. We can assume that we deal with a base field k which is algebraically closed and reductive group G/k . By the Bruhat decomposition with respect to a Killing couple (B, T) an element of $\ker(G(k) \rightarrow \mathrm{GL}(\mathfrak{g})(k))$ can be written up to conjugacy as $g = n_w u$ with $n \in N_G(T)(k)$. By looking at the action on \mathfrak{g} , we see that $w = 1$, that is $g \in B(k)$. We have $Z_G(\mathrm{Lie}(T)) = T$ so that $g \in N_G(T)(k) \cap B(k) = T(k)$. By looking at the roots we conclude that $g \in \mathfrak{Z}(\mathfrak{G})(k)$.

The method for reaching the general case is similar to that for showing Theorem 10.1.4. We denote by \mathfrak{K} the kernel of $G_{ad} \rightarrow \mathrm{GL}(\mathfrak{G})$. We shall show that \mathfrak{K} is proper by using the valuative criterion. Let A/R be a valuation ring and denote by F its fraction field. Since $\mathfrak{K}(F) = 1$ from the field case, we have $\mathfrak{K}(A) = \mathfrak{K}(F)$ and the criterion is fulfilled. Hence \mathfrak{K} is proper. Since \mathfrak{K} is affine, \mathfrak{K} is finite over R [Li, 3.17]. It follows that $R[\mathfrak{K}]$ is a finite R -algebra such that $R/\mathfrak{M}_x \xrightarrow{\sim} R[\mathfrak{K}]/\mathfrak{M}_x R[\mathfrak{K}]$. The Nakayama lemma [St, 18.1.11.(6)] shows that the map $R \rightarrow R[\mathfrak{K}]$ is surjective. By using the unit section $1_{\mathfrak{K}}$ we conclude that $R = R[\mathfrak{K}]$.

(2) We assume that \mathfrak{G} is adjoint and consider the group functor $L : \mathrm{Aut}(\mathfrak{G}) \rightarrow \mathrm{GL}(\mathfrak{G}) = \mathfrak{H}$. Let $\varphi \in \ker(L)(R)$. Then for each S/R and each $g \in \mathfrak{G}(S)$, we have

$$L(\varphi \circ \mathrm{int}(g) \circ \varphi^{-1}) = L(\mathrm{int}(g)) = \mathrm{Ad}(g).$$

But $\varphi \circ \mathrm{int}(g) \circ \varphi^{-1} = \mathrm{int}(\varphi(g))$ so we have $\mathrm{Ad}(\varphi(g)) = \mathrm{Ad}(g)$. Since Ad is a monomorphism by (1), it follows that $\varphi(g) = g$. This shows that $\varphi = \mathrm{id}_{\mathfrak{G}}$. We conclude that L is a monomorphism.

(3) The map $\mathrm{Aut}(\mathfrak{G}) \rightarrow N_{\mathfrak{H}}(\mathfrak{G})$ is then a monomorphism. But this map admits a splitting hence is an isomorphism. By Theorem 14.9.1, we have that $\mathrm{Aut}(\mathfrak{G})$ is representable by a closed subgroup of \mathfrak{H} . \square

19.2. Pinnings.

19.2.1. Definition. Let \mathfrak{G}/R be a split reductive group scheme equipped with a maximal torus \mathfrak{T} . A pinning of \mathfrak{G}/R is a couple $E = (\Delta, (X_{\alpha})_{\alpha \in \Delta})$ where Δ is base of the root datum $\Psi(\mathfrak{G}, \mathfrak{T})$ and each X_{α} is an R -base of the invertible free R -module $\mathrm{Lie}(\mathfrak{G})(R)_{\alpha}$.

If R is connected and $g \in \mathfrak{G}(R)$, then we can talk about the conjugated pinning ${}^g E_p$ relative to the R -torus ${}^g \mathfrak{T}$.

19.2.2. Lemma. Assume that R is connected and let \mathfrak{G}/R be an adjoint split group scheme which splits relatively to a split R -torus \mathfrak{T} .

(1) The group $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$ acts simply transitively on the pinnings relatively to \mathfrak{T} .

(2) The group $\mathfrak{G}(R)$ acts simply transitively on the couples (\mathfrak{G}, E_p) consisting of a maximal split torus which splits \mathfrak{G} and a pinning.

Proof. Let (Δ, Ep) be a pinning. We prove firstly the freeness of the action. Let $g \in \mathfrak{G}(R)$ such that ${}^g(\mathfrak{T}, \Delta, (X_\alpha)) = (\mathfrak{T}, \Delta, (X_\alpha))$. Then $g \in \mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$ and we denote by w its image in $W = W_R(R)$. Then ${}^w\Delta = \Delta$ so that $w = 1$ and $g \in \mathfrak{T}(R)$. For each $\alpha \in \Delta$, we have $X_\alpha = {}^gX_\alpha = \alpha(g)X_\alpha$ so that $g \in \ker(\mathfrak{T} \rightarrow \mathbb{G}_{m,R}^\Delta) = 1$ since \mathfrak{G} is adjoint. We prove now the transitivity.

(1) Let $(\Delta', (X'_{\alpha'}))$ be another pinning relative to \mathfrak{T} . There exists (a unique) $w \in W$ such that $\Delta = {}^w\Delta'$. By Proposition 18.2.1, w lifts to an element $n_w \in \mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$. We can then assume that $\Delta = \Delta'$. For each $\alpha \in \Delta$, we have $X'_\alpha = c_\alpha X_\alpha$ for some $c_\alpha \in R^\times$. Since $\mathfrak{T} \xrightarrow{\sim} \mathbb{G}_{m,R}^\Delta$, there exists then $t \in \mathfrak{T}(R)$ such that ${}^tX'_\alpha = X_\alpha$ for each $\alpha \in \Delta$. Thus $(\Delta', (X'_{\alpha'}))$ is a $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$ -conjugate of Ep .

(2) Let \mathfrak{T}' be another maximal split torus which splits \mathfrak{T}' and let E'_p be pinning. By the unicity, we can reason étale locally so that \mathfrak{T}' is $\mathfrak{G}(R)$ -conjugated to \mathfrak{T} . This boils down to the case $\mathfrak{T}' = \mathfrak{T}$ where (1) applies. \square

19.3. Automorphism group.

19.3.1. Theorem. [SGA3, XXIV.1] *Let \mathfrak{G}/R be an adjoint reductive group scheme and denote by \mathfrak{g} its Lie algebra.*

(1) *The functor $\text{Aut}(\mathfrak{G})$ is representable by a smooth group scheme and the map $\text{Aut}(\mathfrak{G}) \rightarrow \text{GL}(\mathfrak{g})$ is a closed immersion.*

(2) *The quotient sheaf $\text{Out}(\mathfrak{G}) = \text{Aut}(\mathfrak{G})/\mathfrak{G}$ is representable by a finite étale R -group scheme.*

(3) *We assume that \mathfrak{G} is split. Let \mathfrak{T} be a maximal split torus of \mathfrak{G} and let Ep be a pinning. Then $\text{Out}(\mathfrak{G})$ is a finite constant group and the exact sequence*

$$1 \rightarrow \mathfrak{G} \rightarrow \text{Aut}(\mathfrak{G}) \rightarrow \text{Out}(\mathfrak{G}) \rightarrow 1$$

splits. More precisely, we have an isomorphism

$$\left\{ f \in \text{Aut}(\mathfrak{G}, \mathfrak{T})(R) \mid {}^fEp = Ep \right\} \xrightarrow{\sim} \text{Out}(\mathfrak{G})(R).$$

Proof. to be written. \square

19.4. Unicity and existence theorems. We give a weak version of the unicity theorem.

19.4.1. Theorem. *Let $(\mathfrak{G}, \mathfrak{T})$ and $(\mathfrak{G}', \mathfrak{T}')$ be two split reductive R -group schemes equipped with maximal split tori. Then $(\mathfrak{G}, \mathfrak{T})$ and $(\mathfrak{G}', \mathfrak{T}')$ are isomorphic and only if the root data $\Psi(\mathfrak{G}, \mathfrak{T})$ and $\Psi(\mathfrak{G}', \mathfrak{T}')$ are isomorphic.*

We come now to the Chevalley's existence theorem.

19.4.2. Theorem. *Let $\Psi = (M, \mathcal{R}, M^\vee, \mathcal{R}^\vee)$ be a reduced root datum. Then there exists a split \mathbb{Z} -reductive group scheme \mathfrak{G} equipped with a maximal split \mathbb{Z} -torus \mathfrak{T} such that $\Psi(\mathfrak{G}, \mathfrak{T}) \cong \Psi$.*

This \mathbb{Z} -group scheme is called the Chevalley group associated to Ψ . If R is connected, a given reductive R -group scheme \mathfrak{G}/R is isomorphic étale locally to a unique Chevalley group scheme $\mathfrak{G}_0 \times_{\mathbb{Z}} R$. It defines then a class in the non-abelian cohomology set $H^1(R, \text{Aut}(\mathfrak{G}_0))$. This set encodes then the classification of reductive group schemes over R .

20. APPENDIX: SMOOTHNESS

We refer to [BLR, §2.2] and this is equivalent to the presentation given in §8.3. Let \mathfrak{X}/R be an affine scheme and let $x \in \mathfrak{X}$.

We say that \mathfrak{X} is smooth at x of relative dimension d if there exists a neighbourhood \mathfrak{U}/R of x and an R -immersion $j : \mathfrak{U} \rightarrow \mathbb{A}_R^n$ such that the following holds:

- (1) Locally at $y = j(x)$, the ideal defining \mathfrak{U} is generated by $n - d$ sections g_1, \dots, g_{n-d} ;
- (2) The differentials $dg_1(y), \dots, dg_{n-d}(y)$ are linearly independent in the $\kappa(y)$ -vector space $\Omega_{R[t_1, \dots, t_n]/R}^1 \otimes_{R[t_1, \dots, t_n]} k(y)$.

We say that \mathfrak{X}/R is étale at x if \mathfrak{X}/R is smooth at x of relative dimension zero. Smoothness (resp. étaleness) is an open condition on \mathfrak{X} and the relative rank is locally constant.

We say that \mathfrak{X}/R is smooth (resp. étale) if it is smooth (resp. étale) everywhere. Smoothness and étaleness are stable by composition and base change. Smoothness (resp. étaleness) can be characterized by a property of the functor of points.

20.0.1. Definition. *We say that an R -functor is formally smooth (resp. formally étale) if for each R -ring C equipped with an ideal J satisfying $J^2 = 0$, the map*

$$F(C) \rightarrow F(C/J)$$

is surjective (resp. bijective).

Note that this property implies that for each R -ring C equipped with a nilpotent ideal J then the map $F(C) \rightarrow F(C/J)$ is surjective (resp. bijective).

20.0.2. Theorem. *Let \mathfrak{X}/R be an affine scheme of finite presentation. Then \mathfrak{X}/R is smooth (resp. étale) if and only if the R -functor $h_{\mathfrak{X}}$ is formally smooth (resp. formally étale).*

Another important result is the following characterisation of open immersions.

20.0.3. Theorem. [EGA4, th. 17.9.1] *Let $f : \mathfrak{X}/R \rightarrow \mathfrak{Y}/R$ be a morphism of affine R -schemes. The following are equivalent:*

- (1) *f is an open immersion;*
- (2) *f is a monomorphism flat of finite presentation.*

In particular, a smooth (and a fortiori étale) monomorphism is an open immersion.

A useful consequence is the following.

20.0.4. Corollary. [EGA4, th. 17.9.5] *Let $f : \mathfrak{X}/R \rightarrow \mathfrak{Y}/R$ be a morphism of finite presentation between affine R -schemes. Assume that \mathfrak{X}/R is flat. Then the following are equivalent:*

- (1) *f is an open immersion (resp. an isomorphism);*
- (2) *$f_s : \mathfrak{X} \times_R \kappa(s) \rightarrow \mathfrak{Y} \times_R \kappa(s)$ is an open immersion (resp. an isomorphism) for each $s \in \operatorname{Spec}(R)$.*

We recall also the following differential criterion.

20.0.5. Theorem. [EGA4, §17.11] *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a finitely presented R -morphism between two smooth affine R -schemes $\mathfrak{X}, \mathfrak{Y}$. Let $x \in \mathfrak{X}$ and put $y = f(x)$. Then the following statements are equivalent :*

- (1) *f is smooth (resp. étale) at x ;*
- (2) *The $\kappa(x)$ -map $f^* : (\Omega_{\mathfrak{Y}/R}^1 \otimes_{R[\mathfrak{Y}]} \kappa(y)) \otimes_{\kappa(y)} \kappa(x) \rightarrow \Omega_{\mathfrak{X}/R}^1 \otimes_{R[\mathfrak{X}]} \kappa(x)$ is injective (resp. bijective).*

If furthermore, $\kappa(x) = \kappa(y)$, it is also equivalent to

- (3) *The $\kappa(x)$ -map on tangent spaces $T(f) : T_{\mathfrak{X},x} \rightarrow T_{\mathfrak{Y},y}$ is surjective (resp. bijective).*

20.0.6. Corollary. *Let I be an ideal included in $\operatorname{rad}(R)$ (e.g. R is local and I is its maximal ideal). Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a finitely presented R -morphism between two smooth affine R -schemes $\mathfrak{X}, \mathfrak{Y}$. Then the following are equivalent:*

- (1) *f is smooth (resp. étale);*
- (2) *$f_{R/I} : \mathfrak{X} \times_R R/I \rightarrow \mathfrak{Y} \times_R R/I$ is smooth (resp. étale).*

Proof. The direct way follows of the base change property of smooth morphisms. Conversely, we assume that $f_{R/I}$ is smooth (resp. étale) and we consider the smooth (resp. étale) locus \mathfrak{U} of f . It is an open subscheme of \mathfrak{X} . According to the differential criterion, \mathfrak{U} contains then $\mathfrak{X} \times_R R/I$. We claim that $\mathfrak{U} = \mathfrak{X}$. Assume it is not the case. Since \mathfrak{X} is quasi-compact (see [EGA1, §2.1.3]), $\mathfrak{X} \setminus \mathfrak{U}$ contains a closed point x . It maps to a closed point of $\operatorname{Spec}(R)$, contradiction. \square

20.0.7. Corollary. *Let $f : \mathfrak{Y} \rightarrow \operatorname{Spec}(R)$ be an étale morphism where \mathfrak{Y} is an affine scheme. Let s be a section of f . Then f is an clopen immersion.*

In particular, the diagonal map $\mathfrak{Y} \xrightarrow{\Delta} \mathfrak{Y} \times_R \mathfrak{Y}$ is a clopen immersion.

The general underlying statement is [EGA4, cor. 17.9.4].

Proof. We put $\mathfrak{X} = \operatorname{Spec}(R)$ and see the section s as an R -map $\mathfrak{X} \rightarrow \mathfrak{Y}$ between two smooth R -schemes. By considering the cartesian square

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\Delta_{\mathfrak{Y}}} & \mathfrak{Y} \times_R \mathfrak{Y} \\ s \uparrow & & \uparrow id \otimes s \\ \mathfrak{X} & \xrightarrow{s \otimes id} & \mathfrak{Y} \times_R \mathfrak{X} \end{array}$$

we see that s is a closed immersion. It is then enough to show that s is étale. Let $x \in \mathfrak{X}$ with image $y = s(x)$. We have $\kappa(x) = \kappa(y)$ and the $\kappa(x)$ -map on tangent spaces $T(s)_x : T_{\mathfrak{X},x} \rightarrow T_{\mathfrak{Y},y}$ is a section of the bijective map $T(f)_y$, so s is bijective. Therefore s is étale and is an open immersion. \square

21. APPENDIX: UNIVERSAL COVER OF A CONNECTED RING

The interested reader could skip that appendix and read directly at the source [SGA1] or Szamuel's book [Sz]. There is no noetherianity assumption but the reader can assume for simplicity that R is noetherian.

A finite étale cover S/R is flat cover which is finite and étale. Since it is flat, it is projective and then locally free. We state the following nice characterization (not used in the sequel).

21.0.1. Proposition. [EGA4, 18.3.1] *Let S/R be a finite ring. Then S/R is étale if and only if S is a projective R -module and S is a projective $S \otimes_R S$ -module.*

A special case is that of a Galois cover with respect to a finite abstract group Γ . Recall that our convention (and that of [P]) is that a Galois cover is a étale Γ -algebra for a finite group Γ ¹³.

If Γ is a finite group, the pointed set $H^1(R, \Gamma)$ classifies Γ -torsors and then Galois covers of R with group Γ . If $u : \Gamma \rightarrow \Gamma'$ is a group homomorphism of finite groups, we have an induced map $u_* : H^1(R, \Gamma) \rightarrow H^1(R, \Gamma')$. Of course, this map has a description in terms of algebras and by composition, it is enough to deal with the case u surjective and u injective.

If u is surjective, we associate to a Γ -algebra S/R the Γ' -algebra $S^{\ker(u)}$.

If u is injective, we associate to a Γ -algebra S/R its induction

$$\mathrm{Ind}_{\Gamma}^{\Gamma'}(S)$$

which is defined exactly as in the field case [KMRT, 18.17]. In terms of R -modules, $\mathrm{Ind}_{\Gamma}^{\Gamma'}(S) \xrightarrow{\sim} S^{[\Gamma':\Gamma]}$ and

$$\mathrm{Ind}_{\Gamma}^{\Gamma'}(S) = \left\{ \alpha \in \mathrm{Map}(\Gamma', S) \mid \alpha(\gamma\gamma') = \gamma \circ \alpha(\gamma') \quad \forall \gamma, \gamma' \right\}.$$

21.1. Embedding étale covers into Galois covers. Our goal is to make the construction of the universal cover of the base ring R with respect to a F -point $f : R \rightarrow F$ where F is a separably closed field.

21.1.1. Lemma. *Let S/R be a finite étale cover of R . Assume that R is connected and denote by d the rank of S/R .*

(1) *Let $s : S \rightarrow R$ be a (ring) section of $R \rightarrow S$. Then $S = R \times S'$ where S'/R is a finite étale cover and s is the first projection.*

(2) *S has at least d connected components and those are finite étale covers.*

(3) *If S/R is Galois, so are its connected components.*

¹³Warning: in [SGA1] and [Sz], a Galois cover is by definition connected.

Proof. (1) This follows of Corollary 20.0.7.

(2) We reason by induction on the rank d of S/R , the rank 0 case being trivial. If S is connected, there is nothing to do. Else there exists an idempotent $e \in S \setminus \{0, 1\}$ such that $S = Se \times S(1 - e)$. Those are direct summands of S , so are f.g. projective or respective rank $r \geq 1$, $d - r \geq 1$. Then Se and $S(1 - e)$ are finite over R and are étale since $\text{Spec}(Se)$ (resp. $\text{Spec}(S(1 - e))$) is open in $\text{Spec}(S)$. By induction Se and $S(1 - e)$ have at least r and $d - r$ connected components. Thus S has at least d connected components.

Assume firstly that S is noetherian. It has finitely many connected components, hence reads $S = S_1 \times \cdots \times S_l$. Each S_i is locally free over R of rank d_i . Also S_i is étale over R , so $S_i \rightarrow R$ is finite étale and is surjective since R is connected.

The general case is a limit argument. We can write $R = \varinjlim_{\alpha \in I} R_\alpha$ and $S = S_{\alpha_0} \otimes_{R_{\alpha_0}} S$ with the R_α noetherians [St, 135.3]. We can assume furthermore that the R_α 's are connected, then each S_α has at least d connected components. It follows that the number of connected components of S is finite and the components are defined at finite stage.

(3) We assume furthermore than S/R is Galois for a group Γ . Then Γ permutes the connected components so that $S = S_1^{(\Gamma/\Gamma_1)}$ where Γ_1 is the stabilizer of S_1 . Then Γ_1 acts freely on S_1 and $R = S_1^{(\Gamma_1)}$. We conclude that S_1 is a Galois cover of group Γ_1 . \square

21.1.2. Proposition. *Assume that R is connected. Let S/R be a ring extension and d be a non-negative integer. Then the following are equivalent:*

- (i) S/R is finite étale of degree d ;
- (ii) There exists a finite étale connected cover T/R such that $S \otimes_R T \xrightarrow{\sim} T \times \cdots \times T$ (d times);
- (iii) There exists a flat cover R'/R such that $S \otimes_R R' \xrightarrow{\sim} R' \times \cdots \times R'$ (d times).

Proof. The implication (ii) \implies (iii) is obvious and the implication (iii) \implies (i) is faithfully flat descent. We assume then (i) namely S/R is finite étale of degree d . If $d = 0$, there is nothing to do. We assume then $d \geq 1$ and put $T_1 = S$. According to Lemma 21.1.1.(1), the codiagonal map $S \otimes_R T_1 \rightarrow S$ defines a decomposition

$$(21.1.3) \quad S \otimes_R T_1 = T_1 \times S_1$$

where S_1/T_1 is a finite étale cover of degree $d - 1$. By induction, there exists a finite étale cover T_d/T_1 such that

$$S_1 \otimes_R T_d \xrightarrow{\sim} T_d \times \cdots \times T_d \quad ((d - 1) \text{ times}).$$

Then T_d is finite étale over R and by reporting in the identity (21.1.3), we get $S \otimes_R T_d = T_d \times \cdots \times T_d$ (d times). We now take a connected component of T_d and we have $S \otimes_R T = T \times \cdots \times T$ (d times). \square

21.1.4. Remark. The proof of (i) \implies (ii) is a ring version of the construction of a splitting field for a separable polynomial.

21.1.5. Lemma. *We consider the flat sheaf in algebras $S \rightarrow S^d = F_d(S)$. Then the natural map $S_{d,R} \rightarrow \text{Aut}(F_d)$ of R -functors is an isomorphism.*

In other words, the flat sheaf $R\text{-Aut}(F_d)$ is representable by the constant group scheme S_d .

Proof. We can assume that R is noetherian and also connected. We have $S_d \subset \text{Aut}_R(R^d)$. An R -automorphism σ of R^d permutes the idempotent elements, so is given by a permutation. \square

According to Lemma 21.1.1.(3), the yoga of forms 15.2.1 yields

21.1.6. Corollary. *Let d be a positive integer. Then there is a one to one correspondence*

$$\left\{ \text{Étale algebras of degree } d \right\} \longleftrightarrow H^1(R, S_d).$$

In other words, if S is a finite R -étale algebra of degree d , we attach to it the S_d -torsor

$$T \mapsto E_S(T) = \text{Isom}_{T\text{-alg}}(T^d, S \otimes_R T).$$

S defines a flat sheaf in algebras $\mathfrak{W}(S)$ and one has a canonical isomorphism $E_S(F_d) \xrightarrow{\sim} \mathfrak{W}(S)$.

21.1.7. Proposition. (Serre, [Se1, §1.5]) *Let S/R be a finite étale cover.*

(1) *There exists an étale cover \tilde{S}/S such that \tilde{S}/R is a Galois cover.*

(2) *If S is connected, \tilde{S} can be chosen to be connected.*

Proof. (1) Up to localize, we can assume that S/R is locally free of constant rank, say $d \geq 1$. The idea of the proof is to do it in the split case R^d in an S_d -equivariant way and to twist that construction. The group S_d acts by permutation on the finite set $\Omega_d = \{1, \dots, d\} = S_d/S_{d-1}$. We have $R^d = R[S_d/S_{d-1}]$ and it embeds in the group algebra $R[S_d]$ by the norm map $N = N_{S_d/S_{d-1}} : R[S_d/S_{d-1}] \rightarrow R[S_d] \cong R^{(S_d)}$.

Since $R[S_d]$ is the split Galois R -algebra of group S_d , that makes the case of R^d . For the general case of S/R , we can twist this construction by the S_d -torsor above, this provides an embedding of S in a Galois S_d -algebra \tilde{S} . By descent, \tilde{S} is finite étale of degree $(d-1)!$ over S .

(2) This follows of Lemma 21.1.1.(3). \square

21.2. Construction of the universal cover. We assume that R is connected and equipped with a point $f : R \rightarrow F$ where F is a separably closed field. We consider the category \mathcal{C} of pointed connected Galois covers over R . The objects are Galois covers S/R equipped with a map $f_S : S \rightarrow F$ extending f . The morphisms are R -morphisms commuting with the maps f_S . More precisely, a map $h : S_1 \rightarrow S_2$ is an R -ring map such that $f_{S_1} = f_{S_2} \circ h$.

21.2.1. Lemma. *Let $(S_1, f_1), (S_2, f_2) \in \mathcal{C}$. Then $\text{Hom}_{\mathcal{C}}((S_1, f_1), (S_2, f_2))$ is empty or consists in one map which is a finite étale cover.*

Proof. Let $h : S_1 \rightarrow S_2$ be such a map. Proposition 21.1.2 provides a finite connected étale cover T which splits S_1 and S_2 . Hence $S_1 \otimes_R T \xrightarrow{\sim} T^{(d_1)}$ and $S_2 \otimes_R T \xrightarrow{\sim} T^{(d_2)}$ so that h_T is finite étale. By descent, h is finite étale. Since S_2 is connected, h is a finite étale cover.

Now, let h, h' be two such maps. We want to show that $h = h'$. The T -maps $h_T, h'_T : T^{(d_1)} \rightarrow T^{(d_2)}$ are given by matrices with entries 0 or 1. Since the two maps agree after tensoring by $T \otimes_R F$, we get that $h_T = h'_T$. By descent, we conclude that $h' = h$. \square

This permits to equip that category with the preorder $(S_2, f_2) \geq (S_1, f_1)$ if $\text{Hom}_{\mathcal{C}}((S_1, f_1), (S_2, f_2)) \neq \emptyset$. For each relation $(S_2, f_2) \geq (S_1, f_1)$, there is a unique map $h_{1,2} : S_1 \rightarrow S_2$.

This category is directed by the following construction. Given objects $(S_1, f_1), \dots, (S_n, f_n)$ of \mathcal{C} , the tensor product $T = S_1 \otimes_R S_2 \cdots \otimes_R S_n$ is a Galois R -algebra equipped with the codiagonal map $f_T : T \rightarrow F$. Then T splits as $T \xrightarrow{\sim} T_1 \times \dots \times T_r$ where the T_i are connected Galois R -algebras such that $f_T(T_j) = 0$ for $j = 2, \dots, r$. It follows that $(T_1, f_{T_1}) \geq (S_i, f_i)$ for $i = 1, \dots, n$ and this is the upper bound.

We can define then the simply connected étale cover of (R, f) by

$$(21.2.2) \quad R^{sc} = \varinjlim_{(S,f)} S.$$

It comes equipped with a map $f^{sc} : R^{sc} \rightarrow F$ and we define the fundamental group

$$\pi_1(R, f) = \varprojlim_{(S,f)} \text{Gal}(S/R).$$

It is a profinite group, it acts continuously on R^{sc} . For each object (S, f_S) we have then a natural map $S \rightarrow R^{sc}$ which is $\pi_1(R, f)$ -equivariant. Such a map is unique.

21.2.3. Remark. Since the transition maps $\text{Gal}(S_2/R) \rightarrow \text{Gal}(S_1/R)$ are onto, it follows that each map $\pi_1(R, f) \rightarrow \text{Gal}(S/R)$ is onto.

We record the following property of R^{sc} .

21.2.4. Proposition. *R^{sc} is connected and the finite étale covers of R^{sc} are split so that $\pi_1(R^{sc}, f^{sc}) = 1$.*

Proof. The fact R^{sc} is connected follows from the argument as in Lemma 21.1.1.(2). Let L/R^{sc} be a finite étale connected cover. According to [St, 135.3], it is defined at finite stage, that is there exists $S \in \mathcal{C}$ and a finite étale cover T/S such that $T \otimes_R R^{sc} = L$. Also T is connected. By Proposition 21.1.7, L has a Galois closure \tilde{T} which is connected and it enough to show that $\tilde{T} \otimes_R R^{sc}$ splits. We put $\Gamma = \text{Gal}(\tilde{T}/R)$. Since $\tilde{T} \otimes_S F \cong F[\Gamma]$, f extends to a map $\tilde{f} : \tilde{T} \rightarrow F$. It follows that \tilde{T} embeds in R^{sc} . Thus L/R^{sc} splits. \square

One can also give a universal property for connected étale covers, not only the Galois ones, see [Sz, th. 5.4.2]. For Galois covers with a given fixed group, we get

21.2.5. Theorem. *Let Γ be a finite abstract group. There is a natural bijection*

$$\text{Hom}_{\text{cont}}(\pi_1(R, f), \Gamma)/\Gamma \xrightarrow{\sim} H^1(R, \Gamma).$$

Proof. The pointed set $H^1(R, \Gamma)$ classifies Γ -torsors and equivalently Galois R -algebras of group Γ . Given such a Γ -étale algebra S , we decompose it as $A = S_1 \times \cdots \times S_r$ in connected components which are Galois over R . The subgroups $\Gamma_i = \text{Gal}(S_i/R)$'s are conjugated in Γ . Choose an extension $f_1 : S_1 \rightarrow F$ of f , the choice is up to Γ_1 -conjugacy. Then by construction, we have a surjective continuous map

$$\phi_1 : \pi_1(R, f) \rightarrow \Gamma_1.$$

Up to Γ -conjugacy, the composite map $\phi_1 : \pi_1(R, f) \rightarrow \Gamma_1 \subset \Gamma$ does not depend of the choices made. We have then defined a map

$$H^1(R, \Gamma) \rightarrow \text{Hom}_{\text{cont}}(\pi_1(R, f), \Gamma)/\Gamma.$$

Let us define the converse map. Let $\phi : \pi_1(R, f) \rightarrow \Gamma$ be a continuous homomorphism. Then it factorizes at finite level, that is there exists a Galois connected cover $S \subset R^{sc}$ and ϕ_0 such that

$$\phi : \pi_1(R, f) \rightarrow \text{Gal}(S/R) \xrightarrow{\phi_0} \Gamma.$$

Then we can attach to ϕ_0 the class of the Γ -cover $u_*(S)$. We let the reader to check that the two maps are inverse of each other. \square

We discuss quickly other functoriality properties of this construction. If $h : R_0 \rightarrow R$ is a morphism of rings, we put $f_0 = f \circ h : R_0 \rightarrow F$. Given a connected Galois cover S_0/R , we denote by S/R the component of $S_0 \otimes_{R_0} R$ on which $f_{S_0} \otimes f$ is not trivial. By passing to the limit, it yields a natural map

$$R_0^{sc} \rightarrow R^{sc}$$

and a continuous map $\pi_1(R, f) \rightarrow \pi_1(R, f_0)$. This last base change map is onto if and only if $S_0 \otimes_{R_0} R$ is connected for each connected Galois cover

S_0/R_0 . One important special case is when R/R_0 is a Galois cover of group Γ_0 . In that case, one has the fundamental exact sequence of Galois theory

$$(*) \quad 1 \rightarrow \pi_1(R, f) \rightarrow \pi_1(R_0, f_0) \rightarrow \Gamma_0 \rightarrow 1.$$

Also, in some sense, the group $\pi_1(R, f)$ does not depend of the choice of the base point f , but in a non canonical way see [Sz, prop. 5.5.1].

21.2.6. Remark. One needs to be careful with continuity issues. For example, we consider the profinite group $\mathcal{G} = \varprojlim_{n \geq 1} (\mathbb{F}_p)^n$ where the transition maps are the projections on the last coordinates. It is a \mathbb{F}_p -vector space. Since there are \mathfrak{F}_p linear forms mapping the vector $(\cdots, 1, 1, \cdots, 1, 1)$ to 1, \mathcal{G} has plenty of no continuous group homomorphisms onto \mathbb{F}_p .

21.3. Examples.

21.3.1. Case of a normal ring R . We assume that R is normal with fraction field K . In this case, it is convenient to take F as a separable closure of K . Each point $x \in \text{Spec}(R)$ of codimension 1 defines a discrete valuation v_x on K . We denote by \widehat{K}_x its completion.

Let L/K be a Galois subextension of F and put $\Gamma = \text{Gal}(L/K)$. For each x point of $\text{Spec}(R)$ of codimension 1, we have a decomposition

$$L \otimes_K \widehat{K}_x \xrightarrow{\sim} (\widehat{L}_x)^{(\Gamma/\Gamma_x)}$$

where $\widehat{L}_x/\widehat{K}_x$ is a Galois extension of local fields for a subgroup $\Gamma_x \subset \Gamma$. We say that L/K is unramified at x if the extension $\widehat{L}_x/\widehat{K}_x$ is unramified (that is an uniformizing parameter of \widehat{K}_x is an uniformizing parameter of \widehat{L}_x).

We say that L/K is unramified over R if it is unramified at each point of $\text{Spec}(R)^{(1)}$. In this case, it can be shown [Sz, 5.4.9], that the ring of integers R_L of R in L is a finite Galois extension of group Γ . Also R^{sc} is the inductive limit of those R_L , so that $\pi_1(R, f)$ is the maximal unramified quotient of $\pi_1(R, f)$ with respect to $\text{Spec}(R)$.

In particular if R' is a localisation of R , then $\pi_1(R', f)$ maps onto $\pi_1(R, f)$. This applies to the so-called Kummer covers.

21.3.1. Proposition. *Let $n \geq 1$ be an integer such that K contains a primitive root of unity.*

(1) *Let $a \in K^\times$ and assume that $K_a = K[T]/(T^n - a)$ is a field extension of K . Then K_a/K is unramified over R if and only if $\text{div}(u) \in n\text{Div}(R)$.*

(2) *The construction above induces a natural group isomorphism*

$$\ker\left(K^\times / (K^\times)^n \xrightarrow{\text{div}} \mathbb{Z}/n\mathbb{Z}\right) \xrightarrow{\sim} H^1(R, \mathbb{Z}/n\mathbb{Z}).$$

Proof. (1) The field extension K_a/K is a cyclic extension of degree n . For each $x \in \text{Spec}(R)^{(1)}$, we can write $a = a_x^{m_x}$ with $m_x \mid n$ such that

$$K_a \otimes_R \widehat{K}_x \xrightarrow{\sim} \widehat{K}_x(\sqrt[n_x]{a_x}) \times \cdots \times \widehat{K}_x(\sqrt[n_x]{a_x}) \quad (m_x \text{ times})$$

where $n_x = \frac{n}{m_x}$. If $\text{div}(a) = (v_x(a)) \in n\text{Div}(R)$, then we can replace write $a = b_x^n (u_x)^{m_x}$ with $u_x \in \widehat{R}_x^\times$, so that $\widetilde{K}_x(\sqrt[n_x]{a_x})/\widehat{K}_x = \widetilde{K}_x(\sqrt[n_x]{u_x})$ is an unramified extension. In this case, K_a/K is unramified over R .

Conversely, we assume that K_a/K is unramified over R . Then $a_x \in \widehat{R}_x^\times \cdot (\widehat{K}_x)^{n_x}$ so that $a = a_x^{m_x} \in \widehat{R}_x^\times \cdot (\widehat{K}_x)^n$.

(2) To be written. □

21.3.2. Affine line and affine spaces. Let k be a field of characteristic zero and let k_s be a Galois closure. Consider the point $0_s : \text{Spec}(k_s) \rightarrow \text{Spec}(k) \xrightarrow{s_0} \mathbb{A}_k^1$.

21.3.2. Proposition. *The map $s_{0,*} : \pi_1(\mathbb{A}_k^1, 0_s) \rightarrow \text{Gal}(k_s/k)$ is an isomorphism. In particular, if k is algebraically closed, \mathbb{A}_k^1 is simply connected.*

Proof. From the fundamental (split) sequence of Galois theory (*) above

$$1 \rightarrow \pi_1(\mathbb{A}_{k_s}^1, 0_s) \rightarrow \pi_1(\mathbb{A}_k^1, 0_s) \rightarrow \text{Gal}(k_s/k) \rightarrow 1$$

we can assume that k is algebraically closed. Let $f : X \rightarrow \mathbb{A}_k^1$ be a finite étale cover. By normalization, it extends to a map $\widetilde{f} : \widetilde{X} \rightarrow \mathbb{P}_k^1$ where \widetilde{X} is a smooth projective curve. Put $d = [k(X) : k(t)]$ and denote by g the genus of \widetilde{X} . Let p_1, \dots, p_r be the points of $\widetilde{X}(k)$ mapping to ∞ and denote by e_1, \dots, e_r their respective multiplicities. Then we have the formula $d = e_1 + e_2 + \dots + e_r$, in particular $r \leq d$. In the other hand, we have the Hurwitz formula [H, IV.2.4]

$$2g - 2 = d(-2) + (e_1 - 1) + \dots + (e_r - 1).$$

Hence

$$2g - 2 = -d - r < 0$$

so that $g = 0$ and $d = 1$. We conclude that f is an isomorphism. □

21.3.3. Remark. The characteristic zero assumption is essential here (this was used in Hurwitz formula). If k is of characteristic $p > 0$, the Artin-Schreier map

$$f : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1, \quad t \mapsto t^p - t$$

is a connected Galois cover of group $\mathbb{Z}/p\mathbb{Z}$.

By using §21.3.1, one can derive that $\pi_1(X \times \mathbb{A}_k^1, 0_s \times x_s) \rightarrow \pi_1(X, x_s)$ for each geometrically normal variety X/k . In particular, it follows by induction on n that $\pi_1(\mathbb{A}_k^n, 0_s) \xrightarrow{\sim} \text{Gal}(k_s/k)$ for all $n \geq 1$.

21.3.3. Split tori. For a split torus \mathbb{G}_m^n , one has a complete description of the covers.

21.3.4. Proposition. (1) $\pi_1(\mathbb{G}_m^n) \xrightarrow{\sim} \left(\varprojlim_m \mu_m(k_s) \right)^n \rtimes \text{Gal}(k_s/k)$.

(2) Let S be a connected finite étale cover of $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Let $L \subset S$ be the integral closure of k in S . Then there exists $g \in \mathrm{GL}_n(\mathbb{Z})$, $a_1, \dots, a_n \in L^\times$ and positive integers d_1, \dots, d_n such that $d_1 \mid d_2 \cdots \mid d_n$ and

$$S \otimes_{R_n} R_n^g \simeq_{R_n\text{-alg}} (R_n \otimes_k L) \left[\sqrt[d_1]{a_1 t_1}, \dots, \sqrt[d_n]{a_n t_n} \right].$$

In particular, S is k -isomorphic to $R_n \otimes_k L$ and $\mathrm{Pic}(S) = 0$.

By R_n^g/R_n we mean the map $g_* : R_n \rightarrow R_n$ arising from the left action of $\mathrm{GL}_n(\mathbb{Z})$ on R_n . For (1) (resp. (2)), see [GP2, 2.10] (resp. [GP3, §2.8]).

For $k = k_s$, the simply connected cover is then $\varinjlim_m k \left[t_1^{\pm \frac{1}{m}}, \dots, t_n^{\pm \frac{1}{m}} \right]$.

21.3.4. Semisimple algebraic groups. We assume that k is algebraically closed of characteristic zero. Let G'/k be a semisimple algebraic group. Denote by $f : G \rightarrow G'$ its simply connected cover defined by the theory of algebraic groups [Sp, 10.1.4]; for example SL_n is the simply connected cover of PGL_n . Recall that we require that if T is a maximal torus of G , then the cocharacter group \widehat{T}^0 is generated by the coroots of (G, T) (or equivalently that \widehat{T} is the weight lattice). The kernel $\mu = \ker(f)$ is a finite diagonalizable group.

21.3.5. Proposition. (1) $\pi_1(G, 1) = 1$, that is G is simply connected in the sense of [SGA1].

(2) $f : G \rightarrow G'$ is the universal cover of G , so that $\mu(k) \xrightarrow{\sim} \pi_1(G, 1)$.

Proof. (1) Let B and B^- be a pair of opposite Borel subgroups of G such that $B \cap B^- = T$. We denote by U and U^- their respective unipotent radicals. The idea is to use the big cell $V = U n_0 U T$ of G . This is an open subvariety of G so that the map

$$\pi_1(U n_0 U T, n_0) \rightarrow \pi_1(G, n_0)$$

is onto. But $V \xrightarrow{\sim} U \times U \times T$ and U is an affine space. From §21.3.1, it follows that $\pi_1(V, n_0) \cong \pi_1(T, 1)$. We have then shown that the map

$$\widehat{T}^0 \otimes_{\mathbb{Z}} \varprojlim_n \mu_n(k) \xrightarrow{\sim} \pi_1(T, 1) \rightarrow \pi_1(G, 1)$$

is surjective. In particular, $\pi_1(G, 1)$ is abelian so that all finite étale covers are Galois.

The SL_2 -case: In this case, $\pi_1(\mathrm{SL}_2)$ is procyclic. Let $p : X \rightarrow \mathrm{SL}_2$ be a finite Galois cover. Then p is Galois of group $\mathbb{Z}/n\mathbb{Z}$ and its restriction V is a connected Galois of group $\mathbb{Z}/n\mathbb{Z}$. With coordinates $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the big cell is $V = \{c \neq 0\} \cong \mathbb{A}^2 \times \mathbb{G}_m$. It follows that $k(X) = k(\mathrm{SL}_2)(\sqrt[n]{c})$. We consider the divisor of the rational function c on SL_2 .

21.3.6. Claim. (special elementary case of [BD, lemme 2.4]) We have $\mathrm{div}(c) = 1[B] \in \mathrm{Div}(\mathrm{SL}_2)$.

By Proposition 21.3.1.(1), the fact that the extension $k(\mathrm{SL}_2)(\sqrt[n]{c})/k$ is unramified over SL_2 implies that $n = 1$.

The general case. Since the coroots of (G, T) generate the cocharacter group \hat{T}^0 , it is enough to show the triviality of the map $(\alpha^\vee)_* : \varprojlim_m \mu_m(k) \rightarrow \pi_1(G, 1)$ for each root $\alpha \in \Phi(G, T)$. But $\alpha^\vee : \mathbb{G}_m \rightarrow G$ factorizes by $\alpha^\vee : \mathrm{SL}_2 \rightarrow G$, so $(\alpha^\vee)_* : \varprojlim_m \mu_m(k) \rightarrow \pi_1(G, 1)$ factorizes by $\pi_1(\mathrm{SL}_2, 1) = 1$. We conclude that $\pi_1(G, 1) = 1$.

(2) to be written. □

21.3.7. Remark. This statement is wrong in characteristic $p > 0$. We consider the Frobenius map $F : \mathrm{SL}_2 \rightarrow \mathrm{SL}_2$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^p & b^p \\ c^p & d^p \end{bmatrix}$. Generalizing the Artin-Schreier cover, the Lang isogeny

$$h : \mathrm{SL}_2 \rightarrow \mathrm{SL}_2, \quad g \mapsto g^{-1} F(g)$$

is a finite (connected) Galois cover of group $\mathrm{SL}_2(\mathbb{F}_p)$ (e.g. [G, §1]).

21.4. Local rings and henselizations.

21.4.1. Definition. A ring extension S/R is said *standard étale* if there are two polynomials $f, g \in R[X]$ such that

- (i) f is monic and its derived polynomial f' is invertible in $R[X]_g/f(X)$;
- (ii) $S \xrightarrow{\sim} R[X]_g/f(X)$.

The map $R \rightarrow S$ decomposes then as $R \rightarrow R[X]_g \rightarrow R[X]_g/f(X)$ and is then étale by definition. Locally, étale maps are standard [SGA1, §7].

21.4.2. Proposition. [St, 135.16] *Let $R \rightarrow S$ be an étale extension and let \mathfrak{Q} be prime ideal of S . Then there exists $g \notin \mathfrak{Q}$ such that S_g/R is standard étale.*

21.4.3. Definition. Let R be a local ring with maximal ideal \mathfrak{M} and residue field k . Then R is *henselian* if for each monic polynomial $P(X)$, each coprime factorization $\overline{P}(X) = \overline{P}_1(X) \cdots \overline{P}_r(X) \in k[X]$ can be lifted in a factorization $P(X) = P_1(X) \cdots P_r(X)$.

The ring R is strictly henselian if it is henselian and its residue field is separably closed.

There are several characterisations of Henselian rings.

21.4.4. Theorem. [Mi1, §I.4] *Let (R, \mathfrak{M}, k) a local ring. Then the following are equivalent:*

- (1) R is henselian;
- (2) Any finite R -algebra is a product of local rings;
- (3) For each étale map $f : \mathfrak{Y} \rightarrow \mathrm{Spec}(R)$ and each k -point $y \in \mathfrak{Y}$ then f admits a section mapping the closed point x of $\mathrm{Spec}(R)$ to y .

(4) For each smooth map $f : \mathfrak{Y} \rightarrow \operatorname{Spec}(R)$ and each k -point $y \in \mathfrak{Y}$ then f admits a section mapping the closed point x of $\operatorname{Spec}(R)$ to y .

The strict henselization R^{sh} of R is defined by taking a limit. The objects in the category \mathcal{C}^h are the essentially étale rings S/R . It means that there exists an étale ring S_{\sharp}/R and an ideal \mathfrak{Q} of S_{\sharp} such that $S = (S_{\sharp})_{\mathfrak{Q}}$ and such that the morphism $R \rightarrow S$ is local. Given two such objects S_1, S_2 , the morphisms are local R -ring morphisms $S_1 \rightarrow S_2$ and there exists at least one object. This defines a preorder and this category is directed [EGA4, 18.6.3]. This permits to take the inductive limit.

21.4.5. Proposition. R^{sh} is a strictly henselian local ring.

Proof. The ring R^{sh} is local by construction and the construction allows to lift separable field algebras $k[t]/f(t)$ so that the residue field of R^{sh} is a separable closure of k . We need to verify that R^{sh} satisfies the third property of the characterization 21.4.4. Consider an étale map $f : \mathfrak{Y} \rightarrow \operatorname{Spec}(R^{sh})$ and a k_s -point $y \in \mathfrak{Y}$ mapping to the closed point x of $\operatorname{Spec}(R^{sh})$. Let S be an affine neighborhood of y and denote by \mathfrak{M}_y the maximal ideal of regular functions vanishing at y . Then $R^{sh} \rightarrow S_{\mathfrak{M}_y}$ is an essentially étale morphism. Since this morphism is defined at finite level in the tower, it splits. \square

This construction satisfies an universal property, see [R, VIII]. Note also that there is natural map $R^{sc} \rightarrow R^{sh}$.

The construction of the henselization is similar, we require furthermore that k is the residue field of each S/R .

21.4.6. Theorem. [BLR, §2.4, cor. 9] R^h and R^{sh} are faithfully flat extensions of R .

We focus on the case of a normal ring.

21.4.7. Theorem. (Raynaud [R, XI.1] or [BLR, §2.3, prop. 11]) Let (R, \mathfrak{M}, k) be a normal local ring. We denote by K its fraction field, by K_s/K a separable closure and by $\mathcal{G} = \operatorname{Gal}(K_s/K)$. Denote by C the integral closure of R in K_s and let \mathfrak{Q} be a maximal ideal of C . We consider the two following subgroups of \mathcal{G}

Decomposition group $\mathcal{D} = \left\{ \sigma \in \mathcal{G} \mid \sigma(\mathfrak{Q}) = \mathfrak{Q} \right\}; B = C^{\mathcal{D}}.$

Inertia group $\mathcal{I} = \left\{ \sigma \in \mathcal{D} \mid \sigma_{\kappa(\mathfrak{Q})} = \operatorname{id}_{\kappa(\mathfrak{Q})} \right\}; B' = C^{\mathcal{I}}.$

Then $B_{\mathfrak{Q} \cap B}$ (resp. $B'_{\mathfrak{Q} \cap B'}$) is the henselization (resp. the strict henselization) of R .

Furthermore $B_{\mathfrak{Q} \cap B}$ (resp. $B'_{\mathfrak{Q} \cap B'}$) is a normal ring [BLR, §2.3, prop. 10]. If R is a DVR, the construction shows that $B_{\mathfrak{Q} \cap B}$ (resp. $B'_{\mathfrak{Q} \cap B'}$) is a limit of DVR's. So in the DVR case, we have $R^{sc} = R^{sh}$.

21.4.8. Remark. Assume that the R is an excellent DVR which is equivalent to require that the completed fraction field \widehat{K} is separable over K [EGA4, Err_IV.27]. Then R^h consists of the elements of \widehat{R} which are algebraic over R . There exist non excellent DVR , see [Ku, 11.40].

21.5. Artin's approximation theorem.

21.5.1. Theorem. [A] (see also [BLR, §3.16]) *Let R be a ring finitely generated over \mathbb{Z} or over a field. Let F be an R -functor in sets such that F is locally of finite presentation, that is commutes with filtered direct limits of R -rings. Let $x \in \text{Spec}(R)$ be a point, and denote by \mathfrak{P}_x the associated prime ideal. Let R_x^h the henselization of the local ring at R_x and let \widehat{R}_x be the completion of R_x . Then for each $n \geq 1$, we have*

$$\text{Im}\left(F(R_x^h) \rightarrow F(R_x^h/\mathfrak{P}_x^n R_x^h)\right) = \text{Im}\left(F(R_x^h) \rightarrow F(\widehat{R}_x/\mathfrak{P}_x^n \widehat{R}_x)\right).$$

21.5.2. Remark. If \mathfrak{X}/R is an affine scheme, the functor $h_{\mathfrak{X}}$ is of locally of finite presentation and only if \mathfrak{X} is of finite presentation over R . In this case, the statement is that $\mathfrak{X}(R_x^h)$ is dense in $\mathfrak{X}(\widehat{R}_x)$. This special case is actually everything since the general case follows from a formal argument.

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