INTRODUCTION TO REDUCTIVE GROUP SCHEMES OVER RINGS

In construction

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Contents

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1. Introduction

The theory of reductive group schemes is due to Demazure and Grothendieck and was achieved fifty years ago in the seminar SGA 3, see also Demazure's thesis [De]. Roughly speaking it is the theory of reductive groups in family focusing to subgroups and classification issues. It occurs in several areas: representation theory, model theory, automorphic forms, arithmetic groups and buildings, infinite dimensional Lie theory, ...

The story started as follows. Demazure asked Serre whether there is a good reason for the map $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/d\mathbb{Z})$ to be surjective for all $d > 0$. Serre answered it is a question for Grothendieck ... Grothendieck answered it is not the right question !

The right question was the development of a theory of reductive groups over schemes and especially the classification of the "split" ones. The general underlying statement is now that the specialization map $G(\mathbb{Z}) \to G(\mathbb{Z}/d\mathbb{Z})$ is onto for each semisimple group split (or Chevalley) simply connected scheme $G/\mathbb{Z}$. It is a special case of strong approximation.

Demazure-Grothendieck's theory assume known the theory of reductive groups over an algebraically closed field due mainly to C. Chevalley ([Ch], see also [Bo], [Sp]) and we will do the same. In the meantime, Borel-Tits achieved the theory of reductive groups over an arbitrary field [BT65] and Tits classified the semisimple groups [Ti1]. In the general setting, Borel-Tits theory extends to the case of a local base.

Let us warn the reader by pointing out that we do not plan to prove all hard theorems of the theory, for example the unicity and existence theorem of split reductive groups. Our purpose is more to take the user viewpoint by explaining how such results permit to analyse and classify algebraic structures.

It is not possible to enter into that theory without some background on affine group schemes and strong technical tools of algebraic geometry (descent, Grothendieck topologies,...). Up to improve afterwards certain results, half of the lectures avoid descent theory and general schemes.

The aim of the notes is to try to help people attending the lectures. It is very far to be self-contained and quotes a lot in several references starting with [SGA3], Demazure-Gabriel's book [DG], and also the material of the Luminy's summer school provided by Brochard [Br], Conrad [C] and Oesterlé [O].

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Affine group schemes I

We shall work over a base ring R (commutative and unital).

2. Sorites

2.1. **R-Functors.** We denote by $\mathcal{A}ff_R$ the category of affine R-schemes. We are interested in R–functors, i.e. covariant functors from $\mathcal{A}ff_R$ to the category of sets. If $\mathfrak{X}$ an R-scheme, it defines a covariant functor

$$h_{\mathfrak{X}} : \ \mathcal{A}ff_R \ \to \ Sets, \ \ S \mapsto \mathfrak{X}(S).$$

Given a map $f : \mathfrak{Y} \to \mathfrak{X}$ of R-schemes, there is a natural morphism of functors $f_* : h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ of R-functors.

We recall now Yoneda's lemma. Let F be an R-functor. If $\mathfrak{X} = \mathrm{Spec}(R[\mathfrak{X}])$ is an affine R–scheme and $\zeta \in F(R[\mathfrak{X}])$, we define a morphism of R-functors

$$\widetilde{\zeta} : h_{\mathfrak{X}} \to F$$

by $\widetilde{\zeta}(S) : h_{\mathfrak{X}}(S) = \mathrm{Hom}_R(R[\mathfrak{X}], S) \to F(S)$, $f \mapsto F(f)(\zeta)$.

Each morphism $\varphi : h_{\mathfrak{X}} \to F$ is of this shape for a unique $\zeta \in F(R[\mathfrak{X}])$: ζ is the image of $Id_{R[X]}$ by the mapping $\varphi : h_X(R[\mathfrak{X}]) \to F(R[\mathfrak{X}])$.

In particular, each morphism of functors $h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ is of the shape h_v for a unique R–morphism $v : \mathfrak{Y} \to \mathfrak{X}$.

An R-functor F is representable by an affine scheme if there exists an affine scheme $\mathfrak{X}$ and an isomorphism of functors $h_X \to F$. We say that $\mathfrak{X}$ represents F. The isomorphism $h_X \to F$ comes from an element $\zeta \in F(R[\mathfrak{X}])$ which is called the universal element of $F(R[\mathfrak{X}])$. The pair $(\mathfrak{X}, \zeta)$ satisfies the following universal property:

For each affine R-scheme $\mathfrak{T}$ and for each $\eta \in F(R[\mathfrak{T}])$, there exists a unique morphism $u : \mathfrak{T} \to \mathfrak{X}$ such that $F(u^*)(\zeta) = \eta$.

2.2. **Definition.** An affine R–group scheme $\mathfrak{G}$ is a group object in the category of affine R-schemes. It means that $\mathfrak{G}/R$ is an affine scheme equipped with a section $\epsilon : \mathrm{Spec}(R) \to \mathfrak{G}$, an inverse $\sigma : \mathfrak{G} \to \mathfrak{G}$ and a multiplication $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ such that the three following diagrams commute:

Associativity:

$$\begin{tikzcd}
(\mathfrak{G} \times_R \mathfrak{G}) \times_R \mathfrak{G} \arrow[r, "m \times id"] \arrow[d, "can"', "\cong"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[r, "m"] & \mathfrak{G} \\
\mathfrak{G} \times_R (\mathfrak{G} \times_R \mathfrak{G}) \arrow[r, "id \times m"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[ur, "m"'] &
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathrm{Spec}(R) \arrow[r, "id \times \epsilon"] \arrow[dr, "p_2"'] & \mathfrak{G} \times_R \mathfrak{G} \arrow[d, "m"] & \mathrm{Spec}(R) \times \mathfrak{G} \arrow[l, "\epsilon \times id"'] \arrow[dl, "p_1"] \\
& \mathfrak{G} &
\end{tikzcd}$$

Symetry:

$$\begin{tikzcd}
\mathfrak{G} \arrow[r, "id\times\sigma"] \arrow[d, "s_{\mathfrak{G}}"'] & \mathfrak{G}\times_R\mathfrak{G}. \arrow[d, "m"'] \\
\mathrm{Spec}(R) \arrow[r, "\epsilon"] & \mathfrak{G}
\end{tikzcd}$$

We say that $\mathfrak{G}$ is commutative if furthermore the following diagram commutes

$$\begin{tikzcd}
\mathfrak{G}\times_R\mathrm{Spec}(R) \arrow[r, "switch"] \arrow[d, "m"'] & \mathfrak{G}\times_R\mathfrak{G} \arrow[d, "m"'] \\
\mathfrak{G} \arrow[r, phantom, "="] & \mathfrak{G}.
\end{tikzcd}$$

Let $R[\mathfrak{G}]$ be the coordinate ring of $\mathfrak{G}$. We call $\epsilon^* : R[\mathfrak{G}] \to \mathfrak{G}$ the counit (augmentation), $\sigma^* : R[\mathfrak{G}] \to R[G]$ the coinverse (antipode), and denote by $\Delta = m^* : R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ the comultiplication. They satisfies the following rules:

Co-associativity:

$$\begin{tikzcd}
R[\mathfrak{G}] \arrow[r, "m^*"] \arrow[dr, "m^*"'] & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \arrow[r, "m^*\otimes id"] & (R[\mathfrak{G}]\otimes_R R[\mathfrak{G}])\otimes_R R[\mathfrak{G}] \\
 & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \arrow[r, "id\otimes m^*"] & R[\mathfrak{G}]\otimes_R (R[\mathfrak{G}]\otimes_R R[\mathfrak{G}]). \arrow[u, "can"', "\cong"]
\end{tikzcd}$$

Counit:

$$\begin{tikzcd}
R[\mathfrak{G}] \arrow[r, "id\otimes\epsilon^*"] & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] & \arrow[l, "\epsilon^*\times id"'] & R[\mathfrak{G}] \\
 & R[\mathfrak{G}] \arrow[ul, "id"] \arrow[u, "m^*"] \arrow[urr, "id"'] & &
\end{tikzcd}$$

Cosymmetry:

$$\begin{tikzcd}
R[\mathfrak{G}]\otimes R[\mathfrak{G}] \arrow[r, "\sigma^*\otimes id"] & R[\mathfrak{G}]. \\
R[\mathfrak{G}] \arrow[u, "m^*"] \arrow[r, "\epsilon^*"] & R. \arrow[u, "s^*_{\mathfrak{G}}"]
\end{tikzcd}$$

In other words, $(R[\mathfrak{G}], m^*, \sigma^*, \epsilon^*)$ is a commutative Hopf R–algebra[1]. Given an affine R–scheme $\mathfrak{X}$, there is then a one to one correspondence between group structures on $\mathfrak{X}$ and Hopf R–algebra structures on $R[\mathfrak{X}]$.

If $\mathfrak{G}/R$ is an affine R–group scheme, then for each R–algebra S the abtract group $\mathfrak{G}(S)$ is equipped with a natural group structure. The multiplication is $m(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \to \mathfrak{G}(S)$, the unit element is $1_S = (\epsilon \times_R S) \in \mathfrak{G}(S)$ and the inverse is $\sigma(S) : \mathfrak{G}(S) \to \mathfrak{G}(S)$. It means that the functor $h_{\mathfrak{G}}$ is actually a group functor.

[1]This is Waterhouse definition [Wa, §I.4], other people talk abour cocommutative coassociative Hopf algebra.

2.2.1. **Lemma.** *Let $\mathfrak{X}/R$ be an affine scheme. Then the Yoneda lemma induces a one to one correspondence between group structures on $\mathfrak{X}$ and group structures on $h_{\mathfrak{X}}$.*

In other words, defining a group law on $\mathfrak{X}$ is the same that do define compatible group laws on each $\mathfrak{G}(S)$ for S running over the R-algebras.

2.2.2. **Remark.** We shall encounter certain non-affine group R-schemes. A group scheme $\mathfrak{G}/R$ is a group object in the category of R-schemes.

3. Examples

3.1. **Constant group schemes.** Let Γ be an abstract group. We consider the R–scheme $\mathfrak{G} = \bigsqcup_{\gamma\in\Gamma} \mathrm{Spec}(R)$. Then the group structure on Γ induces a group scheme structure on $\mathfrak{G}$ with multiplication

$$\mathfrak{G} \times_R \mathfrak{G} = \bigsqcup_{(\gamma,\gamma')\in\Gamma^2} \mathrm{Spec}(R) \to \mathfrak{G} = \bigsqcup_{\gamma\in\Gamma} \mathrm{Spec}(R)$$

applying the component (γ, γ') to $\gamma\gamma'$; This group scheme is affine iff Γ is finite.

There usual notation for such an object is Γ_R. This group scheme occurs as solution of the following universal problem.

3.2. **Vector groups.** Let N be an R–module. We consider the commutative group functors

$$V_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto \mathrm{Hom}_S(N \otimes_R S, S) = (N \otimes_R S)^\vee,$$
$$W_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto N \otimes_R S.$$

3.2.1. **Lemma.** *The R–group functor V_N is representable by the affine R–scheme $\mathfrak{V}(N) = \mathrm{Spec}(S^*(N))$ which is then a commutative R–group scheme. Furthermore N is of finite presentation if and only if $\mathfrak{V}(N)$ is of finite presentation.*

Proof. It follows readily of the universal property of the symmetric algebra

$$\mathrm{Hom}_{R'-mod}(N \otimes_R R', R') \xleftarrow{\sim} \mathrm{Hom}_{R-mod}(N, R') \xrightarrow{\sim} \mathrm{Hom}_{R-alg}(S^*(N), R')$$

for each R-algebra R'. □

The commutative group scheme $\mathfrak{V}(N)$ is called the vector group-scheme associated to N. We note that $N = \mathfrak{V}(N)(R)$.

Its group law on the R–group scheme $\mathfrak{V}(N)$ is given by $m^* : S^*(N) \to S^*(N) \otimes_R S^*(N)$, applying each $X \in N$ to $X \otimes 1 + 1 \otimes X$. The counit is $\sigma^* : S^*(N) \to S^*(N)$, $X \mapsto -X$.

3.2.2. **Remarks.** (1) If $N = R$, we get the affine line over R. Given a map $f : N \to N'$ of R–modules, there is a natural map $f^* : \mathfrak{V}(N') \to \mathfrak{V}(N)$.

(2) If N is projective and finitely generated, we have $W(N) = V(N^\vee)$ so that $\mathfrak{W}(N)$ is representable by an affine group scheme.

(3) If R is noetherian, Nitsure showed the converse holds [Ni04]. If N is finitely generated projective, then $\mathfrak{W}(N)$ is representable iff N is locally free.

3.2.3. **Lemma.** *The construction of (1) provides an antiequivalence of categories between the category of R-modules and that of vector group R-schemes.*

3.3. **Group of invertible elements, linear groups.** Let A/R be an algebra (unital, associative). We consider the R-functor

$$S \mapsto \mathrm{GL}_1(A)(S) = (A \otimes_R S)^\times.$$

3.3.1. **Lemma.** *If A/R is finitely generated projective, then* $\mathrm{GL}_1(A)$ *is representable by an affine group scheme. Furthermore,* $\mathrm{GL}_1(A)$ *is of finite presentation.*

Proof. We shall use the norm map $N : A \to R$ defined by $a \mapsto \det(L_a)$ constructed by glueing. We have $A^\times = N^{-1}(R^\times)$ since the inverse of L_a can be written L_b by using the characteristic polynomial of L_a. The same is true after tensoring by S, so that

$$\mathrm{GL}_1(A)(S) = \Big\{ a \in (A \otimes_R S) = \mathfrak{W}(A)(S) \quad | \quad N(a) \in R^\times \Big\}.$$

We conclude that $\mathrm{GL}_1(A)$ is representable by the fibered product

$$\begin{tikzcd} \mathfrak{G} \arrow[r] \arrow[d] & \mathfrak{W}(A) \arrow[d, "N"'] \\ \mathfrak{G}_{m,R} \arrow[r] & \mathfrak{W}(R). \end{tikzcd}$$

□

Given an R–module N, we consider the R–group functor

$$S \mapsto \mathrm{GL}_1(N)(S) = \mathrm{Aut}_{S-mod}(N \otimes_R S).$$

So if N is finitely generated projective. then $\mathrm{GL}_1(N)$ is representable by an affine R–group scheme. Furthermore $\mathrm{GL}_1(N)$ is of finite presentation.

3.3.2. **Remark.** If R is noetherian, Nitsure has proven that $\mathrm{GL}_1(N)$ is representable if and only if N is projective [Ni04].

3.4. **Diagonalizable group schemes.** Let A be a commutative abelian (abstract) group. We denote by $R[A]$ the group R–algebra of A. As R-module, we have

$$R[A] = \bigoplus_{a \in A} R\, e_a$$

and the multiplication is given by $e_a\, e_b = e_{a+b}$ for all $a, b \in A$.

For $A = \mathbb{Z}$, $R[\mathbb{Z}] = R[T, T^{-1}]$ is the Laurent polynomial ring over R. We have an isomorphism $R[A] \otimes_R R[B] \xrightarrow{\sim} R[A \times B]$. The R-algebra $R[A]$ carries the following Hopf algebra structure:

Comultiplication: $\Delta : R[A] \to R[A] \otimes R[A]$, $\Delta(e_a) = e_a \otimes e_a$, *Antipode:* $\sigma^* : R[A] \to R[A]$, $\sigma^*(e_a) = e_{-a}$;

Augmentation: $\epsilon^* : R[A] \to R$, $\epsilon(e_a) = 1$.

3.4.1. **Definition.** *We denote by* $\mathfrak{D}(A)/R$ *(or* $\widehat{A}$*) the affine commutative group scheme* $\mathrm{Spec}(R[A])$*. It is called the diagonalizable* R*–group scheme of base* A*. An affine* R*–group scheme is diagonalizable if it is isomorphic to some* $\mathfrak{D}(B)$*.*

We denote by $\mathfrak{G}_m = \mathfrak{D}(\mathbb{Z}) = \mathrm{Spec}(R[T, T^{-1}])$, it is called the multiplicative group scheme. We note also that there is a natural group scheme isomorphism $\mathfrak{D}(A \oplus B) \xrightarrow{\sim} \mathfrak{D}(A) \times_R \mathfrak{D}(B)$. We let in exercise the following fact.

3.4.2. **Lemma.** *The following are equivalent:*

(i) A *is finitely genereated;*

(ii) $\mathfrak{D}(A)/R$ *is of finite presentation;*

(iii) $\mathfrak{D}(A)/R$ *is of finite type.*

If $f : B \to A$ is a morphism of abelian groups, it induces a group homomorphism $f^* : \mathfrak{D}(A) \to \mathfrak{D}(B)$. In particular, when taking $B = \mathbb{Z}$, we have a natural mapping

$$\eta_A : A \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{G}_m).$$

3.4.3. **Lemma.** *If* R *is connected,* η_A *is bijective.*

Proof. Let $f : \mathfrak{D}(A) \to \mathfrak{G}_m$ be a group R–morphisms. Equivalently it is given by the map $f^* : R[T, T^{-1}] \to R[A]$ of Hopf algebra. In other words, it is determined by the function $X = f(T) \in R[A]^\times$ satisfying $\Delta(X) = X \otimes X$. Writing $X = \sum_{a \in A} r_a e_a$, the relation reads as follows $r_a r_b = 0$ if $a \neq b$ and $r_a \, r_a = r_a$. Since the ring is connected, 0 and 1 are the only idempotents so that $r_a = 0$ or $r_a = 1$. Then there exists a unique a such that $r_a = 1$ and $r_b = 0$ for $b \neq a$. This shows that the map η_A is surjective. It is obviously injective so we conclude that η_A is bijective. □

3.4.4. **Proposition.** *Assume that* R *is connected. The above construction induces an anti-equivalence of categories between the category of abelian groups and that of diagonalizable* R*–group schemes.*

Proof. It is enough to contruct the inverse map $\mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{D}(B)) \to \mathrm{Hom}(A, B)$ for abelian groups A, B. We are given a group homomorphism $f : \mathfrak{D}(A) \to \mathfrak{D}(B)$. It induces a map

$$f^* : \mathrm{Hom}_{R-gp}(\mathfrak{D}(B), \mathfrak{G}_m) \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{G}_m),$$

hence a map $B \to A$. □

4. Sequences of group functors

4.1. **Exactness.** We say that a sequence of R–group functors

$$1 \to F_1 \overset{u}{\to} F_2 \overset{v}{\to} F_3 \to 1$$

is exact if for each R–algebra S, the sequence of abstract groups

$$1 \to F_1(S) \overset{u(S)}{\to} F_2(S) \overset{v(S)}{\to} F_3(S) \to 1$$

is exact.

If $w : F \to F'$ is a map of R–group functors, we denote by $\ker(w)$ the R–group functor defined by $\ker(w)(S) = \ker(F(S) \to F'(S))$ for each R–algebra S.

If $w(S)$ is onto for each R–algebra S/R, it follows that

$$1 \to \ker(w) \to F \overset{w}{\to} F' \to 1$$

is an exact sequence of R–group functors.

4.1.1. **Lemma.** *Let $f : \mathfrak{G} \to \mathfrak{G}'$ be a morphism of R–group schemes. Then the R–functor* $\ker(f)$ *is representable by a closed subgroup scheme of $\mathfrak{G}$.*

Proof. Indeed the carthesian product

$$\begin{tikzcd} \mathfrak{N} \arrow[r] \arrow[d] & \mathfrak{G} \arrow[d, "f"'] \\ \mathrm{Spec}(R) \arrow[r, "\epsilon'"] & \mathfrak{G}' \end{tikzcd}$$

does the job. □

We can define also the cokernel of an R–group functor. But it is very rarely representable. The simplest example is the Kummer morphism $f_n : \mathfrak{G}_{m,R} \to \mathfrak{G}_{m,R}$, $x \mapsto x^n$ for $n \geq 2$ for $R = \mathbb{C}$, the field of complex numbers. Assume that there exists an affine $\mathbb{C}$–group scheme $\mathfrak{G}$ such that there is a four terms exact sequence of $\mathbb{C}$–functors

$$1 \to h_{\mu_n} \to h_{\mathfrak{G}_m} \overset{h_{f_n}}{\to} h_{\mathfrak{G}_m} \to h_{\mathfrak{G}} \to 1$$

We denote by T' the parameter for the first $\mathfrak{G}_m$ and by $T = (T')^n$ the parameter of the second one. Then $T \in \mathfrak{G}_m(R[T, T^{-1}])$ defines a non trivial element of $\mathfrak{G}(R[T, T^{-1}])$ which is trivial in $\mathfrak{G}(R[T', {T'}^{-1}])$ It is a contradiction.

4.2. **Semi-direct product.** Let $\mathfrak{G}/R$ be an affine group scheme acting on another affine group scheme $\mathfrak{H}/R$, that is we are given a morphism of R–functors

$$\theta : h_{\mathfrak{G}} \to \mathrm{Aut}(h_{\mathfrak{H}}).$$

The semi-direct product $h_{\mathfrak{H}} \rtimes^{\theta} h_{\mathfrak{G}}$ is well defined as R–functor.

4.2.1. **Lemma.** *$h_{\mathfrak{Y}} \rtimes^{\theta} h_{\mathfrak{X}}$ is representable by an affine R-scheme.*

Proof. We consider the affine R-scheme $\mathfrak{X} = \mathfrak{H} \times_R \mathfrak{G}$. Then $h_{\mathfrak{X}} = h_{\mathfrak{H}} \rtimes^{\theta} h_{\mathfrak{G}}$ has a group structure so defines a group scheme structure on $\mathfrak{X}$. $\square$

4.3. **Monomorphisms of group schemes.** A morphism of R–functors $f : F \to F'$ is a monomorphism if $f(S) : F(S) \to F'(S)$ is injective for each R–algebra S/R. We say that a morphism $f : \mathfrak{G} \to \mathfrak{H}$ of affine R-group schemes is a monomorphism if h_f is a monomorphism. It is a monomorphism iff the kernel R-group scheme $\ker(f)$ is the trivial group scheme.

Over a field F, we know that a monomorphism of algebraic groups is a closed immersion [SGA3, VI_B.1.4.2].

Over a DVR, it is not true in general that an open immersion (and a fortiori a monomorphism) of group schemes of finite type is a closed immersion. We consider the following example [SGA3, VIII.7]. Assume that R is a DVR and consider the constant group scheme $\mathfrak{H} = (\mathbb{Z}/2\mathbb{Z})_R$. Now let $\mathfrak{G}$ be the open subgroup scheme of $\mathfrak{H}$ which is the complement of the closed point 1 in the closed fiber. By construction $\mathfrak{G}$ is dense in $\mathfrak{H}$, so that the immersion $\mathfrak{G} \to \mathfrak{H}$ is not closed.

However diagonalizable groups have a wonderful behaviour with that respect.

4.3.1. **Proposition.** *Let $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ be a group homomorphism of diagonalizable R–group schemes. Then the following are equivalent:*

(i) $f^* : A \to B$ is onto;

(ii) f is a closed immersion;

(iii) f is a monomorphism.

Proof. $(i) \Longrightarrow (ii)$: Then $R[\mathfrak{B}]$ is a quotient of $R[A]$ so that $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ is a closed immersion.

$(ii) \Longrightarrow (iii)$: obvious.

$(iii) \Longrightarrow (i)$: We denote by $B_0 \subset B$ the image of f^*. The compositum of monomorphisms

$$\mathfrak{D}(B/B_0) \to \mathfrak{D}(B) \to \mathfrak{D}(B_0) \to \mathfrak{D}(A)$$

is a monomorphism and is zero. It follows that $\mathfrak{D}(B/B_0) = \mathrm{Spec}(R)$ and we conclude that $B_0 = B$. $\square$

Of the same flavour, the kernel of a map $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ is isomorphic to $\mathfrak{D}(f(A))$. The case of vector groups is more subtle.

4.3.2. **Proposition.** *Let $f : N_1 \to N_2$ be a morphism of finitely generated projective R-modules. Then the morphism of functors $f_* : W(N_1) \to W(N_2)$ is a mononomorphism if and only if f identifies locally N_1 as a direct summand of N_2. If it the case, $f_* : \mathfrak{W}(N_1) \to \mathfrak{W}(N_2)$ is a closed immersion.*

Proof. We can assume that R is local with maximal ideal M, so that N_1 and N_2 are free. If f identifies locally N_1 as a direct summand of N_2, then f_* identifies locally $W(N_1)$ as a direct summand of $W(N_2)$ hence f is a monomorphism. Conversely suppose that f_* is a monomorphism. Then the map $f_*(R/M) : N_1 \otimes_R R/M \to N_2 \otimes_R R/M$ is injective and there exists an R/M-base $(\overline{w}_1, ..., \overline{w}_r, \overline{w}_{r+1}, \ldots, \overline{w}_n)$ of $N_2 \otimes_R R/M$ such that $(\overline{w}_1, ..., \overline{w}_r)$ is a base of $f(N_1 \otimes_R R/M)$. We have $\overline{w}_i = f(\overline{v}_i)$ for $i = 1, .., r$. We lift the $\overline{v}_i$ in an arbitrary way in N_1 and the $\overline{w}_{r+1}, \ldots, \overline{w}_n$ in N_2. Then $(v_1, ..., v_r)$ is an R-base of N_1 and $(f(v_1), \ldots, f(v_r), w_{r+1}, \ldots, w_n)$ is an R–base of N_2. We conclude that f identifies N_1 as a direct summand of N_2. □

This shows that an exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of f.g. modules with N_1, N_2 projective induces an exact sequence of R–functors

$$0 \to W(N_1) \to W(N_2) \to W(N_3) \to 0$$

if and only if the starting sequence splits locally.

Affine group schemes II

5. Flatness

5.1. **The DVR case.** Assume that R is a DVR with uniformizing parameter π and denote by K its field of fractions. We know that an affine scheme $\mathfrak{X}/R$ is flat iff it is torsion free [2]; it rephrases here to require that $R[\mathfrak{X}]$ embeds in $K[\mathfrak{X}]$ [Li, cor. 2.14]. If $\mathfrak{X}/R$ is flat, there is a correspondence between the flat closed R-subschemes of $\mathfrak{X}$ and the closed K–subschemes of the generic fiber $\mathfrak{X}_K$ [EGA4, 2.8.1]. In one way we take the generic fiber and in the way around we take the schematic adherence. Let us explain the construction in terms of rings. If Y/K is a closed K–subscheme of X/K, it is defined by an ideal $I(Y) = \mathrm{Ker}(K[\mathfrak{X}] \to K[Y])$ of $K[\mathfrak{X}]$. The the schematic closure $\mathfrak{Y}$ of Y in $\mathfrak{X}$ is defined by the ideal $I(\mathfrak{Y}) = I \cap R[\mathfrak{X}]$. Since $I(\mathfrak{Y}) \otimes_R K = I(Y)$, we have $R(\mathfrak{Y}) \otimes_R K = K[Y]$, that is $\mathfrak{Y} \times_R K = Y_K$. Also the map $R[\mathfrak{Y}] \to K[Y]$ is injective, i.e $\mathfrak{Y}$ is a flat affine R–scheme.

This correspondence commutes with fibered products over R. In particular, if $\mathfrak{G}/R$ is a flat group scheme, it induces a one to one correspondence between flat closed R-subgroup schemes of $\mathfrak{G}$ and closed K–subgroup schemes of $\mathfrak{G}_K$[3].

We can consider the centralizer closed subgroup scheme of GL_2

$$\mathfrak{Z} = \Big\{ g \in \mathrm{GL}_{2,R} \quad | \quad g\,A = A\,g \Big\}$$

of the element $A = \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix}$. Then $\mathfrak{Z} \times_R R/\pi R \overset{\sim}{\longrightarrow} \mathrm{GL}_{2,R}$ and

$$\mathfrak{Z} \times_R K = \mathfrak{G}_{m,K} \times_K \mathfrak{G}_{a,K} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \right\}$$

Then the adherence of $\mathfrak{Z}_K$ in $\mathrm{GL}_{2,R}$ is $\mathfrak{G}_{m,R} \times_R \mathfrak{G}_{a,R}$, so does not contain the special fiber of $\mathfrak{Z}$. We conclude that $\mathfrak{Z}$ is not flat.

5.2. **A necessary condition.** In the above example, the geometrical fibers were of dimension 4 and 2 respectively. It illustrates then the following general result.

5.2.1. **Theorem.** [SGA3, VI_B.4.3] *Let $\mathfrak{G}/R$ be a flat group scheme of finite presentation. Then the dimension of the geometrical fibers is locally constant.*

---

[2] That is 0 is the only element annihilated by a regular element (non zero-divisor) of R.

[3] Warning: the fact that the schematic closure of a group scheme is a group scheme is specific to Dedekind rings.

5.3. **Examples.** Constant group schemes and diagonalizable groups schemes are flat. If N is a finitely generated projective R-module, the affine groups schemes $\mathfrak{V}(N)$ and $\mathfrak{W}(N)$ are flat.

5.3.1. **Remark.** What about the converse ? If N is of finite presentation, is is true that N is flat iff $\mathfrak{W}(N)$ is flat (?).

The group scheme of invertible elements of an algebra A/R f.g. projective is flat since it is open in $\mathfrak{V}(A)$.

6. Representations

Let $\mathfrak{G}/R$ be an a affine group scheme.

6.0.2. **Definition.** *A (left) $R-\mathfrak{G}$-module (or $\mathfrak{G}$-module for short) is an R–module M equipped with a morphism of group functors*

$$\rho : h_{\mathfrak{G}} \to \mathrm{Aut}(W(M)).$$

We say that the $\mathfrak{G}$–module M is faithful is ρ is a monomorphism.

It means that for each algebra S/R, we are given an action of $\mathfrak{G}(S)$ on $W(M)(S) = M \otimes_R S$. We use again Yoneda lemma. The mapping ρ is defined by the image of the universal point $\zeta \in \mathfrak{G}(R[\mathfrak{G}])$ provides an element called the coaction

$$c_\rho \in \mathrm{Hom}_R\Big(M, M \otimes_R R[\mathfrak{G}]\Big) \xrightarrow{\sim} \mathrm{Hom}_{R[\mathfrak{G}]}\Big(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]\Big).$$

We denote $\widetilde{c_\rho}$ its image in $\mathrm{Hom}_{R[\mathfrak{G}]}\Big(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]\Big)$.

6.0.3. **Proposition.** *(1) Both diagrams*

$$\begin{array}{ccc} M & \xrightarrow{c_\rho} & M \otimes_R R[\mathfrak{G}] \\ {\scriptstyle c_\rho}\downarrow & & \downarrow{\scriptstyle id\times\Delta_{\mathfrak{G}}} \\ M \otimes_R R[\mathfrak{G}] & \xrightarrow{c_\rho \times id} & M \otimes_R R[\mathfrak{G}] \otimes R[\mathfrak{G}], \end{array}$$

$$\begin{array}{ccc} M & \xrightarrow{c_\rho} & M \otimes_R R[\mathfrak{G}] \\ \downarrow & \swarrow{\scriptstyle id\times\epsilon^*} & \\ M & & \end{array}$$

commute.

(2) Conversely, if an R–map $c : M \to M \otimes_R R[\mathfrak{G}]$ satisfying the two rules above, there is a unique representation $\rho_c : h_{\mathfrak{G}} \to \mathrm{GL}(W(M))$ such that $c_{\rho_c} = c$.

A module M equipped with a R–map $c : M \to M \otimes_R R[\mathfrak{G}]$ satisfying the two rules above is called a $\mathfrak{G}$-module (and also a comodule over the Hopf algebra $R[\mathfrak{G}]$). The proposition shows that it is the same to talk about representations of $\mathfrak{G}$ or about $\mathfrak{G}$-modules.

In particular, the comultiplication $R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes R[\mathfrak{G}]$ defines a $\mathfrak{G}$-structure on the R–module $R[\mathfrak{G}]$. It is called the regular representation.

Proof. (1) We double the notation by putting $\mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}$. We consider the following commutative diagram

$$\begin{array}{ccc}
\mathfrak{G}(R[\mathfrak{G}_1]) \times \mathfrak{G}(R[\mathfrak{G}_2]) & \xrightarrow{\rho\times\rho} & \mathrm{Aut}(V(M))(R[\mathfrak{G}_1]) \times \mathrm{Aut}(V(M))(R[\mathfrak{G}_2]) \\
\downarrow & & \downarrow \\
\mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \times \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) & \xrightarrow{\rho\times\rho} & \mathrm{Aut}(V(M))(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \times \mathrm{Aut}(V(M))(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \\
{\scriptstyle m}\downarrow & & {\scriptstyle m}\downarrow \\
\mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) & \xrightarrow{\rho} & \mathrm{Aut}(W(M))(R[\mathfrak{G}_1 \times \mathfrak{G}_2])
\end{array}$$

and consider the image $\eta \in \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2])$ of the couple (ζ_1, ζ_2) of universal elements. Then η is defined by the ring homomorphism $\eta^* : R[G] \overset{\Delta_{\mathfrak{G}}}{\to} R[\mathfrak{G} \times \mathfrak{G}] \xrightarrow{\sim} R[\mathfrak{G}_1 \times \mathfrak{G}_2]$. It provides the commutative diagram

$$\begin{array}{ccc}
M \otimes R[\mathfrak{G}_1 \times \mathfrak{G}_2] & \xrightarrow{\widetilde{c}_{\rho(\eta)}} & M \otimes R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\
{\scriptstyle \widetilde{c}_{\rho,2}}\downarrow & & \nearrow{\scriptstyle \widetilde{c}_{\rho,1}} \\
M \otimes R[\mathfrak{G}_1 \times \mathfrak{G}_2] & &
\end{array}$$

But the map $M \to M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \overset{\widetilde{c}_{\rho(\eta)}}{\to} M \otimes R[\mathfrak{G}_1 \times \mathfrak{G}_2]$ is the compositum $M \overset{c}{\to} R[\mathfrak{G}] \overset{\Delta_{\mathfrak{G}}}{\to} M \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \cong M \otimes_R R[\mathfrak{G}_1] \otimes_R R[\mathfrak{G}_2]$. Hence we get the commutative square

$$\begin{array}{ccc}
M & \xrightarrow{c_\rho} & M \otimes R[\mathfrak{G}] \\
{\scriptstyle c_{\rho,1}}\downarrow & & {\scriptstyle id\times\Delta_{\mathfrak{G}}}\downarrow \\
M \otimes R[\mathfrak{G}_1] & \xrightarrow{c_{\rho,2}} & M \otimes R[\mathfrak{G}_1 \times \mathfrak{G}_2].
\end{array}$$

as desired. The other rule comes from the fact that $1 \in G(R)$ acts trivially on M.

(2) This follows again from Yoneda.

□

6.0.4. **Example.** The regular representation $R[\mathfrak{G}]$ is faithful. Let S be an R–ring and let $g \in \mathfrak{G}(S)$ acting trivially on $S[\mathfrak{G}]$. It means that $f \circ L_g = f$ for each $f \in S[\mathfrak{G}]$ hence $f(g) = f(1)$ for each $f \in S[\mathfrak{G}]$. But $\mathfrak{G}(S) = \mathrm{Hom}_S(S[\mathfrak{G}], S)$, so that $g = 1$. This shows that the regular representation is faithful.

A morphism of $\mathfrak{G}$-modules is an R–morphism $f : M \to M'$ such that $f(S) \circ \rho(g) = \rho'(g) \circ f(S) \in \mathrm{Hom}_S(M \otimes_R S, M' \otimes_R S)$ for each S/R. It is clear that the R–module $\mathrm{coker}(f)$ is equipped then with a natural structure of $\mathfrak{G}$-module. For the kernel $\ker(f)$, we cannot proceed similarly because the mapping $\ker(f)\otimes_R S \to \ker(M\otimes_R S \overset{f(S)}{\to} M'\otimes_R S)$ is not necessarily injective. One tries to use the module viewpoint by considering the following commutative exact diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker(f) & \longrightarrow & M & \xrightarrow{f} & M' \\
 & & & & \Big\downarrow c_\rho & & \Big\downarrow c_{\rho'} \\
 & & \ker(f)\otimes_R R[\mathfrak{G}] & \longrightarrow & M\otimes_R R[\mathfrak{G}] & \xrightarrow{f\otimes id} & M'\otimes_R R[\mathfrak{G}].
\end{array}$$

If $\mathfrak{G}$ is flat, then the left bottom map is injective, and the diagram defines a map $c:\ker(f)\to \ker(f)\otimes_R R[\mathfrak{G}]$. This map c satisfies the two compatibilities and define then a $\mathfrak{G}$-module structure on $\ker(f)$. We have proven the important fact.

6.0.5. **Proposition.** *Assume that $\mathfrak{G}/R$ is flat. Then the category of $\mathfrak{G}$-modules is an abelian category.*

6.1. **Representations of diagonalizable group schemes.** Let $\mathfrak{G}=\mathfrak{D}(A)/R$ be a diagonalizable group scheme. For each $a\in A$, we can attach a character $\chi_a=\eta_A(a):\mathfrak{D}(A)\to\mathfrak{G}_m=\mathrm{GL}_1(R)$. It defines then a $\mathfrak{G}$–structure of the R–module R. If $M=\oplus_{a\in A}M_a$ is a A-graded R–module, the group scheme $\mathfrak{D}(A)$ acts diagonally on it by χ_a on each piece M_a.

6.1.1. **Proposition.** *The category of A–graded R-modules is equivalent to the category of $R-\mathfrak{D}(A)$-modules.*

Proof. Let M be an $R-\mathfrak{D}(A)$-module and consider the underlying map $c:M\to M\otimes_R R[A]$. We write $c(m)=\sum\limits_{m\in M}\varphi_a(m)\otimes e_a$ and the first (resp. second) condition reads

$$\varphi_a\circ\varphi_b=\delta_{a,b}\varphi_a \quad (\text{resp. } \sum_{a\in A}\varphi_a=id_M).$$

Hence the φ_a's are pairwise orthogonal projectors whose sum is the identity. Thus $M=\bigoplus_{a\in A}\varphi_a(M)$ which decomposes a direct summand of eigenspaces. $\square$

6.1.2. **Corollary.** *Let $0\to M_1\to M_2\to M_3\to 0$ be an exact sequence of $R-\mathfrak{D}(A)$–modules.*

(1) For each $a\in A$, it induces an exact sequence $0\to(M_1)_a\to(M_2)_a\to(M_3)_a\to 0$.

(2) The sequence $0\to M_1\to M_2\to M_3\to 0$ splits as sequence of $R-\mathfrak{D}(A)$–modules if and only if it splits as sequence of R–modules.

Proof. (1) It follows from the proposition.

(2) The direct sense is obvious. Conversely, let $s : M_3 \to M_2$ be a splitting. Then for each $a \in A$, the composite $(M_3)_a \to M_3 \overset{s}{\to} M_2 \overset{\varphi_a}{\to} (M_2)_a$ provides the splitting of $(M_2)_a \to (M_3)_a$.

$\square$

We record also the following property.

6.1.3. **Corollary.** *Let M be an $R - \mathfrak{G}$-module. Then for each S/R and for each $a \in A$, we have $M_a \otimes_R S = (M_a \otimes_R S)_a$.*

It is also of interest to know kernels of representations.

6.1.4. **Lemma.** *Let $A^\sharp$ be a finite subset of A and denote by A_0 the subgroup generated by $A^\sharp$. We consider the representation $M = \oplus_{a \in A} R^{n_a}$ of $\mathfrak{G} = \mathfrak{D}(A)$, with $n_a \geq 1$. Then the representation $\rho : \mathfrak{G} \to \mathrm{GL}(M)$ factorizes as*

$$\mathfrak{G} = \mathfrak{D}(A) \to \mathfrak{D}(A_0) \overset{\rho_0}{\to} \mathrm{GL}(M)$$

where ρ_0 is a closed immersion. Furthermore $\ker(\rho) = \mathfrak{D}(A/A_0)$ is a closed subgroup scheme.

6.1.5. **Remark.** If R is a field, all finite dimensional representations are of this shape, so one knows the kernel of each representation.

Proof. Consider first the case $A = A_0$. Then the map $\mathfrak{G} \to \mathrm{GL}(M)$ factorizes by the closed subgroup scheme $T = \prod\limits_{a \in A^\sharp} \mathfrak{G}_{m,R}^{n_a}$. Since the map $\widehat{T} \to A$ is onto, the map $\mathfrak{G} \to \prod\limits_{a \in A^\sharp} \mathfrak{G}_{m,R}^{n_a}$ is a closed immersion (Proposition 4.3.1). A composite of closed immersion being a closed immersion, ρ is a closed immersion. The representation $\rho : \mathfrak{G} \to \mathrm{Aut}(\mathfrak{V}(M))$ factorizes as

$$\mathfrak{G} = \mathfrak{D}(A) \overset{q}{\to} \mathfrak{D}(A_0) \overset{\rho_0}{\to} \mathrm{Aut}(\mathfrak{V}(M))$$

where ρ_0 is a closed immersion. It follows that $\ker(\rho) = \ker(q)$. This kernel $\ker(q)$ is $\mathfrak{D}(A/A_0)$ and is a closed subgroup scheme of $\mathfrak{G}$ (*ibid*). $\square$

6.2. **Existence of faithful finite dimensional representations.** This question is rather delicate for general groups and general rings, see [SGA3, VI_B.13] and the paper [Th] by Thomason. Over a field or a Dedekind ring, faithful representations occur.

6.2.1. **Theorem.** *Assume that R is a Dedekind ring. Let $\mathfrak{G}/R$ be a flat affine group scheme of finite type. Then there exists a faithful $\mathfrak{G}$-module M which is f.g. free as R-module.*

The key thing is the following fact due to Serre [Se4, §1.5, prop. 2].

6.2.2. **Proposition.** *Assume that R is noetherian and let $\mathfrak{G}/R$ be an affine flat group scheme. Let M be a $\mathfrak{G}$-module. Let N be an R-submodule of M of finite type. Then there exists an $R - \mathfrak{G}$-submodule $\widetilde{N}$ of M which contains N and is f.g. as R-module.*

We can now proceed to the proof of Theorem 6.2.1. We take $M = R[\mathfrak{G}]$ seen as the regular representation $\mathfrak{G}$–module, it is faithful (Example 6.0.4). The proposition shows that M is the direct limit of the family of $\mathfrak{G}$-submodules $(M_i)_{i \in I}$ which are f.g. as R-modules. The M_i are torsion–free so are flats. Hence the M_i are projective.

We look at the kernel $\mathfrak{N}_i/R$ of the representation $\mathfrak{G} \to \mathrm{GL}(M_i)$. The regular representation is faithful and its kernel is the intersection of the $\mathfrak{N}_i$. Since $\mathfrak{G}$ is a noetherian scheme, there is an index i such that $\mathfrak{N}_i = 1$. In other words, the representation $\mathfrak{G} \to \mathrm{GL}(M_i)$ is faithful. Now M_i is a direct summand of a free module R^n, i.e. $R^n = M_i \oplus M_i'$. It provides a representation $\mathfrak{G} \to \mathrm{GL}(M_i) \to \mathrm{GL}(M_i \oplus M_i')$ which is faithful and such that the underlying module is free.

In the Dedekind ring case, Raynaud proved a stronger statement which was extended by Gabber in dimension two.

6.2.3. **Theorem.** *(Raynaud-Gabber* [SGA3, VI_B.13.2]*) Assume that R is a regular ring of dimension ≤ 2. Let $\mathfrak{G}/R$ be a flat affine group scheme of finite type. Then there exists a $\mathfrak{G}$-module M isomorphic to R^n as R–module such that $\rho_M : \mathfrak{G} \to \mathrm{GL}_n(R)$ is a closed immersion.*

In the Dedekind case, an alternative proof is §1.4.5 of [BT2].

6.3. **Hochschild cohomology.** We assume that $\mathfrak{G}$ is flat. If M is a $\mathfrak{G}$–module, we consider the R–module of invariants $M^{\mathfrak{G}}$ defined by

$$M^{\mathfrak{G}} = \Big\{ m \in M \ \mid m \otimes 1 = c(m) \in M \otimes_R R[\mathfrak{G}] \Big\}.$$

It is the largest trivial $\mathfrak{G}$-submodules of M and we have also $M^{\mathfrak{G}} = \mathrm{Hom}_{\mathfrak{G}}(R, M)$. We can then mimick the theory of cohomology of groups.

6.3.1. **Lemma.** *The category of $R - \mathfrak{G}$-modules has enough injective.*

We shall use the following extrem case of induction, see [J, §2, 3] for the general theory.

6.3.2. **Lemma.** *Let N be an R–module. Then for each $\mathfrak{G}$-module M the mapping*

$$\psi : \mathrm{Hom}_{\mathfrak{G}}(M, N \otimes_R R[\mathfrak{G}]) \to \mathrm{Hom}_R(M, N),$$

given by taking the composition with the augmentation map, is an isomorphism.

Proof. We define first the converse map. We are given an R-map $f_0 : M \to N$ and consider the following diagram

$$\begin{array}{ccccc}
M & \xrightarrow{c_M} & M \otimes_R R[\mathfrak{G}] & \xrightarrow{f_0 \otimes id} & N \otimes_R R[\mathfrak{G}] \\
{\scriptstyle c_M}\downarrow & & {\scriptstyle id \times \Delta_G}\downarrow & & {\scriptstyle id \times \Delta_G}\downarrow \\
M \otimes_R R[\mathfrak{G}] & \xrightarrow{c_M \otimes id} & M \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] & \xrightarrow{f_0 \otimes id} & N \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}].
\end{array}$$

The right square commutes obviously and the left square commutes since M is a $\mathfrak{G}$-module. It defines then a map $f : M \to N \otimes_R R[\mathfrak{G}]$ of $\mathfrak{G}$-modules. By construction we have $\psi(f) = f_0$. In the way around we are given a $\mathfrak{G}$–map $h : M \to N \otimes_R R[\mathfrak{G}]$ and denote by $h_0 : M \to N \otimes_R R[\mathfrak{G}] \to N \to 0$. We consider the following commutative diagram

$$\begin{tikzcd}
M \arrow[r, "h"] \arrow[d, "c_M"'] & N \otimes_R R[\mathfrak{G}] \arrow[d, "id\times\Delta_{\mathfrak{G}}"'] \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "h\otimes id"] \arrow[d, "\epsilon^*"'] & N \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[d, "id\times\epsilon^*"'] \\
M \arrow[r, "h_0\otimes id"] & N \otimes_R \otimes_R R[\mathfrak{G}].
\end{tikzcd}$$

The vertical maps are the identities so we conclude that $h = h_0 \otimes id$ as desired. $\square$

6.3.3. **Remark.** If M is a $\mathfrak{G}$-module we denote by M_{tr} the underlying R–module seen as trivial R–$\mathfrak{G}$–module. We claim that $M \otimes_R R[\mathfrak{G}]$ is isomorphic to $M_{tr} \otimes_R R[\mathfrak{G}]$. Indeed for each R-module M', we have an is isomorphism $\mathrm{Hom}_{\mathfrak{G}}(M', M \otimes_R R[\mathfrak{G}]) \xrightarrow{\sim} \mathrm{Hom}_R(M', M) = \mathrm{Hom}_R(M', M_{tr})$. Therefore $M \otimes_R R[\mathfrak{G}]$ satisfies the same universal property than $M_{tr} \otimes_R R[\mathfrak{G}]$ and we conclude that those two $\mathfrak{G}$–modules are isomorphic.

We can proceed to the proof of Lemma 6.3.1.

Proof. The argument is similar as Godement's one in the case of sheaves. Let M be a $\mathfrak{G}$–module and let us embed the R–module M in some injective module I. Then we consider the following injective $\mathfrak{G}$-map

$$M \overset{c_M}{\to} M \otimes_R R[\mathfrak{G}] \to I \otimes_R R[\mathfrak{G}] \xrightarrow{\sim} I_{tr} \otimes_R R[\mathfrak{G}]$$

where we use Remark 6.3.3. We claim that $I_{tr} \otimes_R R[\mathfrak{G}]$ is an injective $\mathfrak{G}$-module. We consider a diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "f"'] & N' \\
& I_{tr} \otimes_R R[\mathfrak{G}]. &
\end{tikzcd}$$

From Frobenius reciprocity, we have the following

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{G}}(N', I_{tr} \otimes_R R[\mathfrak{G}]) \arrow[r, "i^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathfrak{G}}(N, I_{tr} \otimes_R R[\mathfrak{G}]) \arrow[d, "\cong"'] \\
\mathrm{Hom}_R(N', I_{tr}) \arrow[r, "i^*"] & \mathrm{Hom}_{\mathfrak{G}}(N, I_{tr}).
\end{tikzcd}$$

Since I_{tr} is an injective R–module, the bottom map is onto. Thus f extends to a $\mathfrak{G}$-map $N' \to I_{tr} \otimes_R R[\mathfrak{G}]$. $\square$

We can then take the right derived functors of the left exact functor $R-\mathfrak{G}-mod \to R-Mod$, $M \to M^{\mathfrak{G}} = H^0_0(\mathfrak{G}, M)$. It defines the Hochschild cohomology groups $H^i_0(\mathfrak{G}, M)$. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of $\mathfrak{G}$–modules, we have the long exact sequence

$$\dots \to H^i_0(\mathfrak{G}, M_1) \to H^i_0(\mathfrak{G}, M_2) \to H^i_0(\mathfrak{G}, M_3) \xrightarrow{\delta_i} H^{i+1}_0(\mathfrak{G}, M_1) \to \dots$$

6.3.4. **Lemma.** *Let M be an $R[\mathfrak{G}]$-module. Then $M \otimes_R R[\mathfrak{G}]$ is acyclic, i.e. satisfies*

$$H^i_0(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0 \qquad \forall i \geq 1.$$

Proof. We embed the M in an injective R-module I and put $Q = I/M$. The sequence of **G**-modules

$$0 \to M \otimes_R R[\mathfrak{G}] \to I \otimes_R R[\mathfrak{G}] \to Q \otimes_R R[\mathfrak{G}] \to 0$$

is exact. We have seen that $I \otimes_R R[\mathfrak{G}]$ is injective, so that $H^i_0(\mathfrak{G}, I \otimes_R R[\mathfrak{G}]) = 0$ for each $i > 0$. The long exact sequence induces an exact sequence

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathfrak{G}}(R, I \otimes_R R[\mathfrak{G}]) & \longrightarrow & \mathrm{Hom}_{\mathfrak{G}}(R, Q \otimes_R R[\mathfrak{G}]) & \longrightarrow & H^1_0(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) \to 0 \\
\cong \big\downarrow & & \cong \big\downarrow & & \\
I & \longrightarrow & Q & \longrightarrow & 0.
\end{array}$$

Therefore $H^1_0(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0$. The isomorphisms $H^i_0(\mathfrak{G}, Q \otimes_R R[\mathfrak{G}]) \xrightarrow{\sim} H^{i+1}(\mathfrak{G}, M \otimes_R R[\mathfrak{G}])$ permits to use the standard shifting argument to conclude that $H^{i+1}(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0$ for each $i \geq 0$. $\square$

As in the usual group cohomology, these groups can be computed by the bar resolution

$$\dots \to R[\mathfrak{G}^3] \to R[\mathfrak{G}^2] \to R[\mathfrak{G}] \to R \to 0.$$

This provides a description of $H^i_0(\mathfrak{G}, M)$ in terms of cocycles, see [DG, II.3] for details. A n–cocycle (resp. a boundary) in this setting is the data of a n-cocycle $c(S) \in Z^n(\mathfrak{G}(S), M \otimes_R S)$ in the usual sense and which agree with base changes.

6.3.5. **Remark.** In particular, there is a natural map $Z^n(\mathfrak{G}, M) \to Z^n(\mathfrak{G}(R), M)$. If $\mathfrak{G} = \Gamma_R$ is finite constant, this map induces an isomophism $H^*_0(\Gamma, M) \xrightarrow{\sim} H^*_0(\Gamma, M)$ (see [DG, II.3.4]).

We can state an important vanishing statement.

6.3.6. **Theorem.** *Let $\mathfrak{G} = \mathfrak{D}(A)$ be a diagonalizable group scheme. Then for each $\mathfrak{G}$-module M, we have $H^i(\mathfrak{G}, M) = 0$ for each $i \geq 1$.*

Proof. According to Corollary 6.1.2, the $\mathfrak{G}$-module M is a direct summand of the flasque $\mathfrak{G}$–module $M \otimes_R R[\mathfrak{G}]$. Hence M is flasque as well. $\square$

6.4. **First Hochschild cohomology group.** We just focus on H^1 and H^2. Then

$$H^1_0(\mathfrak{G}, M) = Z^1_0(\mathfrak{G}, M)/B^1_0(\mathfrak{G}, M)$$

are given by equivalence of Hochshild 1-cocycles. More precisely, a 1–cocycle (or crossed homomorphism) is an R–functor

$$z : h_{\mathfrak{G}} \to W(M)$$

which satisfies the following rule for each algebra S/R

$$z(g_1g_2) = z(g_1) + g_1 \,.\, z(g_2) \quad \forall \ g_1, g_2 \in \mathfrak{G}(S).$$

The coboundaries are of the shape $g \,.\, m \otimes 1 \,-\, m \otimes 1$ for $m \in M$. As in the classical case, crossed homomorphisms arise from sections of the morphism of R–group functors $V(M) \rtimes \mathfrak{G} \to \mathfrak{G}$ and $H^1_0(\mathfrak{G}, M)$ is nothing but the set of M–conjugacy classes of those sections.

6.5. **H^2 and group extensions.** A 2-cocycle for $\mathfrak{G}$ and M is the data for each S/R of a 2-cocycle $f(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \to M \otimes_R S$ in a compatible way. It satisfies the rule

$$g_1 \,.\, f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) = 0$$

for each S/R and each $g_1, g_2, g_3 \in \mathfrak{G}(S)$. The 2-cocycle c is normalized if it satifies furthermore the rule

$$f(g, 1) = f(1, g) = 0.$$

each S/R and each $g \in \mathfrak{G}(S)$. Up to add a coboundary, we can always deal with normalized cocycles. The link in the usual theory between normalized classes and group extensions [We, §6.6] extends mecanically. Given a normalized Hochshild cocycle $c \in Z^2(\mathfrak{G}, M)$, we can define the following group law on the R–functor $V(M) \times \mathfrak{G}$ by

$$(m_1, g_1) \,.\, m_2, g_2) = \Big(m_1 + g_1 \,.\, m_2 + c(g_1, g_2),\, g_1\, g_2 \Big)$$

for each S/R and each $m \in M \otimes_R S$ and $g \in \mathfrak{G}(S)$. In other words, we defined a group extension E_f of R-functors in groups of $h_{\mathfrak{G}}$ by $V(M)$. Now we denote by $\mathrm{Ext}_{R-functor}(\mathfrak{G}, V(M))$ the abelian group of classes of extensions (equipped with the Baer sum) of R–group functors of $h_{\mathfrak{G}}$ by $V(M)$ with the given action $h_{\mathfrak{G}} \to \mathrm{GL}(V(M))$.

The O is the class of the semi-direct product $V(M) \rtimes h_{\mathfrak{G}}$. As in the classical case, it provides a nice description of the H^2.

6.5.1. **Theorem.** [DG, II.3.2] *The construction above induces a group isomorphism $H^2_0(\mathfrak{G}, M) \xrightarrow{\sim} \mathrm{Ext}_{R-functor}(\mathfrak{G}, V(M))$.*

As consequence of the vanishing theorem 6.3.6, we get the following

6.5.2. **Corollary.** *Let A be an abelian group and let M be a $\mathfrak{D}(A)$–module. Let $0 \to V(M) \to F \to \mathfrak{D}(A) \to 1$ be a group R-functor extension. Then F is the semi-direct product of $\mathfrak{D}(A)$ by $V(M)$ and all sections are M–conjugated.*

6.6. **Linearly reductive algebraic groups.** Let k be a field and let G/k be an affine algebraic group. Recall that a $k-G$–module V is simple if 0 and V are its only G-submodules. Note that simple $k-G$–module are finite dimensional according to Proposition 6.2.2. A $k-G$–module is semisimple if it is a direct summand of its simple submodule.

6.6.1. **Definition.** *The k–group $\mathfrak{G}$ is linearly reductive if each finite dimensional representation of $\mathfrak{G}$ is semisimple.*

We have seen that diagonalizable groups are linearly reductive. An important point is that this notion is stable by base change and is geometrical, namely G is linearly reductive iff $G \times_k \overline{k}$ is linearly reductive (see [Mg, prop. 3.2]). As in for the case of diagonalizable groups, we have the following vanishing statement.

6.6.2. **Theorem.** *Assume that the affine algebraic group G/k is linearly reductive. Then for each representation V of G, we have $H^i_0(G, V)$ for each $i > 0$.*

6.6.3. **Corollary.** *Each extension of group functors of G by a vector algebraic group splits. Furthermore $G(k)$ acts transitively on the sections.*

By using a similar method (involving sheaves), Demarche gave a proof of the following classical result [De].

6.6.4. **Theorem.** *(Mostow* [Mo]*) Assume that* $\mathrm{char}(k) = 0$ *and let G/k be a linearly reductive group and let U/k be a split unipotent k-group. Then each extension of algebraic groups of G by U is split and the sections are conjugated under $U(k)$.*

The smooth connected linearly reductive groups are the reductive groups in characteristic zero and only the tori in positive characteristic (Nagata, see [DG, IV.3.3.6]).

Lie algebras, lifting tori

7. Weil restriction

We are given the following equation $z^2 = 1 + 2i$ in $\mathbb{C}$. A standard way to solve it is to write $z = x + iy$ with $x, y \in \mathbb{R}$. It provides then two real equations $x^2 - y^2 = 1$ and $xy = 1$. We can abstract this method for affine schemes and for functors.

We are given a ring extension S/R or $j : R \to S$. Since a S-algebra is an R–algebra, an R-functor F defines a S-functor denoted by F_S and called the scalar extension of F to S. For each S–algebra S', we have $F_S(S') = F(S')$. If X is an R-scheme, we have $(h_X)_S = h_{X \times_R S}$.

Now we consider a S–functor E and define its Weil restriction to S/R denoted by $\prod\limits_{S/R} E$ by

$$\Big(\prod_{S/R} E\Big)(R') = E(R' \otimes_R S)$$

for each R–algebra R'. We note the two following functorial facts.

(I) For an R-map or rings $u : S \to T$, we have a natural map

$$u_* : \prod_{S/R} E \to \prod_{T/R} E_T.$$

(II) For each R'/R, there is natural isomorphism of R'–functors

$$\Big(\prod_{S/R} E\Big)_{R'} \xrightarrow{\sim} \prod_{S \otimes_R R'/R'} E_{S \otimes_R R'}.$$

For other functorial properties, see appendix A.5 of [CGP].

At this stage, it is of interest to discuss the example of vector group functors. Let N be a S–module. We denote by j_*N the scalar restriction of N from S to R [Bbk1, §II.1.13]. The module j_*N is N equipped with the R–module structure induced by the map $j : R \to S$. It satifies the adjunction property $\mathrm{Hom}_R(M, j_*N) \xrightarrow{\sim} \mathrm{Hom}_S(M \otimes_R S, N)$ (*ibid*, §III.5.2) for each R-module M.

7.0.5. **Lemma.** *(1)* $\prod\limits_{S/R} W(N) \xrightarrow{\sim} W(j_*N)$.

(2) If N is f.g. projective and S/R is finite and locally free, then $\prod\limits_{S/R} W(N)$ *is representable by the vector group scheme* $\mathfrak{W}(j_*N)$.

For a more general statement, see [SGA3, I.6.6].

Proof. (1) For each R-algebra R', we have

$$\Big(\prod_{S/R} W(N)\Big)(R') = W(N)(R' \otimes_R S) = N \otimes_S (R' \otimes_R S) = j_*N \otimes_R R' = W(j_*N)(R').$$

(2) The assumptions implies that j_*N is f.g. over R, hence $W(j_*N)$ is representable by the vector R–group scheme $\mathfrak{W}(j_*N)$. □

If F is an R-functor, we have for each R'/R a natural map

$$\eta_F(R') : F(R') \to F(R' \otimes_R S) = F_S(R' \otimes_R S) = \Big(\prod_{S/R} F_S\Big)(R');$$

it defines a natural mapping of R–functor $\eta_F : F \to \prod\limits_{S/R} F_S$. For each S–functor E, it permits to defines a map

$$\phi : \mathrm{Hom}_{S-functor}(F_S, E) \to \mathrm{Hom}_{R-functor}\big(F, \prod_{S/R} E\big)$$

by applying a S–functor map $g : F_S \to E$ to the composition

$$F \overset{\eta_F}{\to} \prod_{S/R} F_S \xrightarrow{\prod\limits_{S/R} g} \prod_{S/R} E.$$

7.0.6. **Lemma.** *The map ϕ is bijective.*

Proof. We apply the compatibility with $R' = S_2 = S$. The map $S \to S \otimes_R S_2$ is split by the codiagonal map $\nabla : S \otimes_R S_2 \to S, s_1 \otimes s_2 \to s_1 s_2$. Then we can consider the map

$$\theta_E : \Big(\prod_{S/R} E\Big)_{S_2} \xrightarrow{\sim} \prod_{S \otimes_R S_2/S_2} E_{S \otimes_R S_2} \overset{\nabla_*}{\to} \prod_{S/S} E = E.$$

This map permits to construct the inverse map ψ of ϕ as follows

$$\psi(h) : F_S \overset{l_S}{\to} \Big(\prod_{S/R} E\Big)_{S_2} \overset{\theta_E}{\to} E$$

for each $l \in \mathrm{Hom}_{R-functor}\big(F, \prod\limits_{S/R} E\big)$. By construction, the maps ϕ and ψ are inverse of each other. □

In conclusion, the Weil restriction from S to R is then right adjoint to the functor of scalar extension from R to S.

7.0.7. **Proposition.** *Let $\mathfrak{Y}/S$ be an affine scheme of finite type (resp. of finite presentation). Then the functor $\prod_{S/R} h_{\mathfrak{Y}}$ is representable by an affine scheme of finite type (finite representation).*

Again, it is a special case of a much more general statement, see [BLR, §7.6].

Proof. Up to localize for the Zariski topology, we can assume that S is free over R, namely $S = \oplus_{i=1,\dots,d} R\,\omega_i$. We see $\mathfrak{Y}$ as a closed subscheme of some affine space $\mathbb{A}^n_S$, that is given by a system of equations $(P_\alpha)_{\alpha \in I}$

with $P_\alpha \in S[t_1,\dots,t_n]$. Then $\prod\limits_{S/R} h_{\mathfrak{Y}}$ is a subfunctor of $\prod\limits_{S/R} W(S^n) \xrightarrow{\sim} W(j_*(S^n)) \xrightarrow{\sim} W(R^{nd})$ by Lemma 7.0.5. For each I, we write

$$P_\alpha\Big(\sum_{i=1,..,d} y_{1,i}\omega_i, \sum_{i=1,..,d} y_{2,i}\omega_i, \dots, \sum_{i=1,..,d} y_{n,i}\Big) = Q_{\alpha,1}\,\omega_1 + \cdots + Q_{\alpha,r}\,\omega_r$$

with $Q_{\alpha,i} \in R\big[y_{k,i}; i=1,..,d; k=1,...,n\big]$. Then for each R'/R, $\Big(\prod\limits_{S/R} h_{\mathfrak{Y}}\Big)(R')$ inside R'^{nd} is the locus of the zeros of the polynomials $Q_{\alpha,j}$. Hence this functor is representable by an affine R-scheme $\mathfrak{X}$ of finite type. Furthermore, if $\mathfrak{Y}$ is of finite presentation, we can take I finite so that $\mathfrak{X}$ is of finite presentation too. $\square$

In conclusion, if $\mathfrak{H}/S$ is an affine group scheme of finite type, then the R–group functor $\prod\limits_{S/R} h_{\mathfrak{H}}$ is representable by an R-affine group scheme of finite type. There are two basic examples of Weil restrictions.

(a) The case of a finite separable field extension k'/k (or more generally an étale k-algebra). Given an affine algebraic k'-group G'/k', we associate the affine algebraic k–group $G = \prod\limits_{k'/k} G'$ which is often denoted by $R_{k'/k}(G)$, see [Vo, §3. 12]. In that case, $R_{k'/k}(G) \times_k k_s \xrightarrow{\sim} (G'_{k_s})^d$. In particular, the dimension of G is $[k':k]\dim_{k'}(G')$; the Weil restriction of a finite algebraic group is a finite group.

(b) The case where $S = k[\epsilon]$ is the ring of dual numbers which is of very different nature. For example the quotient k-group $\prod\limits_{k[\epsilon]/k} (\mathfrak{G}_m)/\mathfrak{G}_m$ is the additive k–group. Also if $p = \mathrm{char}(k) > 0$, $\prod\limits_{k[\epsilon]/k} \mu_{p,k[\epsilon]}$ is of dimension 1.

Let us give an application of Weil restriction.

7.0.8. **Proposition.** *Let $\mathfrak{G}/R$ be an affine group scheme. Assume that there exists a finite locally free extension S/R such that $\mathfrak{G}\times_R S$ admits a faithful representation N f.g. locally free as S–module. Then $\mathfrak{G}$ admits a faithful representation M which is f.g. locally free as R–module.*

Proof. Let $\rho : \mathfrak{G}\times_R S \to \mathrm{GL}(N)$ be a faithful S–representation and denote by M/R the restriction of N from S to R. We consider then the R–map

$$\mathfrak{G} \to \prod_{S/R} \mathfrak{G}\times_R S \overset{\prod_{S/R}\rho}{\to} \prod_{S/R} \mathrm{GL}(N) \to \mathrm{GL}(M)$$

It is a composite of monomorphisms, hence a monomorphism. $\square$

7.0.9. **Remark.** It is natural to ask whether the functor of scalar extension from R to S admits a left adjoint. It is the case and we denote by $\bigsqcup_{S/R} E$ this left adjoint functor, see [DG, §I.1.6]. It is called the Grothendieck restriction.

If $\rho : S \to R$ is an R–ring section of j, it defines a structure R^ρ of S–ring. We have $\bigsqcup_{S/R} E = \bigsqcup_{\rho : S \to R} E(R^\rho)$. If $E = h_{\mathfrak{Y}}$ for a S–scheme $\mathfrak{Y}$, $\bigsqcup_{S/R} \mathfrak{Y}$ is representable by the R–scheme $\mathfrak{Y}$. This is simply the following R-scheme $\mathfrak{Y} \to \mathrm{Spec}(S) \xrightarrow{j^*} \mathrm{Spec}(R)$.

8. Tangent spaces and Lie algebras

8.1. **Tangent spaces.** We are given an affine R–scheme $\mathfrak{X} = \mathrm{Spec}(A)$. Given a point $x \in \mathfrak{X}(R)$, it defines an ideal $I(x) = \ker(A \xrightarrow{s_x} R)$ and defines an A–structure on R denoted R^x. We denote by $R[\epsilon] = R[t]/t^2$ the ring of R-dual numbers. We claim that we have a natural exact sequence of pointed set

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Der}_A(A, R^x) & \xrightarrow{i_x} & \mathfrak{X}(R[\epsilon]) & \longrightarrow & \mathfrak{X}(R) \to 1 \\ & & & & || & & \\ & & & & \mathrm{Hom}_R\big(A, R[\epsilon]\big). & & \end{array}$$

where the base points are $x \in \mathfrak{X}(R) \subset \mathfrak{X}(R[\epsilon])$. The map i_x applies a derivation D to the map $f \mapsto s_x(f) + \epsilon\, D(f)$. It is a ring homomorphism since for $f, g \in A$ we have

$$\begin{aligned} i_x(fg) &= s_x(fg) + \epsilon\, D(fg) \\ &= s_x(f)\, s_x(g) + \epsilon\, D(f)\, s_x(g) + \epsilon s_x(f)\, D(g) \qquad \text{[derivation rule]} \\ &= (s_x(f) + \epsilon\, D(f)) \,.\, (s_x(g) + \epsilon\, D(g)) \qquad [\epsilon^2 = 0]. \end{aligned}$$

Conversely, one sees that a map $u : A \to R[\epsilon]$, $f \mapsto u(f) = s_x(f) + \epsilon\, v(f)$ is a ring homomorphism iff $v \in \mathrm{Der}_A(A, R^x)$.

8.1.1. **Remark.** The geometric interpretation of $\mathrm{Der}_A(A, R^x)$ is the tangent space at x of the scheme $\mathfrak{X}/R$ (see [Sp, 4.1.3]). Note there is no need of smoothness assumption to deal with that.

We have a natural A-map

$$\mathrm{Hom}_{A-mod}(I(x)/I^2(x), R^x) \to \mathrm{Der}_A(A, R^x);$$

it maps a A–map $l : I(x)/I^2(x) \to R$ to the derivation $D_l : A \to R$, $f \mapsto D_l(f) = l(f - f(x))$. This map is clearly injective but is split by mapping a derivation $D \in \mathrm{Der}_A(A, R^x)$ to its restriction on $I(x)$. Hence the map above is an isomorphism. Furthermore $I(x)/I^2(x)$ is an R^x–module hence the forgetful map

$$\mathrm{Hom}_{A-mod}(I(x)/I^2(x), R^x) \xrightarrow{\sim} \mathrm{Hom}_{R-mod}(I(x)/I^2(x), R)$$

is an isomorphism. We conclude that we have the fundamental exact sequence of pointed sets

$$1 \longrightarrow \big(I(x)/I^2(x)\big)^\vee \xrightarrow{i_x} \mathfrak{X}(R[\epsilon]) \longrightarrow \mathfrak{X}(R) \to 1.$$

We record that the R–module structure on $I(x)/I(x)^2$ is induced by the change of variable $\epsilon \mapsto \lambda\,\epsilon$. This construction behaves well with fibred products.

8.1.2. **Lemma.** *Let $\mathfrak{Y} = \mathrm{Spec}(B)$ be an affine R–scheme and $y \in \mathfrak{Y}(R)$. The dual of the R–module map $v : I(x)/I^2(x) \oplus I(y)/I^2(y) \to I(x,y)/I^2(x,y)$ is an isomorphism and fits in the following commutative diagram*

$$\begin{array}{ccccccc} 1 \longrightarrow & \big(I(x)/I^2(x)\big)^\vee \oplus \big(I(y)/I^2(y)\big)^\vee & \xrightarrow{i_x \times i_y} & \mathfrak{X}(R[\epsilon]) \times \mathfrak{Y}(R[\epsilon]) & \longrightarrow & \mathfrak{X}(R) \times \mathfrak{Y}(R) \to 1 \\ & v^\vee \Big\uparrow \cong & & || & & || \\ 1 \longrightarrow & \big(I(x,y)/I^2(x,y)\big)^\vee & \xrightarrow{i_{(x,y)}} & (\mathfrak{X} \times_R \mathfrak{Y})(R[\epsilon]) & \longrightarrow & (\mathfrak{X} \times_R \mathfrak{Y})(R) \to 1. \end{array}$$

commutes.

We note that an R-module, $I(x)$ is a direct summand of $R[\mathfrak{X}]$. If we consider an R-ring S, il follows that $I(x) \otimes_R S$ is the kernel of $R[\mathfrak{X}] \overset{s_x \otimes id}{\to} S$. In conclusion, we have then defined a (split) exact sequence of pointed R–functors

$$1 \longrightarrow \mathfrak{V}(I(x)/I(x)^2) \xrightarrow{i_x} \prod_{R[\epsilon]/R} \mathfrak{X}_{R[\epsilon]} \longrightarrow \mathfrak{X} \longrightarrow 1.$$

If $\mathfrak{X}/R$ is smooth of dimension d (see appendix), $I(x)/I(x)^2$ is locally free of rank d, so that $\mathfrak{V}(I(x)/I(x)^2) = \mathfrak{W}\big(I(x)/I(x)^2\big)$ is a nice vector R–group-scheme.

8.2. **Lie algebras.** Now $\mathfrak{G}/R$ is an affine group scheme. We denote by $\mathrm{Lie}(\mathfrak{G})(R)$ the tangent space at the origin $1 \in \mathfrak{G}(R)$. This is the dual of I/I^2 where $I \subset R[\mathfrak{G}]$ is the kernel of the augmentation ideal. We define the "Lie algebra of $\mathfrak{G}$" vector R–group scheme by

$$\mathrm{Lie}(\mathfrak{G}) = \mathfrak{V}(I/I^2).$$

It fits in the sequence

$$\begin{array}{ccccc} 0 \longrightarrow & \mathrm{Lie}(\mathfrak{G})(R) & \longrightarrow & \mathfrak{G}(R[\epsilon]) & \longrightarrow \mathfrak{G}(R) \to 1 \\ & X & \mapsto & \exp(\epsilon X) & \end{array}$$

which is a split exact of abstract groups where $\mathrm{Lie}(\mathfrak{G})(R)$ is equipped with the induced group law.

8.2.1. **Lemma.** *That induced group law is the additive law on* $\mathrm{Lie}(\mathfrak{G})(R)$*, namely* $\exp(\epsilon X + \epsilon Y) = \exp(\epsilon X)\,.\,\exp(\epsilon Y)$ *for each* $X, Y \in \mathrm{Lie}(\mathfrak{G})(R)$.

Proof. We apply Lemma 8.1.2 and use the product map $m : \mathfrak{G} \times_R \mathfrak{G} \to \mathfrak{G}$ to construct the following commutative diagram

$$\begin{array}{ccccccc}
1 \longrightarrow & \big(I/I^2\big)^\vee \oplus \big(I/I^2\big)^\vee & \xrightarrow{\exp\times\exp} & \mathfrak{G}(R[\epsilon])\times\mathfrak{G}(R[\epsilon]) & \longrightarrow & \mathfrak{G}(R)\times\mathfrak{G}(R)\to 1 \\
& v^\vee \Big\uparrow \cong & & || & & || \\
1 \longrightarrow & \big(I(\mathfrak{G}\times_R\mathfrak{G})/I(\mathfrak{G}\times_R\mathfrak{G})^2\big)^\vee & \xrightarrow{\exp_{\mathfrak{G}\times_R\mathfrak{G}}} & (\mathfrak{G}\times_R\mathfrak{G})(R[\epsilon]) & \longrightarrow & (\mathfrak{G}\times_R\mathfrak{G})(R)\to 1. \\
& m_* \Big\downarrow & & m\Big\downarrow & & m\Big\downarrow \\
1 \longrightarrow & \big(I/I^2\big)^\vee & \xrightarrow{\exp} & \mathfrak{G}(R[\epsilon]) & \longrightarrow & \mathfrak{G}(R)\to 1.
\end{array}$$

Since the composite $\mathfrak{G} \overset{id\times\epsilon}{\to} \mathfrak{G}\times_R\mathfrak{G} \overset{m}{\to} \mathfrak{G}$ is the identity, the composite map $\big(I/I^2\big)^\vee \overset{id\times 0}{\to} \big(I/I^2\big)^\vee \oplus \big(I/I^2\big)^\vee \to \big(I/I^2\big)^\vee$ is the identity. It is the same for the second summand, so we conclude that that the left vertical composite map is the addition. $\square$

8.2.2. **Remark.** The natural map $\mathrm{Lie}(\mathfrak{G})(R)\otimes_R S \to \mathrm{Lie}(\mathfrak{G})(S)$ is not bijective in general. It is the case if I/I^2 is a projective R–module of finite type, and in particular if $\mathfrak{G}$ is smooth over R. The condition "f.g. projective" is actually necessary for having this property in general, see [DG, II.4.4].

8.2.3. **Example.** Let M be an R-module and consider the R-vector group scheme $\mathfrak{V}(M)$. For each S/R, we have

$$\mathfrak{V}(M)(S[\epsilon]) = \mathrm{Hom}_{S[\epsilon]}(M\otimes_R S[\epsilon], S[\epsilon]) = \mathrm{Hom}_R(M, S[\epsilon]) = \mathfrak{V}(M)^2(S),$$

hence an R-isomorphism $\mathfrak{V}(M) \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{V}(M))(R)$.

The exact sequence defines an action of $\mathfrak{G}(R)$ on $\mathrm{Lie}(\mathfrak{G})(R)$ and actually a representation $\mathrm{Ad} : \mathfrak{G} \to \mathrm{Aut}(\mathrm{Lie}(\mathfrak{G})) = \mathrm{Aut}(\mathfrak{V}(I/I^2))$. It is called the adjoint representation and denoted by Ad.

8.2.4. **Remark.** If I/I^2 is f.g. projective then $\mathfrak{V}(I/I^2) = \mathfrak{W}((I/I^2)^v ee)$ and then $(I/I^2)^v ee$ is an $R-\mathfrak{G}$-module.

Denoting by $s : \mathfrak{G}(R) \to \mathfrak{G}(R[\epsilon])$ the section, we have

$$\mathrm{Ad}(g)\,\exp(\epsilon X) = s(g)\,\exp(\epsilon X)\,s(g^{-1}) \in \mathfrak{G}(R[\epsilon]).$$

If $f : \mathfrak{G} \to \mathfrak{H}$ is a morphism of affine R-group schemes, we have a map $\mathrm{Lie}(f) : \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathfrak{H})$ of R-vector groups and the commutativity property $f\big(\exp(\epsilon X)\big) = \exp\Big(\epsilon\,.\,\mathrm{Lie}(f)(X)\Big)$.

8.2.5. **Lemma.** *Let M be a f.g. projective R-module and put $\mathfrak{G} = \mathrm{GL}(M)$. Then $\mathrm{End}_R(M) = \mathrm{Lie}(\mathfrak{G})(R)$ and the adjoint action is*

$$\mathrm{Ad}(g)\,.\,X = g\,X\,g^{-1}.$$

Proof. The R–group scheme $\mathfrak{G}$ is open in $W(\mathrm{End}_R(M))$ so that the tangent space at 1 in $\mathfrak{G}$ is the same than in $W(\mathrm{End}_R(M))$. By example 8.2.3, we get then an R–isomorphism $\mathrm{End}_R(M) \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{G})(R)$. We perform now the computation of $\mathrm{Ad}(g)\, \exp(\epsilon X)$ in $\mathfrak{G}(R[\epsilon]) \subset \mathrm{End}_R(M) \otimes_R R[\epsilon]$. We have $\mathrm{Ad}(g)\, \exp(\epsilon X) = g\,(Id + \epsilon X)\, g^{-1} = Id + \epsilon g X g^{(1} = \exp(\epsilon\, g\, X\, g^{-1})$. □

More generally, we can define the Lie algebra R–functor of a group R–functor F by putting

$$\mathrm{Lie}(F)(S) = \ker\Big(F(S[\epsilon]) \to F(S)\Big).$$

It is a subgroup equipped with a map $S \times \mathrm{Lie}(F)(S) \to \mathrm{Lie}(F)(S)$ coming from the base change $\epsilon \mapsto \lambda\epsilon$. Also there is an adjoint action of the R–functor F on $\mathrm{Lie}(F)$. In that generality, we are actually mainly interested in the following examples.

8.2.6. **Lemma.** *(1) Let M be a R–module. Then $W(M) \xrightarrow{\sim} \mathrm{Lie}(W(M))$ and $\mathrm{End}_S(M \otimes_R S) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{GL}(W(M))(S)$ for each S/R.*

(2) Let N be an R–module. Then $\mathrm{End}_S\big(V(N)(S)\big) \xrightarrow{\sim} \mathrm{Lie}\big(\mathrm{GL}(V(N)\big)(S)$ for each S/R.

Proof. (1) The first thing is similar as example 8.2.3. For each S/R, we have indeed a split exact sequence of abstract groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{End}_S(M \otimes_R S) & \longrightarrow & \mathrm{GL}(\mathfrak{W}(M))(R[\epsilon]) & \longrightarrow & \mathrm{GL}(M)(R) & \longrightarrow & 1. \\ & & f & \mapsto & Id + \epsilon f & & & & \end{array}$$

(2) The proof is the same. □

We come back to the case of the affine R-group $\mathfrak{G}$. We see the adjoint representation as a morphism of R–group functors

$$\mathrm{Ad} : \mathfrak{G} \to \mathrm{GL}(\mathrm{Lie}(\mathfrak{G})) = \mathrm{GL}\big(\mathfrak{V}(I/I^2)\big)$$

By applying the Lie functor, it induces then a morphism of vector R-group schemes

$$\mathrm{ad} : \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}\Big(\mathrm{GL}\big(\mathfrak{V}(I/I^2)\big)\Big).$$

For each S/R, we have then a S–map

$$\mathrm{ad}(S) : \mathrm{Lie}(\mathfrak{G})(S) \to \mathrm{Lie}\Big(\mathrm{GL}\big(\mathfrak{V}(I/I^2)\big)\Big)(S) = \mathrm{End}_S(\mathrm{Lie}(\mathfrak{G})(S)).$$

For each $X, Y \in \mathrm{Lie}(\mathfrak{G})(S)$, we denote by

$$[X, Y] = \mathrm{ad}(S)(X).\, Y \in \mathrm{Lie}(\mathfrak{G})(S)$$

the Lie bracket of X and Y.

8.2.7. **Lemma.** *(1) Let $f : \mathfrak{G} \to \mathfrak{H}$ be a morphism of affine R–group schemes. For each $X, Y \in \mathrm{Lie}(\mathfrak{G})(R)$, we have*

$$\mathrm{Lie}(f)\,.\,[X, Y] = [\mathrm{Lie}(f)\,.\,X,\ \mathrm{Lie}(f)\,.\,Y\,] \in \mathrm{Lie}(\mathfrak{G})(R).$$

(2) In the case $\mathfrak{G} = \mathrm{GL}(M)$ *with* M *f.g. projective, the Lie bracket* $\mathrm{End}_R(M) \times \mathrm{End}_R(M) \to \mathrm{End}_R(M)$ *reads* $[X, Y] = XY - YX$.

Proof. (1) Up to replace f by $id \times f : \mathfrak{G} \to \mathfrak{G} \times \mathfrak{H}$, we may assume that f is a monomorphism. It follows that the R-functor $\mathrm{Lie}(f) : \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathfrak{H})$ is a monomorphism. We consider the following diagram of R–groups

$$\begin{tikzcd}
\mathfrak{G} \arrow[r, "Ad_{\mathfrak{G}}"] \arrow[dr, "f\circ \mathrm{Ad}_{\mathfrak{H}}"] \arrow[dd, "f"] & \mathrm{GL}(\mathrm{Lie}(\mathfrak{G})) \\
 & \mathrm{GL}(\mathrm{Lie}(\mathfrak{H}), \mathrm{Lie}(\mathfrak{G})) \arrow[u] \arrow[d] \\
\mathfrak{H} \arrow[r, "Ad_{\mathfrak{H}}"] & \mathrm{GL}(\mathrm{Lie}(\mathfrak{H})).
\end{tikzcd}$$

We derive it and get

$$\begin{tikzcd}
\mathrm{Lie}(\mathfrak{G}) \arrow[r, "ad_{\mathfrak{G}}"] \arrow[dr, "\mathrm{Lie}(f\circ \mathrm{Ad}_{\mathfrak{H}})"] \arrow[dd, "\mathrm{Lie}(f)"] & \mathrm{End}(\mathrm{Lie}(\mathfrak{G})) \\
 & \mathrm{End}(\mathrm{Lie}(\mathfrak{H}), \mathrm{Lie}(\mathfrak{G})) \arrow[u] \arrow[d] \\
\mathrm{Lie}(\mathfrak{H}) \arrow[r, "ad_{\mathfrak{H}}"] & \mathrm{End}(\mathrm{Lie}(\mathfrak{H}))
\end{tikzcd}$$

whence the desired compatibility.
(2) We consider the adjoint representation $\mathrm{Ad}(R) : \mathrm{GL}(M)(R) \to \mathrm{GL}(\mathrm{End}_R(M))(R)$ known to be $\mathrm{Ad}(g).X = gXg^{-1}$. We consider $\mathrm{Ad}(R[\epsilon]) : \mathrm{GL}(M)(R[\epsilon]) \to \mathrm{GL}(\mathrm{End}_R(M))(R[\epsilon])$; for $X, Y \in \mathrm{End}_R(M)$ we compute inside $(\mathrm{End}_R(M))(R[\epsilon])$ using Lemma 8.2.5

$$\begin{aligned}
\mathrm{Ad}(R[\epsilon])(\exp(\epsilon X)) \,.\, Y &= (1 + \epsilon X)Y(1 + \epsilon X)^{-1} \\
&= (1 + \epsilon X)Y(1 - \epsilon X) \\
&= Y + \epsilon(XY - YX).
\end{aligned}$$

We conclude that $[X, Y] = XY - YX$. $\square$

8.2.8. **Proposition.** *The Lie bracket defines a Lie* R*-algebra structure on the* R*–module* $\mathrm{Lie}(\mathfrak{G})(R)$*, that is*

(i) the bracket is R*–bilinear and alternating;*

(ii) (Jacobi identity) For each $X, Y, Z \in \mathrm{Lie}(\mathfrak{G})(R)$*, we have*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

We give here a short non orthodox proof specific to affine group schemes; for a more general setting, see [DG, II.4.4.3] and [SGA3, Exp. II].

Proof. Let us start with the case where $\mathfrak{G}$ admits a faithful representation in $\mathrm{GL}(R^n)$. Then the R–map $\mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathrm{GL}(M))$ is a monomorphism. From Lemma 8.2.7, it is then enough to check it for the linear group GL_n. That case is straightforward, we have $\mathrm{Lie}(\mathrm{GL}_n)(R) = M_n(R)$ and the bracket is $[X, Y] = XY - XY$(lemma 8.2.7).

For handling the general case, we consider a faithful $\mathfrak{G}$–module M, e.g. the regular representation. Generalizing the previous strategy, it lasts to study the case of the R–group functors $\mathrm{GL}(W(M))$ where $\mathrm{Lie}(\mathrm{GL}(W(M)))(R) = \mathrm{End}_R(M)$. One needs to define the Lie bracket on $\mathrm{End}_R(M)$ and identify it as $XY - YX$... □

8.2.9. **Remark.** If $j : R \to S$ is a finite locally free morphism and $\mathfrak{H}/S$ a group scheme over S, it is a natural question to determine the Lie algebra of $\mathfrak{G}$. It is done in [CGP, A.7.6]. and we have $\mathrm{Lie}(\mathfrak{G}) = j_* \mathrm{Lie}(\mathfrak{H})$, that is $\mathrm{Lie}(\mathfrak{G})(R') = \mathrm{Lie}(\mathfrak{H})(S \otimes_R R')$ for each R'/R.

8.2.10. **Examples.** If k is a field of characteristic $p > 0$, $\mathrm{Lie}(\mu_p)(k) = k$ with trivial Lie structure.

8.3. **More infinitesimal properties.** Our goal is to generalize the exact sequences of §8.1. Let $\mathfrak{X} = \mathrm{Spec}(A)$ be an affine scheme. Let C be an R–ring equipped equipped with an ideal J satisfying $J^2 = 0$. Let $x \in X(C)$. We denote by $\overline{x}$ the image of x in $\mathfrak{X}(C/J)$. We put $I(x) = \ker(C[\mathfrak{X}] \to C)$. We claim that we have an exact sequence of pointeds sets

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Hom}_{C-mod}(I(x)/I^2(x), J) & \xrightarrow{i_x} & \mathfrak{X}(C) & \longrightarrow & \mathfrak{X}(C/J) \\ & & l & \mapsto & x_l & & \end{array}$$

pointed at 0, x and $\overline{x}$. More precisely, the point x_l is defined by the morphism of rings

$$\begin{array}{ccc} C[\mathfrak{X}] & \xrightarrow{s_{x_l}} & C \\ f & \mapsto & f(x) + l\big(f - f(x)\big). \end{array}$$

It extends indeed the case of §8.1 when taking $C = R[\epsilon]$ and $J = \epsilon R[\epsilon]$. Let us check that the mapping is well defined. The only thing is the multiplicativity. Given $f, g \in C[\mathfrak{X}]$, we compute

$$\begin{aligned} (fg)(x) + l\big(fg - (fg)(x)\big) &= f(x)\, g(x) + l\big(fg - f(x)g(x)\big) \\ &= f(x)\, g(x) + l\big(f(g - g(x)) + (f - f(x))g(x)\big) \\ &= f(x)\, g(x) + f(x)\, l(g - g(x)) + g(x)\, l(f - f(x)) \quad [l \text{ is an } R\text{-map}] \\ &= \big(f(x) + l(f - f(x))\big)\, \big(g(x) + l(g - g(x))\big) \quad [J^2 = 0]. \end{aligned}$$

Conversely if $s : C[\mathfrak{X}] \to C$ is a ring homomorphism which coincide modulo I, we sput $l_s(f) = s(f)$ for each $f \in I$. Then l_s is C^x-linear and satisfies $l_s(I^2) = 0$.

If $\mathfrak{X}/R$ is smooth of relative dimension d, I/I^2 is locally free of rank d. Also the map $\mathfrak{X}(C) \to \mathfrak{X}(C/J)$ is onto (theorem 20.0.4).

If C'/C is a ring extension, putting $J' = J \otimes_R C'$ and[4] $I'(x) = I(x) \otimes_R R'$, we have then an isomorphism [Bbk1, §II.5.3, prop. 7]

$$\mathrm{Hom}_{C-mod}(I(x)/I^2(x), J) \otimes_R C' \xrightarrow{\sim} \mathrm{Hom}_{C'-mod}(I'(x)/I'^2(x), J').$$

In this case we have then an exact sequence of C-functors

$$1 \longrightarrow \mathfrak{W}(M) \xrightarrow{i_x} \mathfrak{X}_C \longrightarrow \prod_{C/J} \mathfrak{X}_C \to 1$$

where $M = \mathrm{Hom}_{C-mod}(I(x)/I^2(x), J)$.

9. Fixed points of diagonalizable groups

9.1. **Representatibility.**

9.1.1. **Proposition.** *Let $\mathfrak{X}$ be an affine R-scheme equipped with an action of a diagonalizable group scheme $\mathfrak{G}/R = \mathfrak{D}(A)$. Then the R–functor of fixed points F defined by*

$$F(S) = \Big\{ x \in \mathfrak{X}(S) \mid G(S') \,.\, x_{S'} = x_{S'} \;\; \forall S'/S \Big\}$$

is representable by a closed subscheme of $\mathfrak{X}$.

It is denoted by $\mathfrak{X}^{\mathfrak{G}}/R$. The proof below is inspired by [CGP, Lemma 2.1.4].

Proof. The R–module $R[\mathfrak{X}]$ decomposes in eigenspaces $\bigoplus_{a \in A} R[\mathfrak{X}]_a$. We denote by $J \subset R[\mathfrak{X}]$ the ideal generated by the $R[\mathfrak{X}]_a$ for a running over $A \setminus \{0\}$. We denote by $\mathfrak{Y}$ the closed subscheme of $\mathfrak{X}$ defined by J. Since J is a $\mathfrak{D}(A)$-submodule of $R[\mathfrak{X}]$, $R[\mathfrak{Y}]$ is $\mathfrak{D}(A)$-module with trivial structure. Hence the R-map $h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ factorizes by F, and we have a monomorphism $h_{\mathfrak{Y}} \to F$. Again by Yoneda, we have

$$F(R) = \Big\{ x \in \mathfrak{X}(R) \mid \zeta \, x_{R[\mathfrak{G}]} = x_{R[\mathfrak{G}]} \Big\}$$

where $\zeta \in \mathfrak{G}(R[\mathfrak{G}])$ stands for the universal element of $\mathfrak{G}$. Let $x \in F(R)$ and denote by $s_x : R[\mathfrak{G}] \to R$ the underlying map. Then the fact $\zeta \, x_{R[\mathfrak{G}]} = x_{R[\mathfrak{G}]} \in \mathfrak{X}(R[\mathfrak{G}])$ translates as follows

$$\begin{array}{ccc} R[\mathfrak{X}] & \xrightarrow{c} & R[\mathfrak{X}] \otimes_R R[A] \\ {\scriptstyle s_x}\downarrow & & \downarrow{\scriptstyle s_x \otimes id} \\ R & \longrightarrow & R[A] \\ & & \\ r & \mapsto & r. \end{array}$$

If $f \in R[\mathfrak{X}]_a$, $a \neq 0$, we have $c(f) = f \otimes e_a$ which maps then to $f(x) \otimes e_a = f(x)$. Therefore $f(x) = 0$. It follows that $J \subset \ker(s_x)$, that is x defines an R–point of $\mathfrak{Y}(R)$. The same holds for any S/R, hence we conclude that $h_{\mathfrak{Y}} = F$. $\square$

[4]Again we use that I is a direct summand of the R-module $R[\mathfrak{X}]$.

9.2. **Smoothness of the fixed point locus.**

9.2.1. **Theorem.** *Assume that R is noetherian. Let $\mathfrak{X}/R$ be an affine smooth R–scheme equipped with an action of the diagonalizable group scheme $\mathfrak{G} = \mathfrak{D}(A)$. Then the scheme of fixed points $\mathfrak{X}^{\mathfrak{G}}/R$ is smooth.*

For more general statements, see [SGA3, XII.9.6], [CGP, A.8.10] and [De, th. 5.4.4].

9.2.2. **Corollary.** *Assume that R is noetherian. Let $\mathfrak{H}/R$ be an affine smooth group scheme equipped with an action of the diagonalizable group scheme $\mathfrak{G} = \mathfrak{D}(A)$. Then the centralizer subgroup scheme $\mathfrak{H}^{\mathfrak{G}}$ is smooth.*

We proceed to the proof of Theorem 9.2.1.

Proof. Since R is noetherian, the closed affine subscheme $\mathfrak{X}^{\mathfrak{G}}$ of $\mathfrak{X}$ is of finite presentation. According to Theorem 20.0.4, it is enough to show that $\mathfrak{X}^{\mathfrak{G}}$ is formally smooth. We are given an R-ring C equipped with an ideal J satisfying $J^2 = 0$. We want to show that the map $\mathfrak{X}^{\mathfrak{G}}(C) \to \mathfrak{X}^{\mathfrak{G}}(C/J)$ is surjective. We start then with a point $\overline{x} \in \mathfrak{X}^{\mathfrak{G}}(C/J)$. Since $\mathfrak{X}$ is smooth, $\overline{x}$ lifts to a point $x \in \mathfrak{X}(C)$. We denote by $I(x) \subset C[\mathfrak{X}]$ the ideal of the regular functions vanishing at x. According to §8.3, we have an exact sequence of pointed C-functors

$$1 \longrightarrow \mathfrak{W}(M) \xrightarrow{i_x} \mathfrak{X}_C \longrightarrow \prod_{C/J} \mathfrak{X}_{C/J} \to 1.$$

where $M = \mathrm{Hom}_{C-mod}(I(x)/I^2(x), J)$. Since $\mathfrak{X}_C$ is equipped with an action of $\mathfrak{G}_C$, it induces an action on $\mathfrak{W}(M)$. In other words, M comes equipped with a $\mathfrak{G}_C$–module structure. For each $g \in \mathfrak{G}(C)$, we have $\overline{g.x} = \overline{g} \,.\, \overline{x} = \overline{x}$ since $\overline{x}$ is $\mathfrak{G}$-invariant. Hence $g.x = i_x(c(g)) = x + c(g)$ for a unique $c(g) \in M$. Now we take $g_1, g_2 \in \mathfrak{G}(C)$ and compute

$$g_1 \,.\, (g_2 \,.\, x) = g_1 \,.\, (x + c(g_2)) = g_1 \,.\, x + g_1 \,.\, c(g_2) = x + (c(g_1) + g_1.c(g_2)).$$

By unicity, we have the 1-cocycle formula $c(g_1 g_2) = c(g_1) + g_1.c(g_2) \in M$ for We can do the same for each C–ring C' and obtain then a 1–cocycle for the Hochschild cohomology. Since $H^1_0(\mathfrak{G}, M) = 0$ (theorem 6.3.6), there exists $m \in M$ such that $c(g) = g.m - m$ for each C'/C and each $g \in G(C')$. It means exactly that the point $i_x(m) \in \mathfrak{X}(R)$ is $\mathfrak{G}$–invariant. It maps to $\overline{x}$, so we conclude that $\mathfrak{X}^{\mathfrak{G}}(\mathfrak{C}) \to \mathfrak{X}^{\mathfrak{G}}(C/J)$ is onto. □

9.2.3. **Remark.** If $x \in \mathfrak{X}(R)$, the tangent space at x of $\mathfrak{X}^{\mathfrak{G}}$ is

$$T_{\mathfrak{X}^{\mathfrak{G}},x} = H^0_0(\mathfrak{G}, T_{\mathfrak{X},x}).$$

10. Lifting homomorphisms

10.1. **Rigidity principle.** Let $\mathfrak{G} = \mathfrak{D}(A)/R$ be a diagonalizable group scheme. The following fact illustrates the "rigidity" of $\mathfrak{G}$.

10.1.1. **Lemma.** *Let I be a nilpotent ideal of R.*

(1) Let M be an $R-\mathfrak{G}$-module. Then M is a trivial $R-\mathfrak{G}$-module if and only if $M\otimes_R R/I$ is a trivial $R/I-\mathfrak{G}_{R/I}$-module.

(2) Assume that $\mathfrak{G}$ acts on an affine R–scheme $\mathfrak{X}$. Then $\mathfrak{G}$ acts trivially on $\mathfrak{X}$ if and only if $\mathfrak{G}\times_R R/I$ acts trivially on $\mathfrak{X}\times_R R/I$.

Proof. (1) The direct way is obvious. Conversely, we assume that $M\otimes_R R/I$ is a trivial $R/I-\mathfrak{G}_{R/I}$-module. We have $M_0/IM_0 = (M\otimes_R R/I)_0 = (M\otimes_R R/I)$ by Corollary 6.1.3. By the nilpotent Nakayama lemma [Sta, 18.1.11], the map $M_0\to M$ is onto hence an isomorphism.

(2) We apply (1) to the $\mathfrak{G}$-module $R[\mathfrak{X}]$.

□

Here is a variation on the same theme not used in the sequel.

10.1.2. **Lemma.** *Let I be a nilpotent ideal of R.*

(1) Let M, M' be two $R-\mathfrak{G}$-modules which are projective R–modules. Then $M\xrightarrow{\sim} M'$ as $\mathfrak{G}$-modules if and only if the $\mathfrak{G}_{R/I}$-modules $M\otimes_R R/I$ and $M'\otimes_R R/I$ are isomorphic.

(2) Assume that $\mathfrak{G}$ acts on an affine R–scheme $\mathfrak{X}$ in two ways $u,v:\mathfrak{G}\to$ $\mathrm{Aut}(\mathfrak{X})$. Assume that $R[\mathfrak{X}]$ is projective. Then $u=v$ if and only if $u\times_R R/I = v\times_R R/I$.

Proof. (1) The direct way is obvious. In the way around, we fix an isomorphism $f: M\otimes_R I/M \xrightarrow{\sim} M'\otimes_R I/M$ of $R/I-\mathfrak{G}$–modules. Let $a\in A$. Then M_a and M'_a are projective. We have $M_a\otimes R/I\xrightarrow{\sim} M'_a\otimes R/I$ hence this map lifts in an isomorphism $\widetilde{f}_a : M_a\xrightarrow{\sim} M'_a$ by the Nakayama fact below. By summing up the $M'_a s$, we get and isomorphism of $\mathfrak{G}$-modules $M\xrightarrow{\sim} M'$.

(2) We apply (1) to $M=R[\mathfrak{X}]$.

□

10.1.3. **Lemma.** *Let I be a nilpotent ideal of R. Let M, M' be projective R-modules. Then M and M' are isomorphic if and only if M/IM and M'/IM' are isomorphic.*

Proof. The direct way is obvious. Conversely, we are given an isomorphism $f: M/IM\xrightarrow{\sim} M'/IM'$. Since M is projective the map $M\to M\otimes R/I\xrightarrow{f} M'\otimes R/I$ lifts to a map $f^\sharp: M\to M'$. In the other hand, f^{-1} lifts in a morphism $f^\dagger: M'\to M$. By construction $f^\dagger\circ f^\sharp = id_M + h$ with $h\in \mathrm{End}_R(M)$ and $h(M)\subset IM$. Then h is nilpotent so $f^\dagger\circ f^\sharp$ is invertible in $\mathrm{End}_R(M)$. Similarly $f^\sharp\circ f^\dagger$ is invertible in $\mathrm{End}_R(M')$, so we conclude that f is an isomorphism. □

The next statement also illustrates the rigidity principle.

10.1.4. **Theorem.** [SGA3, §IX.6] *We assume that A is finitely generated. Let $f : \mathfrak{G} \to \mathfrak{H}$ be a finitely presented group homomorphism to an affine R–group scheme $\mathfrak{H}$ of finite presentation. Let $x \in \mathrm{Spec}(R)$ be a point such that the homomorphism $f_x : \mathfrak{G}_{\kappa(x)} \to \mathfrak{H}_{\kappa(x)}$ is a monomorphism. Then there exists a Zariski neighbourhood $\mathrm{Spec}(R')$ of $\mathrm{Spec}(R)$ of x such that $f_{R'}$ is a monomorphism.*

We present an alternative proof.

Proof. We can assume that R is the local ring at x. We denote by $\mathfrak{K} = \ker(f)$. From §4.3, we have to show that $\mathfrak{K}_R = 1$. Our hypothesis reads $\mathfrak{K}_{\kappa(x)} = 1$.

First case: $\mathfrak{H}/R$ admits a faithful linear representation. We have then only to deal with the case of $\mathfrak{H} = \mathrm{GL}_d$, that is with a $\mathfrak{G}$-module M such that $M \cong R^d$ such that the associated representation ρ_x is a monomorphism. Denote by $A^\sharp$ the (finite set) of weights of $\rho_{\kappa(x)}$. According to Lemma 6.1.4, $A^\sharp$ spans the abelian group A. For $a \in A^\sharp$, M_a is a non-zero module which is projective since it is a direct summand of the free module $M = R^d$. By Nakayama lemma, $M' := \bigoplus_{a \in A^\sharp} M_a$ is isomorphic to M so that $\ker(\rho_M) = 1$.

General case. We shall show that $\mathfrak{K}$ is proper by using the valuative criterion. Let A/R be a valuation ring and denote by F its fraction field. The point is that $\mathfrak{H}_A$ admits a faithful representation (th. 6.2.1). Also the closed point of $\mathrm{Spec}(A)$ maps to the closed point of $\mathrm{Spec}(R)$ so that $\mathfrak{K}_A = 1$ by the first case. Therefore $\mathfrak{K}(A) = \mathfrak{K}(F)$ and $\mathfrak{K}$ is proper. Since $\mathfrak{K}$ is affine, $\mathfrak{K}$ is finite over R [Li, 3.17]. Hence $R[\mathfrak{K}]$ is a finite R–algebra such that $R/\mathfrak{M}_x \xrightarrow{\sim} R[\mathfrak{K}]/\mathfrak{M}_x\, R[\mathfrak{K}]$. The Nakayama lemma [Sta, 18.1.11.(6)] shows that the map $R \to R[\mathfrak{K}]$ is surjective. By using the unit section $1_{\mathfrak{K}}$ we conclude that $R = R[\mathfrak{K}]$. □

10.1.5. **Remark.** We shall see later (i.e. Cor. 16.3.1) that a monomorphism $\mathfrak{D}(A)_R \to \mathfrak{H}$ is a closed immersion.

10.2. **Formal smoothness.** Let $\mathfrak{G}/R$, $\mathfrak{H}/R$ be two affine group schemes. We define the following R–functors $\mathbf{Hom}(\mathbf{G},\mathbf{H})$, $\overline{\mathbf{Hom}}(\mathbf{G},\mathbf{H})$ by

$$\mathbf{Hom}(\mathbf{G},\mathbf{H})(S) = \mathrm{Hom}_{S-gr}(\mathbf{G}_S, \mathbf{H}_S),$$
$$\overline{\mathbf{Hom}}(\mathbf{G},\mathbf{H})(S) = \mathrm{Hom}_{S-gr}(\mathbf{G}_S, \mathbf{H}_S)/\mathbf{H}(S)$$

for each S/R.

10.2.1. **Theorem.** *Assume that $\mathfrak{G} = \mathfrak{D}(A)$ is diagonalizable and that $\mathfrak{H}$ is smooth.*

(1) The R–functor $\mathbf{Hom}(\mathbf{G},\mathbf{H})$ is formally smooth.

(2) The R–functors $\overline{\mathbf{Hom}}(\mathbf{G},\mathbf{H})$ and $\mathbf{Homcent}(\mathbf{G},\mathbf{H})$ are formally étale.

Proof. (1) Let C be an R–ring equipped equipped with an ideal J satisfying $J^2 = 0$. We are given a C/J-homomorphism $f_0 : \mathfrak{G}_{C/J} \to \mathfrak{H}_{C/J}$ and want to lift it. We put $I = \ker(C[\mathfrak{H}] \xrightarrow{\epsilon} C)$. We have $\mathrm{Lie}(\mathfrak{H})(C) = (I/I^2)^\vee$. Since $\mathfrak{H}$ is smooth we have an exact sequence of group C–functors

$$1 \longrightarrow \mathfrak{V}(\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J) \xrightarrow{\exp} \mathfrak{H}_C \longrightarrow \prod_{(C/J)/C} \mathfrak{H}_{C/J} \to 1.$$

Note that the $\prod_{(C/J)/C} \mathfrak{H}_{C/J}$-structure on $\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{H})(C/J) \otimes_{C/J} J$ arises from the adjoint representation of $\mathfrak{H}_{C/J}$. Now we pull-back this extension by the map of R–functors

$$u : \mathfrak{G}_C \prod_{(C/J)/C} \mathfrak{G}_{C/J} \overset{\prod_{(C/J)/C} f_0}{\longrightarrow} \prod_{(C/J)/C} \mathfrak{H}_{C/J}.$$

It defines a C–group functor E which fits in the commutative exact diagram of C–group functors

$$\begin{tikzcd}
1 \arrow[r] & \mathfrak{V}(\mathrm{Lie}(\mathfrak{H}) \otimes_C J) \arrow[r, "\exp"] & \mathfrak{H}_C \arrow[r] & \prod_{(C/J)/C} \mathfrak{H}_{C/J} \arrow[r] & 1 \\
1 \arrow[r] & \mathfrak{V}(\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J) \arrow[r] \arrow[u, equal] & E \arrow[r] \arrow[u, "v"] & \mathfrak{G} \arrow[r] \arrow[u, "u"] & 1.
\end{tikzcd}$$

According to Corollary 6.5.2, the bottom extension splits and a splitting defines then an R-group map $\mathfrak{G}_C \to \mathfrak{H}_C$ which lifts f_0.

(2) Exactly as in the abstract group setting, the choice of a lifting is the same that the choice of a splitting of the bottom extension. Up to $\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J$–conjugacy, this choice is encoded by the Hochschild cohomology group $H^1(\mathfrak{G}, \mathrm{Lie}(\mathfrak{H})(C) \otimes_C J)$. But this group vanishes (Th. 6.5.2), hence all liftings are $\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J$-conjugated. This shows that $\overline{\mathbf{Hom}}(\mathbf{G}, \mathbf{H})(C) \xrightarrow{\sim} \overline{\mathbf{Hom}}(\mathbf{G}, \mathbf{H})(C/J)$ and we conclude that $\overline{\mathbf{Hom}}(\mathbf{G}, \mathbf{H})$ is a formally étale functor.

Now assume that f_0 is central. According to the rigidity principle 10.1.1.(2), any lifting f of f_0 is central as well. If f_1, f_2 lift f_0, they are $\mathrm{Lie}(\mathfrak{H})(C) \otimes_C J$-conjugated, hence equal. It yields that $\mathbf{Homcent}(\mathbf{G}, \mathbf{H})$ is a formally étale functor. □

10.3. **Algebraization.**

10.3.1. **Theorem.** [SGA3, IX.7.1] *Assume that R is noetherian and equipped with an ideal I such that A is separated and complete for the I-adic topology. We put $R_n = R/I^{n+1}$ for each $n \geq 0$. Let $\mathfrak{G} = \mathfrak{D}(A)$ be a diagonalizable group scheme and let $\mathfrak{H}/R$ be a smooth affine group scheme. Then the natural map*

$$\mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R) \to \varprojlim_n \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R_n)$$

is bijective.

10.3.2. **Remarks.** (1) Injectivity is the easy thing there. Let $u, v : \mathfrak{G} \to \mathfrak{H}$ be two homomorphisms such that $u_n = u_{R_n}$ and $v_n = v_{R_n}$ agree. We consider then the R-module map $u^* - v^* : R[\mathfrak{H}] \to R[\mathfrak{G}]$. Our hypothesis implies that $\mathrm{Im}(u^* - v^*) \subset I^{n+1} R[\mathfrak{G}]$ for each $n \geq 0$. Since $\bigcap_n I^n = 0$ and $R[\mathfrak{G}]$ is a free R-module, we conclude that $\mathrm{Im}(u^* - v^*) = 0$ and $u = v$.

(2) The case $\mathfrak{G}$ is finite over R (i.e. A is finite) is easy. Let $u_n : \mathfrak{G} \to \mathfrak{H}$ be a coherent family of group homomorphisms. Then we have a commutative diagram

$$\begin{tikzcd} R[\mathfrak{H}] \arrow[d] & R[\mathfrak{G}] \arrow[d, "\cong"] \\ \varprojlim_n R_n[\mathfrak{H}] \arrow[r, "\varprojlim_n u_n"] & \varprojlim_n R_n[\mathfrak{H}]. \end{tikzcd}$$

The point is that $R[\mathfrak{G}] = R[A]$ is a finite free R–module whence the right vertical map is an isomorphism. The diagram defines then a map $u : \mathfrak{H} \to \mathfrak{G}$ which is a group homomorphism and lifts the u_n.

Theorem 10.2.1 implies that the transition maps $\mathrm{Hom}(\mathfrak{G}, \mathfrak{H})(R_{n+1}) \to \mathrm{Hom}(\mathfrak{G}, \mathfrak{H})(R_n)$ are surjective. It yields the first assertion in the next statement.

10.3.3. **Corollary.** *(1) The map* $\mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R) \to \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R/I)$ *is surjective.*

(2) If $f, f' \in \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R)$ *coincide in* $\mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R/I)$*, then there exists* $h \in \ker\big(\mathfrak{H}(R) \to \mathfrak{H}(R/I)\big)$ *such that* $f = g\, f'\, h^{-1}$.

(3) $\mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R) \xrightarrow{\sim} \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R/I)$.

(4) If $f \in \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R)$*,* f *is a monomorphism iff* $f_{R/I}$ *is a monomorphism.*

Proof. (2) We have $\mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R_1) \xrightarrow{\sim} \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R_0)$. More precisely we have seen that f_1 and f_1' are conjugated under an element of $\ker\big(\mathfrak{H}(R_1) \to \mathfrak{H}(R_0)\big)$. Since $\mathfrak{H}$ is smooth, $\mathfrak{H}(R)$ maps onto $\mathfrak{H}(R_1)$, so there exists $h_1 \in \ker\big(\mathfrak{H}(R) \to \mathfrak{H}(R_0)\big)$ such that $f_1' = {}^{h_1} f_1$. We continue and construct by induction a sequence of elements $h_n \in \mathfrak{H}(R)$ such that $h_n \in \ker\big(\mathfrak{H}(R) \to \mathfrak{H}(R_n)\big)$ and $f_n' = {}^{h_n h_{n-1} \dots h_1} f_n$. The sequence $h_n h_{n-1} \dots h_1$ converges to an element $h \in \mathfrak{H}(R)$ such that h and h_n agree in $\mathfrak{H}(R_n)$ for each $n \geq 1$ It follows that f and ${}^h f'$ agree in $\mathrm{Hom}(\mathfrak{G}, \mathfrak{H})(R_n)$ for each $n \geq 0$, so are equal.

(3) Using that $\mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R_{n+1}) \xrightarrow{\sim} \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R_n)$, we see that the map $\mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R) \to \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R/I)$ is injective. For the surjectivity a central homomorphism $u_0 : \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R/I)$ gives rise to coherent system of central homomorphims $u_n \in \mathbf{Homcent}(\mathfrak{G}, \mathfrak{H})(R_n)$. This system lifts uniquely in $u \in \mathbf{Hom}(\mathfrak{G}, \mathfrak{H})(R)$ and we have to show that u is central. We consider then the adjoint action of $\mathfrak{G}$ on $\mathfrak{H}$. By Theorem 9.2.1, $\mathfrak{H}^{\mathfrak{G}}$ is a closed group subscheme $\mathfrak{H}$ which is then of finite presentation. The closed immersion i satisfies $\mathfrak{H}^{\mathfrak{G}} \times_R R/I \xrightarrow{\sim} \mathfrak{H} \times_R R/I$ and $I = \mathrm{rad}(R)$

[Ma, th. 8.2]. Corollary 20.0.8 yields that i is étale, hence $\mathfrak{H}^{\mathfrak{G}}$ is open in $\mathfrak{H}$. Since it contains $\mathfrak{H} \times_R R/I$, we have $\mathfrak{H}^{\mathfrak{G}} = \mathfrak{H}$. Thus u is a central homomorphism.

(4) This is a special case of Theorem 10.1.4. □

10.4. **Rank in family.**

10.4.1. **Definition.** *Let H/k be an affine algebraic group defined over a field k. Denote by $\overline{k}$ an algebraic closure of k. We denote by* $\mathrm{rank}_{red}(H)$ *the (absolute) reductive rank of H, namely the maximal dimension of a split $\overline{k}$–torus of $H \times_k \overline{k}$.*

Similarly, we denote by $\mathrm{rank}_{red,cent}(H)$ *the (absolute) central reductive rank of H, namely the maximal dimension of a central split $\overline{k}$–torus of $H \times_k \overline{k}$.*

This definition does not depend of the choice of the closure; both ranks remain the same after an arbitrary field extension F/k.

10.4.2. **Theorem.** *Let $\mathfrak{H}/R$ be an affine smooth group scheme. Assume that R is noetherian. Then the map*

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longrightarrow & \mathbb{Z}_{\geq 0} \\ x & \mapsto & \mathrm{rank}_{red}(H \times_R \kappa(x)) \end{array}$$

is lower semi-continuous and idem for $\mathrm{rank}_{red-cent}$.

Proof. Firstly, we notice that we are authorised to make an extension R'/R such that $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective (and R' noetherian). Also the statement is of local nature, hence we can suppose that R is local with maximal ideal $\mathfrak{M}$ and residue field k. Let r be the rank of $\mathfrak{H} \times_R k$. Our assumption reads that there exists a finite field extension k'/k such that $\mathfrak{H} \times_R k'$ contains a k'–torus $\mathfrak{G}^r_{m,k'}$. There exists a finite flat local morphism of noetherian local rings $R \to R'$ inducing $k \to k'$ [EGA3, 10.3.1, 10.3.2]. Then R'/R is finite locally free and faithfully flat. Hence without lost of generality, we can assume that $\mathfrak{H}_k$ contains a k–torus $\mathfrak{G}^r_{m,k'}$. The completion $\widehat{R} = \varprojlim_n R/\mathfrak{M}^n$ is complete and separated for the $\mathfrak{M}$-adic topology, is noetherian and faithfully flat over R [Li, §1.3.3]. We are then allowed to replace R by $\widehat{R}$.

We fix the closed immersion $f : \mathfrak{G}^r_{m,k} \to \mathfrak{H}_k$. By Corollary 10.3.3.(1), it lifts to an homomorphism $\widetilde{f} : \mathfrak{G}^r_{m,\widehat{R}} \to \mathfrak{H} \times_R \widehat{R}$ which is a monomorphism (Cor. 10.3.3.(4)). For each $y \in \mathrm{Spec}(\widehat{R})$, we have then

$$\mathrm{rank}_{red}(\mathfrak{H} \times_R \kappa(y)) \geq r$$

as desired.

The second statement follows similarly of Corollary 10.3.3.(3). □

Reductive group schemes

11. Reductive group schemes

If k is an algebraically closed field, an affine algebraic group G/k is redutive if it is smooth connected and if its unipotent radical is trivial [Sp, §8].

11.0.3. **Definition.** *An affine R–group scheme $\mathfrak{G}$ is reductive if it satisfies the two following requirements:*

(1) $\mathfrak{G}/R$ is smooth;

(2) For each $x \in \mathrm{Spec}(R)$, the geometric fiber $\mathfrak{G} \times_R \overline{\kappa}(x)$ is reductive where $\overline{\kappa}(x)$ stands for an algebraic closure of the residue field $\kappa(x)$.

11.0.4. **Remark.** A naive approach could be to consider the unipotent radical of $\mathfrak{G}$ but this object does not exist ! The problem occurs already over a non-perfect field, see the introduction of [CGP]. However we shall see later an equivalent definition.

11.0.5. **Examples.** (1) The diagonalizable group $\mathfrak{D}(\mathbb{Z}^r) = \mathfrak{G}_{m,R}^r$ is a reductuve group scheme, the linear group GL_n/R is a reductive group scheme and SL_n as well.
(2) A fibered R–product of reductive group schemes is reductive.

Reductivity is stable under base change of the base ring. Also it is an open property among the smooth affine groups with connected fibers [SGA3, XIX.2.6]. We can already prove a useful stability fact.

11.0.6. **Proposition.** *Let $\mathfrak{H}/R$ be a reductive group scheme and let $f : \mathfrak{T} = \mathfrak{G}_m^r \to \mathfrak{H}$ be a homomorphism. Then the centralizer $\mathfrak{H}^{\mathfrak{T}}/R$ is a reductive group scheme.*

Proof. We know that $\mathfrak{G}^{\mathfrak{T}}/R$ is a smooth group scheme (Th. 9.2.1) so satisfies the first requirement. For the second one, we are reduced to the case of an algebraically closed field. In this case, see [Bo, §13.17]. $\square$

11.0.7. **Definition.** *Let $\mathfrak{H}/R$ be an affine group scheme and let $i : \mathfrak{T} = (\mathfrak{G}_{m,R})^r \to \mathfrak{H}$ be a monomorphism. We say that $\mathfrak{T}$ is a maximal (resp. central maximal) R–torus of $\mathfrak{H}$ is for each $x \in \mathrm{Spec}(R)$, $\mathfrak{T} \times_R \overline{\kappa}(x)$ is a maximal (resp. central maximal) $\overline{\kappa}(x)$–torus of $\mathfrak{H} \times_R \overline{\kappa}(x)$.*

As in the field case, we have the following characterization of maximal tori.

11.0.8. **Proposition.** *Assume that R is noetherian. Let $\mathfrak{H}/R$ be an affine reductive group scheme and let $i : \mathfrak{T} = (\mathfrak{G}_{m,R})^r \to \mathfrak{H}$ be a monomorphism. Then the following are equivalent:*

(1) $i : \mathfrak{T} \to \mathfrak{H}$ is a maximal torus;

(2) $\mathfrak{T} \xrightarrow{\sim} \mathfrak{H}^{\mathfrak{T}}$.

Note in particular that i is a closed immersion.

Proof. From the field case, the map $f : \mathfrak{T} \to \mathfrak{H}^{\mathfrak{T}}$ is such that for each $x \in \mathrm{Spec}(R)$, $f_x = f \times_R \kappa(x)$ is an isomorphism. Since $\mathfrak{T}$ is smooth, the fiberwise criterion 20.0.6 yields that f is an isomorphism. $\square$

We continue with the following local statement.

11.0.9. **Proposition.** *Let R be a noetherian ring equipped with an ideal I. We assume that R is complete and separated for the I-adic topology. We assume that $\mathfrak{H} \times_R R/I$ admits a maximal R/I–torus $\mathfrak{G}^r_{m,R/I}$ (resp. central maximal R/I–torus $\mathfrak{G}^s_{m,R/I}$).*

(1) $\mathfrak{H} \times_R R$ admits a maximal (resp. central maximal) torus $\mathfrak{G}^r_{m,R}$ (resp. $\mathfrak{G}^s_{m,R}$).

(2) For each $x \in \mathrm{Spec}(R)$, we have $\mathrm{rank}_{red}(\mathfrak{H} \times_R \kappa(x)) = r$ and $\mathrm{rank}_{red}(\mathfrak{H} \times_R \kappa(x)) = s$.

In other words, $\mathfrak{H} \times_R R/I$ is split iff $\mathfrak{H}$ is split.

Proof. We do only the case of the reductive rank since the other case is similar.

(1) We are given a monomorphism $f_0 : \mathfrak{G}^r_{m,R/I} \to \mathfrak{H} \times_R R/I$ which is a maximal R/I–torus of $\mathfrak{G}$. According to Corollary 10.3.3.(1), it lifts to an R-homomorphism $f : \mathfrak{T} = \mathfrak{G}^r_m \to \mathfrak{H}$. We consider the R–subgroup centralizer $\mathbf{Cent}_{\mathfrak{G}}(\mathfrak{T}) = \mathfrak{H}^{\mathfrak{T}}$ which is reductive according to Proposition 11.0.6.

Now the R–map $f : \mathfrak{T} \to \mathfrak{G}^{\mathfrak{T}}$ is such that $f_{R/I}$ is an isomorphism by Proposition 11.0.8. Both schemes are smooth and again we notice that $I = \mathrm{rad}(R)$. We can apply then the trick 20.0.8, it yields that f is étale. But f is a monomorphism, hence f is an open immersion. Its image contains $\mathfrak{H}^{\mathfrak{T}} \times_R R/I$, so is $\mathfrak{H}^T$. Thus $\mathfrak{T} \xrightarrow{\sim} \mathfrak{H}^T$ and $\mathfrak{T}$ is then a maximal R–torus.

(2) For each $x \in \mathrm{Spec}(R)$, $\mathfrak{T} \times_R \kappa(x)$ is a maximal $\kappa(x)$–torus of $\mathfrak{H} \times_R \kappa(x)$ whence the result. $\square$

This enables us to improve the "lower continuity" theorem (i.e. th. 10.4.2) in the reductive case.

11.0.10. **Corollary.** *Let $\mathfrak{H}/R$ be an affine smooth group scheme. Assume that R is noetherian. Then the map*

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longrightarrow & \mathbb{Z}_{\geq 0} \\ x & \mapsto & \mathrm{rank}_{red}(H \times_R \kappa(x)) \end{array}$$

is continuous and idem for $\mathrm{rank}_{red-cent}$.

The proof goes along the same lines.

12. Limit groups

This part is mainly taken from [CGP, §2.1] and [GP3, §15].

12.1. **Limit functors.** Let $\mathfrak{X}/R$ be a affine scheme equipped with an action $\lambda : \mathfrak{G}_m \to \mathrm{Aut}(\mathfrak{X})$. We define the R–subfunctor $\mathfrak{X}_\lambda$ of $h_{\mathfrak{X}}$ by

$$\mathfrak{X}_\lambda(S) = \Big\{ x \in \mathfrak{X}(S) \mid \lambda(t) \, . \, x \in \mathfrak{X}(S[t]) \, \subset \, \mathfrak{X}(S[t, t^{-1}]) \Big\}$$

for each S/R. It is called the limit functor of $\mathfrak{X}$ with respect to λ since $\mathfrak{X}_\lambda(R)$ consists in the elements $x \in \mathfrak{X}(R)$ such that $\lambda(t).x$ has a limit when $t \mapsto 0$.

12.1.1. **Lemma.** *The R–functor $\mathfrak{X}_\lambda$ is representable by a closed R–subscheme of $\mathfrak{X}$.*

Proof. It is similar to that of Proposition 9.1.1. We consider the decomposition in eigenspaces

$$R[\mathfrak{X}] = \bigoplus_{n \in \mathbb{Z}} R[\mathfrak{X}]_n.$$

We denote by I the ideal of $R[\mathfrak{X}]$ generated by $\bigoplus_{n<0} R[\mathfrak{X}]_n$. We let the reader to check that the closed subscheme $\mathrm{Spec}(R[\mathfrak{X}]/I)$ does the job. $\square$

We denote by $\mathfrak{X}^\lambda$ the fixed point locus for the action. Clearly $\mathfrak{X}^\lambda$ is a $\mathfrak{G}_m$–subscheme of $\mathfrak{X}$ and $\mathfrak{X}^\lambda$ is an R–subscheme of $\mathfrak{X}_\lambda$. The specialization at 0 induces an R-map $q^\dagger : \mathfrak{X}_\lambda \to \mathfrak{X}$.

12.1.2. **Lemma.** *(1) $\mathfrak{X}^\lambda = \mathfrak{X}_\lambda \times_{\mathfrak{X}} \mathfrak{X}^{-\lambda}$.*

(2) The map $q^\dagger$ factorizes by X^λ. It defines then an R–map $q : \mathfrak{X}_\lambda \to \mathfrak{X}^\lambda$ and the composite $\mathfrak{X}^\lambda \to \mathfrak{X}_\lambda \xrightarrow{q} \mathfrak{X}^\lambda$ is the identity.

Proof. (1) If $x \in \mathfrak{X}_\lambda(R) \cap \mathfrak{X}^{-\lambda}(R)$, we have $\lambda(t).x \in \mathfrak{X}(R[t]) \cap \mathfrak{X}(R[t^{-1}]) = \mathfrak{X}(R)$. Hence $\lambda(t).x = x$ and $x \in \mathfrak{X}^\lambda(R)$.

(2) Let $x \in \mathfrak{X}_\lambda(R)$ and put $x' = q(x) \in \mathfrak{X}(R)$. For each $a \in R^\times$, we have $\lambda(at).x = \lambda(a)(\lambda(t).x)$. By doing $t \mapsto 0$, we get that $x' = \lambda(a).x'$. Hence $R^\times.x' = x'$. The same holds for each R–extension S/R, so we conclude that $x' \in \mathfrak{X}^\lambda(R)$. $\square$

12.2. **The group case.** We consider now the case of a group homomorphism $\lambda : \mathfrak{G}_m \to \mathfrak{G}$ where $\mathfrak{G}$ is an affine R–group scheme. We denote by $\mathfrak{P}_{\mathfrak{G}}(\lambda) = \mathfrak{G}_\lambda$. We have then an R-homomorphism $\mathfrak{P}_{\mathfrak{G}}(\lambda) \to \mathfrak{Z}_{\mathfrak{G}}(\lambda)$ which is split. We denote by $\mathfrak{U}_{\mathfrak{G}}(\lambda) = \ker(q)$ and we have then

$$\mathfrak{P}_{\mathfrak{G}}(\lambda) = \mathfrak{U}_{\mathfrak{G}}(\lambda) \rtimes \mathfrak{Z}_{\mathfrak{G}}(\lambda).$$

For each ring S/R, $\mathfrak{P}_{\mathfrak{G}}(\lambda)(S)$ (resp. $\mathfrak{U}_{\mathfrak{G}}(\lambda)(S)$) consists in the $g \in \mathfrak{G}(S)$ such that $\lambda(t) \, g \, \lambda(t^{-1})$ admits a limit (resp. converges to 1) when $t \mapsto 0$. The group scheme $\mathfrak{P}_{\mathfrak{G}}(\lambda)/R$ is called the limit group scheme attached to λ.

12.2.1. **Example.** If we take the diagonal map $\lambda(t) = (t^{a_1}, \dots, t^{a_1}, t^{a_2} \dots t^{a_2}, \dots, t^{a_r}, \dots, t^{a_r})$ in $\mathrm{GL}_{m_1+\dots+m_r}$ with respective multiplicities $m_1, .., m_r$ and $a_1 < a_2 \dots < a_r$, we find

$$\mathfrak{P}_{\mathrm{GL}_d}(\lambda) = \left(\begin{array}{c|c|c|c|c} A_{1,1} & A_{1,2} & \dots & \dots & A_{1,r} \\ \hline 0 & A_{2,2} & \dots & \dots & A_{2,r} \\ \hline 0 & 0 & A_{3,3} & \dots & . \\ \hline \dots & \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & 0 & A_{r,r} \end{array} \right).$$

From the functor viewpoint, we have

$$\mathrm{Lie}(\mathfrak{P}_\lambda(\mathfrak{G})) = \mathrm{Lie}(\mathfrak{G})(R)_{\geq 0}; \ \ \mathrm{Lie}(\mathfrak{U}_\lambda(\mathfrak{G})) = \mathrm{Lie}(\mathfrak{G})(R)_{>0}.$$

Hence $\mathrm{Lie}(\mathfrak{P}_\lambda(\mathfrak{G})) = \mathrm{Lie}(\mathfrak{Z}_{\mathfrak{G}}(\lambda))(R)_{\geq 0} \oplus \ \mathrm{Lie}(\mathfrak{U}_\lambda(\mathfrak{G}))$. Note also that the product R- map

$$i_\lambda : \mathfrak{U}_{\mathfrak{G}}(-\lambda) \times \mathfrak{P}_{\mathfrak{G}}(\lambda) \to \mathfrak{G}$$

is a monomorphism since $\mathfrak{P}_{\mathfrak{G}}(-\lambda) \times_{\mathfrak{G}} \mathfrak{P}_{\mathfrak{G}}(\lambda) = \mathfrak{Z}_{\mathfrak{G}}(\lambda)$. This map plays an important role in the theory. In the same flavour as Theorem 9.2.1, we have the following fact.

12.2.2. **Theorem.** [CGP, 2.1.8] *Assume that $\mathfrak{G}$ is smooth.*

(1) The R-group schemes $\mathfrak{P}_\lambda(\mathfrak{G})$ and $\mathfrak{U}_\lambda(\mathfrak{G})$ are smooth. Furthermore for each $s \in \mathrm{Spec}(R)$, $\mathfrak{U}_\lambda(\mathfrak{G})_{\kappa(s)}$ is a split unipotent group.

(2) The monomorphism i_λ above is an open immersion.

We skip the proof which is quite technical.

12.3. **Parabolic and Borel subgroup schemes.**

12.3.1. **Definition.** *Let $\mathfrak{G}/R$ be a reductive group scheme. An R-subgroup scheme $\mathfrak{P}$ of $\mathfrak{G}$ is parabolic subgroup (resp. a Borel subgroup) if it satifies the two requirements:*

(1) $\mathfrak{P}$ is smooth;

(2) for each $s \in \mathrm{Spec}(R)$, $\mathfrak{G} \times_R \overline{\kappa(s)}$ is a parabolic (resp. a Borel) subgroup of $\mathfrak{G} \times_R \overline{\kappa(s)}$.

12.3.2. **Corollary.** *Let $\lambda : \mathfrak{G}_m \to \mathfrak{G}$ be a homomorphism in a reductive group scheme. Then $\mathfrak{P}_{\mathfrak{G}}(\lambda)$ is an R–parabolic subgroup.*

This follows from the field case [Sp, §15.1] since $\mathfrak{P}_{\mathfrak{G}}(\lambda)$ is smooth. By the way, in the field case, each parabolic subgroup is of this shape and this can be extended.

12.3.3. **Lemma.** *Assume that $(R, \mathfrak{M}, k)$ is noetherian local and let $\mathfrak{G}/R$ be a reductive group scheme and let $\mathfrak{P}/R$ be a parabolic subgroup scheme.*

(1) Let $\lambda : \mathfrak{G}_m \to \mathfrak{P}$ be a homomorphism. Then the following are equivalent:

(i) $\mathfrak{P}_{\mathfrak{G}}(\lambda) = \mathfrak{P}$;

(ii) $\mathfrak{P}_{\mathfrak{G}}(\lambda) \times_R k = \mathfrak{P} \times_R k$.

(2) There exists $\lambda \in \mathrm{Hom}_{\widehat{R}}(\mathfrak{G}_{m,\widehat{R}}, \ \mathfrak{P}_{\widehat{R}})$ such that $\mathfrak{P}_{\mathfrak{G}_{\widehat{R}}}(\lambda) = \mathfrak{P}_{\widehat{R}}$.

Proof. (1) We consider the closed immersion $i : \mathfrak{P}_{\mathfrak{P}}(\lambda) \to \mathfrak{P}$. The map i_k is an isomorphism and both group schemes are smooth. By Corollary 20.0.8, we get that $i_{\widehat{R}}$ is an isomorphism. By faithfully flat descent, i_R is then an isomorphism. In the same way, one show that $\mathfrak{P}_{\mathfrak{P}}(\lambda) \xrightarrow{\sim} \mathfrak{P}_{\mathfrak{G}}(\lambda)$. We conclude that $\mathfrak{P} = \mathfrak{P}_{\mathfrak{G}}(\lambda)$ as desired.

(2) There exists $\lambda_0 \in \mathrm{Hom}_{k-gp}(\mathbb{G}_{m,k}, \mathfrak{P}_k)$ such that $\mathfrak{P}_k = \mathfrak{P}_{\mathfrak{G}_k}(\lambda_0)$. Since $\mathrm{Hom}_{\widehat{R}}(\mathbb{G}_{m,\widehat{R}}, \mathfrak{P}_{\widehat{R}}) \to \mathrm{Hom}_{k-gp}(\mathbb{G}_{m,k}, \mathfrak{P}_k)$ is onto (Cor. 10.3.3.(1)), we can pick a lift $\lambda : \mathbb{G}_{m,\widehat{R}} \to \mathfrak{P}_{\widehat{R}}$ of λ_0. We apply then (1). $\square$

12.3.4. **Remarks.** (1) Assertion (2) is a special case of the same statement without completion, see [GP3, 15.5]. This comes later in the theory.

(2) If $\mathfrak{P}(\lambda)$ is a Borel subgroup, then $\mathfrak{Z}_{\mathfrak{G}}(\lambda)$ is a maximal R–torus. It is true that a Borel subgroup of $\mathfrak{G}/R$ contains a maximal R-torus [SGA3, XXVI.2.3].

(3) The method of the lemma can be used also for lifting parabolic subgroups from the residue field to $\widehat{R}$.

13. Root data, type of reductive group schemes

Root systems come from the study of reductive Lie algebras and for studying reductive groups, we need a richer datum which permits to distinguish for example $\mathrm{SL}_{n,\mathbb{C}}$ of $\mathrm{GL}_{n,\mathbb{C}}$ or $\mathrm{PGL}_{n,\mathbb{C}}$ We follow here verbatim [Sp, §7.4], see also [SGA3, XXI].

13.1. **Definition.** A root datum is a quadruple $\Psi = \Psi(A, \mathcal{R}, A^\vee, \mathcal{R}^\vee)$, where

(a) A and $A^\vee$ are free abelian groups of finite rank, in duality by a pairing $A \times A^\vee \to \mathbb{Z}$, denote by $\langle\, , \,\rangle$;

(b) $\mathcal{R}$ (the roots) and $\mathcal{R}^\vee$ (the coroots) are finite subsets of A and $A^\vee$ and we are given a bijection $\alpha \to \alpha^\vee$ of $\mathcal{R}$ onto $\mathcal{R}^\vee$.

For each $\alpha \in \mathcal{R}$, we define endomorphims s_α and $s_\alpha^\vee$ of A and $A^\vee$ by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \,.\, \alpha; \qquad s_\alpha^\vee(x) = y - \langle \alpha, y \rangle \,.\, \alpha^\vee.$$

The following axioms are imposed.

(RD1) For each $\alpha \in R$, $\langle \alpha, y \rangle = 2$;

(RD2) For each $\alpha \in R$, then $s_\alpha \mathcal{R} = \mathcal{R}$ and $s_\alpha^\vee \mathcal{R}^\vee = \mathcal{R}^\vee$.

The first axiom implies that $s_\alpha^2 = 1$ and $s_\alpha(\alpha) = -\alpha$. The Weyl group $W(\Psi)$ is the subgroup of $\mathrm{GL}(A)$ generated by the s_α ($\alpha \in \mathcal{R}$). Let us give here some terminology.

(a) We say that Ψ is *reduced* if for each $\alpha \in \mathcal{R}$, $c \in \mathbb{Q}$ and $c\alpha \in \mathcal{R}$, then $c = \pm 1$.

(b) If $\Psi = \Psi(A, \mathcal{R}, A^\vee, \mathcal{R}^\vee)$ is a root datum, $\Psi^\vee = \Psi(A^\vee, \mathcal{R}^\vee, A, \mathcal{R})$ is a root datum called the dual root datum (or the Langlands dual root datum).

(c) A root datum $\Psi(A, \mathcal{R}, A^\vee, \mathcal{R}^\vee)$ is semisimple if $\mathcal{R}$ generates the vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Furthermore, it is adjoint (resp. simply connected) if $\mathcal{R}$ generates A (resp. $\mathcal{R}^\vee$ generates $A^\vee$).

(d) Morphisms of root data: to be written.

13.1.1. **Remark.** Denote by Q the subgroup of A generated by $\mathcal{R}$. In $\mathcal{R} \neq \emptyset$, then $\mathcal{R}$ is a root system of $Q \otimes_{\mathbb{Z}} \mathbb{R}$ in the sense of [Bbk3, VI.1]. Furthermore W is a subgroup of $\mathrm{GL}(Q \otimes_{\mathbb{Z}} \mathbb{R})$ and is then a finite group.

13.2. **Geometric case.** Let k be an algebraically closed field. Let G/k be a reductive group. Let $T \subset G$ be a maximal torus. Recall that we can attach to a root datum $\Psi(G,T) = (\widehat{T}, \mathcal{R}, (\widehat{T})^0, \mathcal{R}^\vee)$ where $\widehat{T}$ is the character lattice of T and $(\widehat{T})^0$ its dual.

The root datum $\Psi(G,T)$ is reduced. Since the maximal tori of G are conjugated, $\Psi(G,T)$ is independent of the choice of T and we denote it simply by $\Psi(G)$. The main results (showed in Chernousov's lectures) are the following:

(1) (Unicity theorem) Two reductive k–groups G, G' are isomophic if and only their root data $\Psi(G)$ and $\Psi(G')$ are isomorphic.

(2) (Existence theorem) If Ψ is a reduced root datum, there exists a reductive k-group G such that $\Psi(G) \cong \Psi$.

13.3. **Root datum $\Psi(\mathfrak{G}, \mathfrak{T})$.** Over a ring, it is technically speaking more delicate to define a root datum with a maximal torus $i : \mathfrak{T} = \mathfrak{G}^r_{m,R} \to \mathfrak{G}$. For simplicity, we assume R connected.

We consider the adjoint action of $\mathfrak{G}$ on the R–module $\mathfrak{g} = \mathrm{Lie}(\mathfrak{G})(R)$ (which is f.g. projective). Its restriction to the torus $\mathfrak{T}$ decomposes as

$$\mathfrak{g} = \bigoplus_{\alpha \in \widehat{\mathfrak{T}}} \mathfrak{g}_\alpha.$$

13.3.1. **Definition.** *Assume that $R \neq 0$. A root α for $(\mathfrak{H}, \mathfrak{T})$ is a character $\alpha : \mathfrak{T} \to \mathfrak{G}_m$ such that*

(i) α is everywhere non trivial, that is $\alpha_x \neq 0$ for each $x \in \mathrm{Spec}(R)$.

(ii) The eigenspace $\mathrm{Lie}(\mathfrak{H})(R)_\alpha$ is an invertible R–module (i.e. projective of rank one).

13.3.2. **Lemma.** *Let α be a root for $(\mathfrak{G}, \mathfrak{T})$. We define $\mathfrak{T}_\alpha = \ker(\alpha)$ and $\mathfrak{Z}_\alpha = \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{T}_\alpha)$. We have*

$$\mathrm{Lie}(\mathfrak{Z}_\alpha)(R) = \mathrm{Lie}(\mathfrak{T})(R) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

and $-\alpha$ is a root as well.

Proof. The group scheme $\mathfrak{Z}_\alpha$ is smooth and its Lie algebra is $H^0_0(\mathfrak{T}_\alpha, \mathfrak{g})$ by Remark 9.2.3. From the field case [Bo, 13.18], for each point $x \in \mathrm{Spec}(R)$, we have a decomposition

$$\mathrm{Lie}(\mathfrak{Z}_\alpha)(\kappa(x)) = \mathrm{Lie}(\mathfrak{T})(\kappa(x)) \oplus \mathfrak{g}_\alpha \otimes_R \kappa(x) \oplus \mathfrak{g}_{-\alpha} \otimes_R \kappa(x)$$

and $\mathfrak{g}_{-\alpha} \otimes_R \kappa(x)$ is one dimensional. By the Nakayama lemma, the natural map of f.g. projective R–modules

$$\mathrm{Lie}(\mathfrak{T})(R) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \to \mathrm{Lie}(\mathfrak{Z}_\alpha)(R)$$

is an R–isomorphism. Furthermore $\mathfrak{g}_{-\alpha}$ is locally free of rank one. □

The next hard thing is the "integration" of the Lie algebra $\mathfrak{g}_\alpha$.

13.3.3. **Theorem.** *Let α be a root for $(\mathfrak{G}, \mathfrak{T})$.*

(1) There exists a unique R-group homomorphism

$$\exp_\alpha : \mathfrak{W}(\mathfrak{g}_\alpha) \to \mathfrak{G}$$

inducing the canonical inclusion $\mathfrak{g}_\alpha \to \mathfrak{g}$ and which is $\mathfrak{T}$-equivariant.

(2) The map $\exp_\alpha$ is a closed immersion, factors trough $\mathfrak{Z}_\alpha$, and its formation commutes with base change.

(3) The multiplication map $\mathfrak{W}(\mathfrak{g}_{-\alpha}) \times_R \mathfrak{T} \times_R \mathfrak{W}(\mathfrak{g}_\alpha) \to \mathfrak{Z}_\alpha$ is an open immersion.

We postpone in §17.3 the proof of the following characterisation of rank one vector group scheme since it involves descent techniques.

13.3.4. **Proposition.** *Let $\mathfrak{U}/R$ be an affine smooth group scheme whose geometric fibers are rank one additive groups. We assume that $\mathfrak{U}/R$ is equipped with an action of $\mathfrak{G}_m$ such that the $\mathfrak{G}_m$–module $\mathrm{Lie}(\mathfrak{U})(R)$ is non trivial everywhere*[5]*. Then there exists an invertible R-module L/R such that $\mathfrak{W}(L) \cong \mathfrak{U}$.*

We can sketch the existence part of the proof of Theorem 13.3.3 (see [C, §4.1]). Up to localize, we can assume that $\alpha : \mathfrak{T} \to \mathfrak{G}_m$ is "constant" namely is given by a (non trivial) element of $\mathbb{Z}^r$. The idea is to choose $\lambda \in (\mathbb{Z}^r)^\vee$ such that $\langle \alpha, \lambda \rangle > 0$ and to consider the homomorphism $\lambda : \mathfrak{G}_m \to \mathfrak{T} \to \mathfrak{Z}_\alpha$. It gives rise to the limit R–groups $\mathfrak{P}_{\mathfrak{Z}_\alpha}(\pm\lambda)$ and the R-subgroups $\mathfrak{U}_{\mathfrak{Z}_\alpha}(\pm\lambda)$. By taking into account Lemma 13.3.2, we have

$$\mathrm{Lie}(\mathfrak{P}_{\mathfrak{Z}_\alpha}(\pm\lambda))(R) = \mathrm{Lie}(\mathfrak{T})(R) \oplus \mathfrak{g}_{\pm\alpha},$$
$$\mathrm{Lie}(\mathfrak{U}_{\mathfrak{Z}_\alpha}(\pm\lambda))(R) = \mathfrak{g}_{\pm\alpha}.$$

Furthermore $\mathfrak{U}_{\mathfrak{Z}_\alpha}(\pm\lambda)$ is equipped with an action of $\mathfrak{G}_m$ within λ hence Proposition 13.3.4 yields that there are both rank one vector group schemes. Note that fact (3) follows from Theorem 12.2.2.(2).

The image $\mathfrak{U}_\alpha/R$ of $\exp_\alpha$ is called the root subgroup relative to α. We come to the definition of coroots.

13.3.5. **Theorem.** *Let $\alpha \in \widehat{T}$ be a root.*

(1) There exists a morphism $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to R$, $(X, Y) \mapsto XY$ and a cocharacter $\alpha^\vee : \mathfrak{G}_{m,R} \to \mathfrak{H}$ such that for each S/R, each $X \in \mathfrak{g}_\alpha \otimes_E S$, $Y \in \mathfrak{g}_\alpha \otimes_E S$ we have

$$\exp_\alpha(X)\, \exp_{-\alpha}(Y) \in \Omega_a(S) \Longleftrightarrow 1 - XY \in S^\times$$

[5]that is $\mathrm{Lie}(\mathfrak{U})(\kappa(x))_0 = 0$ for each $x \in \mathrm{Spec}(R)$.

and under this condition we have

$$\exp_\alpha(X)\,\exp_{-\alpha}(Y) = \exp_{-\alpha}\Big(\frac{Y}{1-XY}\Big)\,\alpha^\vee(1-XY)\,\exp_\alpha\Big(\frac{X}{1-XY}\Big)$$

(2) The morphism $(X,Y) \to XY$ and $\alpha^\vee$ are uniquely determined by these conditions.

(3) The morphism $(X,Y) \to XY$ is an R–isomorphism and $\langle \alpha^\vee, \alpha\rangle = 2$.

This statement define the coroot $\alpha^\vee$ attached to α. We denote by $\mathcal{R}$ the set of roots and by $\mathcal{R}^\vee$ the set of coroots. Both are non necessarily constant morphisms, but are locally constant (we have to be careful with connectness issues).

13.3.6. **Definition.** *We say that the reductive group scheme $\mathfrak{G}/R$ is split if it admits a maximal R-torus $\mathfrak{G}_{m,R}^r$ such that the roots and the coroots are constant morphisms and also such that each eigenspace $\mathfrak{g}_\alpha$ is a free R–module of rank one.*

13.3.7. **Remark.** (1) If R is connected and $\mathrm{Pic}(R) = 1$, a reductive group scheme $\mathfrak{G}/R$ is split if it admits a maximal R-torus $\mathfrak{G}_{m,R}^r$.

(2) In the definition, we say that $\mathfrak{G}_{m,R}^r$ is a splitting torus of $\mathfrak{G}$. For a ring R general enough, GL_n contains maximal split tori which are not splitting it, see Remark 18.4.3.

We assume that $\mathfrak{G}/R$ is split. We see immediately that $\Psi(\mathfrak{G},\mathfrak{T}) = (\widehat{\mathfrak{T}}, \mathcal{R}, (\widehat{\mathfrak{T}})^0, \mathcal{R}^\vee)$ is a root data.

13.3.8. **Lemma.** *Assume that $\mathfrak{G}/R$ is split. Then the isomorphism class of $\Psi(\mathfrak{G},\mathfrak{T})$ does not depend of the choice of $\mathfrak{T}$.*

Proof. It is true for fields, so we have only to specialise at some maximal ideal of R. □

Hence we can attach to a split group scheme over R a root datum. The unicity and existence questions analogous with the field case were achieved of Demazure's thesis [D] and need descent techniques to be discussed.

13.4. **Center.** We record that the center of a split group scheme has the expected shape.

13.4.1. **Proposition.** *Let $\mathfrak{G}/R$ be a reductive split group scheme and let $\mathfrak{T} = \mathfrak{G}_m^r$ be a maximal torus of $\mathfrak{G}$. Then the center of $\mathfrak{G}$ is representable, it is the diagonalizable R-group*

$$\ker\Big(\mathfrak{T} \longrightarrow \prod_{\alpha\in\mathcal{R}} \mathfrak{G}_m\Big).$$

In particular, $\mathfrak{Z}(\mathfrak{G}) = 1$ iff the root datum $\Psi(\mathfrak{G},\mathfrak{T})$ is adjoint.

Proof. We can assume that R is noetherian. We define $\mathfrak{D}/R = \ker\Big(\mathfrak{T} \longrightarrow \prod_{\alpha\in R} \mathfrak{G}_m\Big)$. We define the "center of $\mathfrak{G}$" functor

$$C(S) = \mathrm{Ker}\Big(\mathfrak{G}(S) \overset{int}{\to} \mathrm{Aut}(\mathfrak{G})(S)\Big).$$

We have seen that $\mathfrak{T}$ is his own centralizer (prop. 11.0.8), so that C is a subfunctor of $\mathfrak{T}$. Also $C(S)$ acts trivially on $\mathrm{Lie}(\mathfrak{G})(S)$ for each S/R, so that the action of $C(S) \subset \mathfrak{D}(S)$.

We have proven that C is a subfunctor of $\mathfrak{D}$. For the converse, we need to prove that $\phi : \mathfrak{D} \to \mathfrak{G}$ is a central homomorphism. We shall use that the result holds over fields, see [Bo, §14.2].

Let $x \in \mathrm{Spec}(R)$ and denote by $\widehat{R}_x$ the completion. Since $\mathfrak{D} \times_{k(x)} R_x$ is central in $\mathfrak{G}_{k(x)}$, it lifts to a central homomorphism $\psi_x : \mathfrak{D} \times_R \widehat{R}_x \to \mathfrak{G} \times_R \widehat{R}_x$. according to Theorem 10.3.3.(3). But by assertion (2) of the same statement, ψ_x is $\mathfrak{G}(\widehat{R}_x)$–conjugated to $\phi_{\widehat{R}_x}$, so that $\mathfrak{D} \times_R \widehat{R}_x$ is central in $\mathfrak{G} \times_x \widehat{R}_x$. Since $\widehat{R}_x$ is faithfully flat over R_x, we conclude that $\mathfrak{D} \times_R R_x$ is central in $\mathfrak{G} \times_x R_x$. Thus $\mathfrak{D}$ is central in $\mathfrak{G}$. □

13.4.2. **Remark.**

Descent techniques

We do a long interlude for developping descent and sheafifications techniques. We use mainly the references [DG, Ro, Wa].

14. Flat sheaves

Our presentation is that of Demazure-Gabriel [DG, III] which involves only rings.

14.1. **Covers.** A fppf (flat for short) cover of the ring R is a ring S/R which is faithfully flat and of finite presentation[6] "fppf" stands for "fidèlement plat de présentation finie".

14.1.1. **Remarks.** (1) If $1 = f_1 + \dots + f_s$ is a partition of 1_R with $f_1, ... f_r \in R$, the ring $R_{f_1} \times \dots \times R_{f_r}$ is a Zariski cover of R and a fortiori a flat cover.

(2) If S_1/R and S_2/R are flat covers of R, then $S_1 \otimes_R S_2$ is a flat cover of R.

(3) If S/R is a flat cover of S and S'/S is a flat cover of S, then S'/R is a flat cover of R.

(4) Finite locally free extensions S/R are flat covers, in particular finite étale surjective maps are flat covers.

14.2. **Definition.** We consider an R-functor $F : \{R - Alg\} \to Sets$ It is called additive if the natural map $F(S_1 \times S_2) \to F(S_1) \times F(S_2)$ is bijective for all R–rings S_1, S_2.

For each R–ring morphism $S \to S'$, we can consider the sequence

$$F(S) \longrightarrow F(S') \underset{d_{2,*}}{\overset{d_{1,*}}{\rightrightarrows}} F(S' \otimes_S S') \quad .$$

A functor of $F : \{R - Alg\} \to Sets$ is a fppf sheaf (or flat sheaf) for short if it is additive, if $F(0) = \{\bullet\}$ and if for each R–ring S and each flat cover S'/S, the sequence

$$F(S) \longrightarrow F(S') \underset{d_{2,*}}{\overset{d_{1,*}}{\rightrightarrows}} F(S' \otimes_S S')$$

is exact. It means that the restriction map $F(S) \to F(S')$ is injective and its image consists in the sections $\alpha \in F(S')$ satisfying $d_{1,*}(\alpha) = d_{2,*}(\alpha) \in F(S' \otimes_S S')$.

Given an R–module M and S'/S as above, the theorem of faithfully flat descent states that we have an exact sequence of S–modules

$$0 \to M \otimes_R S \to (M \otimes_R S) \otimes_S S' \overset{d_{1,*} - d_{2,*}}{\longrightarrow} (M \otimes_R S) \otimes_S S' \otimes_S S' \quad .$$

[6] One may consider also not finitely presented covers, it is called *fpqc*, see [SGA3, IV] and [Vi].

This rephases by saying that the vector group functor $V(M)/R$ (which is additive) is a flat sheaf over $\mathrm{Spec}(R)$. A special case is the exactness of the sequence

$$0 \to S \to S' \overset{d_{1,*}-d_{2,*}}{\longrightarrow} S' \otimes_S S'.$$

If N is an R-module, it follows that the sequence of R–modules

$$0 \to \mathrm{Hom}_R(N, S) \to \mathrm{Hom}_R(N, S') \overset{d_{1,*}-d_{2,*}}{\longrightarrow} \mathrm{Hom}_R(N, S' \otimes_S S')$$

is exact. This shows that the vector R–group scheme $\mathfrak{W}(M)$ is a flat sheaf. More generally we have

14.2.1. **Proposition.** *Let $\mathfrak{X}/R$ be an affine scheme. Then the R–functor of points $h_{\mathfrak{X}}$ is a flat sheaf.*

Proof. The functor $h_{\mathfrak{X}}$ is additive. We are given an R–ring S and a flat cover S'/S. We write the sequence above with the R-module $R[\mathfrak{X}]$. It reads

$$0 \to \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S) \to \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S') \overset{d_{1,*}-d_{2,*}}{\to} \mathrm{Hom}_{R_mod}(R[\mathfrak{X}], S'\otimes_S S').$$

It follows that $\mathfrak{X}(S)$ injects in $\mathfrak{X}(S')$ and identifies with $\mathrm{Hom}_{R-rings}(R[\mathfrak{X}], S') \cap \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S)$. Hence the exact sequence

$$\mathfrak{X}(S) \longrightarrow \mathfrak{X}(S') \underset{d_{2,*}}{\overset{d_{1,*}}{\rightrightarrows}} \mathfrak{X}(S' \otimes_S S') \quad .$$

□

14.2.2. **Remark.** More generally, the proposition holds with a scheme $\mathfrak{X}/R$, see [Ro, 2.4.7] or [Vi, 2.5.4].

14.2.3. **Examples.** If E, F are flat sheaves over R, the R–functor $\mathrm{Hom}(E, F)$ of morphisms from E to F is a flat sheaf. Also the R–functor $\mathrm{Isom}(E, F)$ is a flat sheaf and as special case, the R–functor $\mathrm{Aut}(F)$ is a flat sheaf.

14.3. **Monomorphisms and epimorphisms.** A morphism $u : F \to E$ of flat sheaves over R is a monomorphism if $F(S) \to E(S)$ is injective for each S/R. It is an epimorphism if for each S/R and each element $e \in E(S)$, there exists a flat cover S'/S and an element $f' \in F(S')$ such that $e_{|S'} = u(f')$.

A morphism of flat sheaves which is a monomorphism and and an epimorphism is an isomorphism (exercise, solution [SGA3, IV.4.4]).

We say that a sequence of flat sheaves in groups over R $1 \to F_1 \to F_2 \to F_3 \to 1$ is exact if the map of sheaves $F_2 \to F_3$ is an epimorphism and if for each S/R the sequence of abstract groups $1 \to F_1(S) \to F_2(S) \to F_3(S)$ is exact.

14.3.1. **Examples.** (1) For each $n \geq 1$, the Kummer sequence $1 \to \mu_{n,R} \to \mathfrak{G}_{m,R} \overset{f_n}{\to} \mathfrak{G}_{m,R} \to 1$ is an exact sequence of flat sheaves where f_n is the n–power map. The only thing to check is the epimorphism property. Let S/R be a ring and $a \in \mathfrak{G}_m(S) = S^\times$. We put $S' = S[X]/(X^n - a)$, it is finite

free over S, hence is faithfully flat of finite presentation. Then $f_n(X) = a_{|S'}$ and we conclude that f_n is an epimorphism of flat sheaves.
(2) More generally, let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of f.g. abelian groups. Then the sequence of R–group schemes

$$1 \to \mathfrak{D}(A_3) \to \mathfrak{D}(A_2) \to \mathfrak{D}(A_1) \to 0$$

is exact.

14.4. **Sheafification.** Given an R-functor F, there is natural way to sheafify it in a flat functor $\widetilde{F}$. The first thing is to make the functor additive. For each Zariski cover $(S_j)_{j\in J}$ (J finite, eventually empty), we have a map

$$F(S) \to \prod_{j\in J} F(S_j)$$

with the convention $\prod\limits_{j\in\emptyset} = \{\bullet\}$. We define

$$F_{add}(S) = \prod_{j\in J} F(S_j)$$

where the limit is taken on Zariski covers of S. By construction, F_{add} is an additive functor and there is a natural map $F \to F_{add}$

Now, for each S/R, we consider the "set" $\mathrm{Cov}(S)$ of flat covers[7]. Also if $f : S_1 \to S_2$ is an arbitrary R-ring map, the tensor product defines a natural map $f_* : \mathrm{Cov}(S_1) \to \mathrm{Cov}(S_2)$. We define then

$$\widetilde{F}(S) = \varinjlim_{I\subset \mathrm{Cov}(S)} \ker\Big(\quad \prod\nolimits_{i\in I} F_{add}(S_i) \quad \overset{d_{1,*}}{\underset{d_{2,*}}{\rightrightarrows}} \quad F_{ad}(S_i \otimes_S S_j) \Big)$$

where the limit is taken on finite subsets I of $\mathrm{Cov}(S)$. It is an R-functor since each map $f : S_1 \to S_2$ defines $f_* : \widetilde{F}(S_1) \to \widetilde{F}(S_2)$. We have also a natural mapping $u_F : F \to F_{add} \to \widetilde{F}$.

14.4.1. **Proposition.** *(1) For each R–functor F, the R–functor $\widetilde{F}$ is a flat sheave.*

(2) The functor $F \to \widetilde{F}$ is left adjoint to the forgetful functor applying a flat sheaf to its underlying R–functor. For each R–functor F and each flat sheaf E, the natural map

$$\mathrm{Hom}_{flat\, sheaves}(\widetilde{F}, E) \xrightarrow{\sim} \mathrm{Hom}_{R-functor}(F, E)$$

(applying a morphism $u : \widetilde{F} \to E$ to the composite $F \to \widetilde{F} \to E$) is bijective.

(1) follows essentially by construction [DG, III.1.8]. Note that in this reference, the two steps are gathered in one. For (2) one needs to define the inverse mapping. Observe that the sheafification of E is itself, so that the sheafification of $F \to E$ yields a natural morphism $\widetilde{F} \to E$.

[7] We do not enter in set-theoric considerations but the reader can check there is no problem there.

Given a morphism of flat R-sheaves $f : E \to F$, we can sheafify the functors

$$S \mapsto E(S)/R_f(S), \ \ S \mapsto \mathrm{Im}\big(E(S) \to F(S)\big),$$

where $R_f(S)$ is the equivalence relation defined by $f(S)$. We denote by $\mathrm{Coim}(f)$ and $\mathrm{Im}(f)$ their respective sheafifications. We have an induced mapping

$$f_* : \mathrm{Coim}(f) \to \mathrm{Im}(f)$$

between the coimage sheaf and the image sheaf. We say that f is strict when f_* is an isomorphism of flat sheaves.

14.4.2. **Lemma.** *If f is a monomorphism (resp. an epimorphism), then f is strict.*

In the first case, we have $E \xrightarrow{\sim} \mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$; in the second case, we have $\mathrm{coker}(f) \xrightarrow{\sim} \mathrm{Im}(f) \xrightarrow{\sim} F$, see [DG, III.1.2].

14.5. **Group actions, quotients sheaves and contracted products.** Let G be an R–group flat sheaf and let F be a flat sheaf equipped with a right action of G. The quotient functor is $Q(S) = F(S)/G(S)$ and its sheafification is denoted by F/G. It is called the quotient sheaf[8].

When G and F are representable, the natural question is to investigate whether the quotient sheaf Q is representable. It is quite rarely the case. A first evidence to that is the following fact.

14.5.1. **Proposition.** *We are given an affine group scheme $\mathfrak{G}$ and a monomorphism $\mathfrak{G} \to \mathfrak{H}$ into an affine group scheme. Assume that the quotient sheaf $\mathfrak{H}/\mathfrak{G}$ is representable by an R–scheme $\mathfrak{X}$. We denote by $p : \mathfrak{H} \to \mathfrak{X}$ the quotient map and by $\epsilon_X = p(1_{\mathfrak{G}}) \in \mathfrak{X}(R)$.*

(1) The R–map $\mathfrak{H} \times_R \mathfrak{G} \to \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H}$ is an isomorphism.

(2) The diagram

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{i} & \mathfrak{H} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{\epsilon_{\mathfrak{X}}} & \mathfrak{X} \end{array}$$

is carthesian.

(3) The map i is an immersion. It is a closed immersion iff $\mathfrak{X}/R$ is separated.

(4) $\mathfrak{G}/R$ is flat iff p is flat.

(5) $\mathfrak{G}/R$ is smooth iff p is smooth.

The general statement is [SGA3, VI_B.9.2].

[8] One can work in a larger setting, that of equivalence relations and groupoids, see [DG, §III.2].

Proof. (1) The map $\mathfrak{H}\times_R\mathfrak{G} \to \mathfrak{H}\times_{\mathfrak{X}}\mathfrak{H}$ is an monomorphism. Let us show that it is an epimorphism of flat sheaves. We are given S/R and $(h_1,h_2) \in \mathfrak{H}(S)^2$ such that $p(h_1) = p(h_2)$. There exists a flat cover S'/S and $g \in \mathfrak{G}(S')$ such that $h_{1|S'} = h_{2|S'}\, g$. Hence $g \in \mathfrak{G}(S') \cap \mathfrak{H}(S)$. Since i is a monomorphism, we conclude by descent that $g \in \mathfrak{G}(S)$ whence (h_1,h_2) comes from (h_1,g).

(2) It follows that the diagram

$$\begin{array}{ccccc}
\mathfrak{G} & \xrightarrow{1_{\mathfrak{H}}\times id} & \mathfrak{H}\times_R\mathfrak{G} \cong \mathfrak{H}\times_{\mathfrak{X}}\mathfrak{H} & \longrightarrow & \mathfrak{H} \\
\downarrow & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\
\mathrm{Spec}(R) & \xrightarrow{1_{\mathfrak{H}}} & \mathfrak{H} & \xrightarrow{p} & \mathfrak{X}
\end{array}$$

is carthesian as desired.

(3) If $\mathfrak{X}$ is separated, $\epsilon_{\mathfrak{X}}$ is a closed immersion and so is i.

(4) and (5). If p is flat (resp. smooth), so is i by base change. The converse is a consequence of "permanences properties" of failthfully flat descent. □

One very known case of representatiblity result is the following.

14.5.2. **Theorem.** *Let k be a field. Let H/k be an affine algebraic group and G/k be a closed subgroup. Then the quotient sheaf H/G is representable by a k–scheme of finite type X.*

One needs the following

14.5.3. **Proposition.** [DG, III.3.5.2] *Let G acts on a quasi-projective k–variety X. Let $x \in X(k)$ and denote by G_x the stabilizer of x.*

(1) The quotient G/G_x is representable by a quasi-projective k-variety.

(2) The orbit map induces an immersion $G/G_x \to X$.

It can be suitably generalized over rings, see [SGA3, XVI.2], by means of the theorem of Grothendieck-Murre.

Sketch of proof: We assume for simplicity that G is smooth, that is absolutely reduced. By faithfully flat descent, one can assume that k is algebraically closed.

(1) We know denote by X_0 the reduced subscheme of the schematic image of f_x. Since G is smooth, it is (absolutely) reduced and acts then on X_0. We know that the $X_0 \setminus G.x$ consists in orbits of smaller dimensions so that $G.x$ is an open subset of X_0. We denote it by U_x. We claim that the map $h_x : G \to U_x$ is faithfully flat. The theorem of generic flatness [DG, I.3.3.7] shows that the flat locus of h_x is not empty. By homogeneity, it is U_x, hence h_x is faithfully flat. Let us show now that it implies that U_x represents the orbit of x. The morphism $h_x : G \to U_x$ gives rise to a morphism of

k–sheaf $h_x^\dagger : G/G_x \to U_x$. Since the map $h_x : G \to U_x$ is faithfully flat, the morphism h_x is an epimorphism of flat sheaves[9].

$h_x^\dagger$ *is a monomorphism.* Let S be an R–ring and let $y_1, y_2 \in (G/G_x)(S)$ having same image u in $U_x(S)$. There exists a flat cover S'/S such that y_1 (resp. y_2) comes from some $g_1 \in G(S')$ (resp. g_2). Then $g_1 \,.\, x = g_2 \,.\, x \in U_x(S')$ so that $g_2^{-1} g_1 \in G_x(S')$. Thus $u_1 = u_2 \in (G/G_x)(S)$.

(2) By construction, U_x is locally closed in X. □

Theorem 14.5.2 follows then of the fact that G admits a representation V such that there exists a point $x \in \mathbf{P}(V)(k)$ such that $G = H_x$ [DG, II.2.3.5].

14.5.4. **Remark.** One interest of the Chevalley quotient is the fact it is universal. That is for each k–algebra R, $(H/G) \times_k R$ represents the quotient R–sheaf $(H \times_k R)/(G \times_k R)$. It can use as follows (see [CTS2, 6.12]). Assume we are given a closed immersion $\iota : \mathfrak{G} \to \mathfrak{H}$ of R-group schemes, a flat cover R'/R and a commutative square

$$\begin{array}{ccc} \mathfrak{G} \times_R R' & \xrightarrow{\iota_{R'}} & \mathfrak{H} \times_R R' \\ u \big\downarrow \cong & & v \big\downarrow \cong \\ G \times_k R' & \xrightarrow{i \times_k R'} & H \times_k R' \end{array}$$

where u, v are isomorphisms. We claim then that the quotient sheaf $\mathfrak{H}/\mathfrak{G}$ is representable by an R-scheme. According to Theorem 14.5.2, $\mathfrak{H}_{R'}/\mathfrak{G}_{R'}$ is representable by a quasi-projective R'-scheme which indeed descends to R.

14.6. **Contracted products.** We are given two flat R–sheaves in sets F_1, F_2 and and a flat sheaf G in groups. If F_1 (resp. F_2) is equipped with a right (resp. left) action of G, we have a natural right action of G on the product $F_1 \times F_2$ by $(f_1, f_2).g = (f_1 g, g^{-1} f_2)$. The sheaf quotient of $F_1 \times F_2$ under this action by G is denoted by $F_1 \wedge^G F_2$ and is called the contracted product of F_1 and F_2 with respect to G.

14.6.1. **Remark.** This construction occurs for group extensions. Let $1 \to A \to E \to G \to 1$ be an exact sequence of flat sheaves in groups with A abelian. Given a map $A \to B$ of abelian flat sheaves equipped with compatible G–actions, the contracted product $B \wedge^A E$ is a sheave in groups and is an extension of G by B.

14.7. **Sheaf torsors.** Let G/R be a flat sheaf in groups (e.g. an affine group scheme over R).

14.7.1. **Definition.** *A sheaf G–torsor over R is a flat sheaf E/R equipped with a right action of G submitted to the following requirements:*

[9] We are given S/R and a point $u \in U_x(S)$. Then $h_x^{-1}(u) = \mathrm{Spec}(S')$ is a flat cover of S and there is a point $v \in G(S)$ mapping to u.

(T1) The R-map $E \times G \to E \times E$, $(e,g) \mapsto (e, e.g)$ is an isomorphism of flat sheaves over R.

(T2) There exists a flat cover S/R such that $E(S) \neq \emptyset$.

The basic example of such an object is the trivial G-torsor sheaf G equipped with the right action. For avoiding confusions, we denote it sometimes E_{tr}. Condition (T2) implies that E_S is G_S-isomorphic to $E_{tr,S}$. In other words a G-torsor sheaf is locally trivial with respect to the flat topology.

Now if F is a flat sheaf over R equipped with a right G-action and E/R is a G-torsor, we call the contracted product $E \wedge^G F$ the twist of F by E. It is denoted sometimes ${}^E F$ or ${}_E F$. We record the two special cases:

(1) The action of G on E_{tr} by left translations, we get then $E = {}^E E_{tr}$.

(2) The action of G on itself by inner automorphisms, the twist ${}^E G$ is called the inner twisted form of G associated to E.

(3) We can twist the left action (by translation) $G \times E_{tr} \to E_{tr}$, where G acts on itself by inner automorphisms. It provides a left action ${}^E G \times E \to E$.

In the case G is representable by an affine group scheme $\mathfrak{G}/R$, then descent theory shows that sheaf G-torsors are representable as well and we we say that the relevant schemes are G–torsors/ Furthermore if $\mathfrak{G}/R$ is flat (resp. smooth), so are the G–torsors. We give some examples of torsors.

14.7.2. **Examples.** (1) Galois covers $\mathfrak{Y} \to \mathfrak{X}$ under a finite group Γ, see below 14.8.1.

(2) The Kummer cover $\mathfrak{G}_m \to \mathfrak{G}_{m,n}$.

(3) The Chevalley quotient 14.5.2 gives rise to the H–torsor $G \to G/H$.

14.7.3. **Lemma.** *(1) Let S/R be a flat cover which splits E. Then $({}^E F)_S \xrightarrow{\sim} F_S$.*

(2) ${}^E(\mathrm{Aut}(F)) = \mathrm{Aut}({}^E F)$.

(3) If F is representable by an R–affine scheme $\mathfrak{X}$, so is ${}^E F$. Furthermore if $\mathfrak{X}$ is finitely presented (resp. faithfully flat, smooth), so are F.

(4) If G is representable by an affine R–scheme $\mathfrak{G}$, so are E and ${}^E G$. Furthermore if $\mathfrak{G}/R$ is finitely presented (resp. faithfully flat, smooth), so are E and ${}^E\mathfrak{G}$.

Note that te R–functor $\mathrm{Aut}(F)$ is a flat sheaf, see 14.2.3.

Proof. (1) The formation of contracted products commute with arbitrary base change, hence $({}^E F)_S = {}^{E_S}F_S \xrightarrow{\sim} {}^{E_{tr,S}}F_S = F_S$.
(2) Twisting the morphism of flat sheaves $\mathrm{Aut}(F) \times F \to F$ by E yields a morphism ${}^E\,\mathrm{Aut}(F) \times F \to F$. It defines then a map ${}^E\,\mathrm{Aut}(F) \to \mathrm{Aut}({}^E F)$. It is an isomorphism since it is after making the base change S/R.
(3) It is a special case of faithfully flat descent.

(4) It comes from the permanence properties kept by faithfully flat descent. □

Statement (1) says that ${}^E F$ is a S/R–form of F, that is a flat sheaf F' such that $F'_S \xrightarrow{\sim} F_S$.

14.8. **Quotient by a finite constant group.** An important case of torsor and quotients is the following

14.8.1. **Theorem.** [DG, §III.6] *Let Γ be a finite abtract group. We assume that Γ_R acts freely on the right on an affine R-scheme $\mathfrak{X}$. It means that the graph map $\mathfrak{X} \times_R \Gamma_R \to \mathfrak{X} \times_R \mathfrak{X}$ is a monomorphism. We put $\mathfrak{Y} = \mathrm{Spec}(R[\mathfrak{X}]^\Gamma)$.*

(1) The map $\mathfrak{X} \to \mathfrak{Y}$ is a Γ_R–torsor, i.e. a Galois cover of group Γ;

(2) The scheme $\mathfrak{Y}/R$ represents the quotient sheaf $\mathfrak{X}/\mathfrak{G}$.

See also [R, X, p. 108] for another proof.

14.9. **Quotient by a normalizer.** A more advanced result is the following representability theorem used only at the end of the lectures.

14.9.1. **Theorem.** [SGA3, XVI.2.4] *(see also* [Br, §3.8]*) Let $i : \mathfrak{H} \to \mathfrak{G}$ be a monomorphism of affine group schemes. We assume that $\mathfrak{G}$ is finitely presented and that $\mathfrak{H}$ is smooth with connected geometric fibers.*

(1) Then the normalizer functor N defined by

$$N(S) = \Big\{ g \in \mathfrak{G}(S) \mid g\mathfrak{H}(S')g^{-1} = \mathfrak{H}(S') \quad \forall S'/S \Big\}$$

for each S/R is representable by a closed subscheme of $\mathfrak{G}/R$ of finite presentation.

(2) We assume that N is flat. Then the quotient sheaf G/N is representable by a scheme which is of finite presentation over R and quasi-projective.

15. Non-abelian cohomology, I

15.1. **Definition.** We denote by $H^1(R, G)$ the set of isomorphism classes of G–torsors over R. It is a pointed set pointed by the class of the trivial G-torsor. If S/R is a cover, we denote by $H^1(S/R, G)$ the subset consisting of G–torsors split by S/R. This set $H^1(S/R, G)$ can be computed by means of cocycles modulo coboundaries [P]. More precisely, a 1–cocycle is an element $g \in G(S \otimes_R S)$ satisfying the rule

$$d_{2,3,*}(g)\, d_{1,2,*}(g) = d_{1,3,*}(g) \in G(S \otimes_R S \otimes S).$$

Two 1–cocycles $g_1, g_2 \in G(S \times_R S)$ are equivalent if there exists $g \in G(S)$ such that

$$g_2 = d_{2,*}(g)^{-1}\, g_1\, d_{1,*}(g) \quad \in G(S \otimes_R S).$$

15.1.1. **Remark.** If S/R is a Galois covering for an abstract group Γ, then $S \otimes_S R \xrightarrow{\sim} S^\Gamma$ and this leads to non-abelian Galois cohomology, see [P].

15.2. **Twisting.** If G is not abelian, there is no natural group structure on $H^1(E,G)$. We have however the torsion operation (change of origin)

$$\tau_E : H^1(R, {}^E G) \xrightarrow{\sim} H^1(R,G)$$

for a G-torsor E. Its definition (and also of the converse map) requires some preparation. Firstly the left action of G on itself gives rise to an action pof ${}^E G$ to E. We have then a bitorsor structure

$${}^E G \times E \times G \to E.$$

Given a ${}^E G$-torsor F, the contracted product $F \wedge^{{}^E G} E$ is equipped with a right G–action and is indeed a G–torsor. We put $\tau_E(F) = [F \wedge^{{}^E G} E]$.

The opposite torsor E^{op} of E is the right ${}^E G$–torsor obtained by taking the opposite actions above. It comes then with a left action of G. Now, given a G–torsor L, the contracted product $L \wedge^G E^{op}$ is similarly a right ${}^E G$–torsor. It defines the converse of the torsion bijection map.

Also the contracted product permits to define $H^1(R,G) \to H^1(R,H)$ for a map $u : G \to H$.

15.2.1. **Proposition.** *There is one to one correspondence*

$$\Big\{ S/R\text{-}\textit{forms of } F \Big\} \quad \xrightarrow{\sim} \quad H^1(S/R, \mathrm{Aut}(F)).$$

Proof. We explain only the maps. Given a S/R-form F' of F, we observe that $\mathrm{Aut}(F)$ acts on the right on the flat sheaf $\mathrm{Isom}(F,F')$ which is a $\mathrm{Aut}(F)$-torsor since it is so after extension to S/R. Conversely, given a $\mathrm{Aut}(F)$-torsor E, the twisted sheaf ${}^E F$ is a S/R–form of F. $\square$

A special case is the following, see [P].

15.2.2. **Theorem.** *(Hilbert-Grothendieck 90) Let M be a R–module which is locally free of rank d.*

(1) The set $H^1(R, \mathrm{GL}(M))$ classifies the isomorphism classes of R–modules of rank d.

(2) If R is semilocal, $H^1(R, \mathrm{GL}(M)) = 1$.

Another nice example is that of the even orthogonal group, see [DG, III.5.2].

15.3. **Weil restriction II.** Let S be an R–ring. Let H/S be a flat sheaf in groups and consider the R–functor $G = \prod\limits_{S/R} H$, that is the Weil restriction of H from S to R. We note that G is a flat R–sheaf in groups. The adjunction map ; $\psi : G_S \to H$ defines a natural map

$$H^1(R,G) \to H^1(S,G_S) \stackrel{\psi_*}{\to} H^1(S,H).$$

15.3.1. **Proposition.** [SGA3, XXIV.8.2]
(1) The map $H^1(R,G) \to H^1(S,H)$ is injective and its image consists in H–torsors which are split after a flat cover coming from R.

(2) If S/R is a flat cover, we have $H^1(R,G) \xrightarrow{\sim} H^1(S,H)$.

Proof. (1) We denote by $H^1_R(S,H)$ the subset of $H^1(H,S)$ consisting in classes of H–torsors split after a flat cover coming from R. Let R'/R be a flat cover of R and put $S' = S \otimes_R R'$. Then $G(R') = H(S')$ and $G(R' \otimes_R R') = H(S' \times_S S')$ so that $H^1(R'/R, G) \xrightarrow{\sim} H^1(S'/S, H)$. By passing to the limit we get the desired bijection $H^1(R,G) \xrightarrow{\sim} H^1_R(S,H)$.

Assertion (2) follows.

□

16. Quotients by diagonalizable groups

Let A be a finitely generated abelian group and consider the diagonalizable R-group scheme $\mathfrak{G}/R = \mathfrak{D}(A) = \mathrm{Spec}(R[A])$. We assume it acts on the right on an affine R–scheme $\mathfrak{X}$. We have then the decomposition in eigenspaces

$$R[\mathfrak{X}] = \bigoplus_{a \in A} R[\mathfrak{X}]_a.$$

16.1. **Torsors.** We are interested in understanding when $\mathfrak{X} \to \mathrm{Spec}(R)$ is a $\mathfrak{G}$–torsor.

16.1.1. **Proposition.** *Assume that $\mathfrak{X}$ is of finite presentation. Then $\mathfrak{X}/R$ is a $\mathfrak{G}$-torsor if and only if the two following conditions hold*

(i) For each $a \in A$, $R[\mathfrak{X}]_a$ is an invertible R–module;

(ii) For each pair $(a,b) \in A^2$, the multiplication homomorphism $R[\mathfrak{X}]_a \otimes_R R[\mathfrak{X}]_b \to R[\mathfrak{X}]_{a+b}$ is an isomorphism.

Furthermore, these two conditions are equivalent to the next conditions

(iii) $R \xrightarrow{\sim} R[\mathfrak{X}]_0$;

(iv) $R[\mathfrak{X}]_a\, R[\mathfrak{X}] = R[\mathfrak{X}]$ for each $a \in A$.

Proof. We observe first that the trivial torsor $\mathfrak{G}/R$ satisfies conditions (i) and (ii). Assume that $\mathfrak{X}$ is a $\mathfrak{G}$–torsor. There exists a flat cover S/R such that $\mathfrak{X} \times_R S \xrightarrow{\sim} \mathfrak{G}_S$ in an equivariant way. so that $\mathfrak{X}_S$ satisfies (i) and (ii). By faithfully flat descent, $\mathfrak{X}$ satisfies (i) and (ii).

Conversely, we assume that $\mathfrak{X}$ satisfies (i) and (ii). Then $R[\mathfrak{X}]$ is a projective module. The cograph map $h : R[\mathfrak{X}] \otimes_R R[\mathfrak{X}] \to R[\mathfrak{X}] \otimes_R R[A]$ applies an homogeneous element $f_a \otimes f_b$ to $(f_a\, f_b) \otimes e_a$. Hence h splits in a direct summand

$$\begin{array}{rrcl} h_a : & R[\mathfrak{X}]_a \otimes_R R[\mathfrak{X}] & \longrightarrow & R[\mathfrak{X}] \\ & f_a \otimes f & \mapsto & f_a f \end{array}$$

Condition (ii) ensures that h_a is an isomorphism and so is h. This shows that $\mathfrak{X}$ is a pseudo $\mathfrak{G}$-torsor.

Since $R[\mathfrak{X}]$ is a projective R–module, it is then faithfully flat over R. Then $R \to R[\mathfrak{X}]$ is a flat cover which splits $\mathfrak{X} \to \mathrm{Spec}(R)$, therefore $\mathfrak{X}/R$ is a $\mathfrak{G}$-torsor.

Conditions (i) and (ii) imply (iii) and (iv). Conversely assume (iii) and $iv)$. Let $a \in A$. There are elements $f_1, ..., f_r$ of $R[\mathfrak{X}]_{-a}$ and $h_1, ..., h_r$ of $R[\mathfrak{X}]_a$

such that $1 = f_1h_1 + \cdots + h_r f_r$. Then the family $R_{f_ih_i}$ is a Zariski cover of R and up to localize, we can assume that there exists $f \in R[\mathfrak{X}]_{-a} \cap R[\mathfrak{X}]^\times$. For each $b \in A$, it follows that the homomorphism $R[\mathfrak{X}]_b \to R[\mathfrak{X}]_{a+b}$, $u \mapsto fu$ is an isomorphism. In particular, $R = R[\mathfrak{X}]_0 \xrightarrow{\sim} R[\mathfrak{X}]_a$ and the multiplication $R[\mathfrak{X}]_a \otimes_R R[\mathfrak{X}]_b \to R[\mathfrak{X}]_{a+b}$ is an isomorphism. □

16.1.2. **Example.** The case of $A = \mathbb{Z}$, that is of $\mathfrak{G} = \mathfrak{G}_{m,R}$. In this case, we know from the yoga of forms that a $\mathfrak{G}_m$–torsor $\mathfrak{X}/R$ is the same thing than an invertible R-module M. Another way to see it is to consider the invertible module $R[\mathfrak{X}]_1$.

16.2. **Quotients.**

16.2.1. **Theorem.** *We assume that $\mathfrak{G}$ acts freely on $\mathfrak{X}$, that is the map $\mathfrak{X} \times_R \mathfrak{G} \to \mathfrak{X} \times_R \mathfrak{X}$ is a monomorphism. We put $\mathfrak{Y} = \mathrm{Spec}(R[\mathfrak{X}]_0)$.*

(1) The R–map $p : \mathfrak{X} \to \mathfrak{Y}$ is a $\mathfrak{G}_{\mathfrak{Y}}$–torsor;

(2) $\mathfrak{Y}/R$ represents the flat quotient sheaf $\mathfrak{X}/\mathfrak{G}$.

Proof. (1) Without lost of generality, we can assume that $R = R[\mathfrak{X}]_0$. The morphism $\mathfrak{X} \to \mathfrak{Y} = \mathrm{Spec}(R)$ is $\mathfrak{G}$-invariant. From Proposition 16.1.1, we need to check that $R[\mathfrak{X}]_a\, R[\mathfrak{X}] = R[\mathfrak{X}]$ for each $a \in A$.

Let $\mathfrak{M}$ be a maximal ideal of R and consider the subset $A_\sharp$ of A consisting in the elements $a \in A$ such that $R[\mathfrak{X}]_a R[\mathfrak{X}]_{-a} \not\subset \mathfrak{M}$. We note that $A_\sharp$ is a subgroup of A and consider the ideal

$$\mathcal{I} = \sum_{a \notin A_\sharp} R[X]_a\, R[X]$$

of $R[\mathfrak{X}]$. We have $\mathcal{I} \cap R \subset \mathfrak{M}$.

16.2.2. **Claim.** $A_\sharp = A$.

The point is that $\mathcal{I}$ is a graded ideal of $R[\mathfrak{X}]$ so that $\mathrm{Spec}(R[\mathfrak{X}]/\mathcal{I})$ carries an induced $\mathfrak{G}$-action which is fixed by the closed subgroup R–scheme $\mathfrak{D}(A/A_\sharp)$ of $\mathfrak{G}$. Since the action is free, we conclude that $A_\sharp = A$.

From the claim, we get that for each $a \in A$, the ideal $R[\mathfrak{X}]_a R[\mathfrak{X}]_{-a}$ of R is R.

(2) Denote by Q the quotient sheaf $\mathfrak{X}/\mathfrak{G}$. The map $p : \mathfrak{X} \to \mathfrak{Y}$ factorizes by Q, that is defines a map of flat sheaves $\widetilde{p} : Q \to \mathfrak{Y}$. Since p is faithfully flat, q is an epimorphism (same argument as at the end of proof of Proposition 14.5.3). Let us show that q is a monomorphism. We are given an R–ring S and two elements $q_1, q_1 \in Q(S)$ such that $\widetilde{p}(q_1) = \widetilde{p}(q_2) = y \in \mathfrak{Y}(S)$. Let S'/S be a cover such that q_1 and q_2 come from $x_1, x_2 \in \mathfrak{X}(S')$. Since $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{G}_{\mathfrak{Y}} \xrightarrow{\sim} \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$, there exists $g \in \mathfrak{G}(S')$ such that $x_1 g = x_2$. Therefore $q_1 = q_2 \in Q(S)$. □

16.2.3. **Corollary.** *(1) The graph morphism $\mathfrak{X} \times_R \mathfrak{G} \to \mathfrak{X} \times_R \mathfrak{X}$ is a closed immersion.*

(2) For each $x \in \mathfrak{X}(R)$, the orbit map $\mathfrak{G} \to \mathfrak{X}$, $g \mapsto x\,.\,g$ is a closed immersion.

16.3. **Homomorphisms to a group scheme.**

16.3.1. **Corollary.** *Let $f : \mathfrak{G} = \mathfrak{D}(A) \to \mathfrak{H}$ be an R–group monomorphism where $\mathfrak{H}/R$ is an affine group scheme. Then f is a closed immersion.*

Proof. The action of $\mathfrak{G}$ on $\mathfrak{H}$ is free. Then $\mathfrak{G}$ arises as the fiber at 1 of the quotient map $\mathfrak{H} \to \mathfrak{H}/\mathfrak{G}$. □

More difficult is the following

16.3.2. **Theorem.** [SGA3, IX.6.4] *Let $f : \mathfrak{G} = \mathfrak{D}(A) \to \mathfrak{H}$ be a group homomorphism where $\mathfrak{H}/R$ is a smooth affine group scheme. Assume that R is noetherian and connected. Then the kernel of f is a closed subgroup scheme $\mathfrak{D}(A/B)$ of $\mathfrak{G}$ and f factorizes in an unique way as*

$$\mathfrak{G} = \mathfrak{D}(A) \to \mathfrak{D}(B) \xrightarrow{\widetilde{f}} \mathfrak{H}$$

where $\widetilde{f}$ is a closed immersion.

Proof. We can assume that R is local with maximal ideal $\mathfrak{M}$, residue field κ.

We denote by $\widehat{R}$ the completion of R with respect to the ideal $\mathfrak{M}$. Since $\mathfrak{H} \times_R k$ admits a faithful representation, the kernel of f_k is $\mathfrak{D}(A/B)$ for some B. By the rigidity principle 10.1.1, $f_{R/\mathfrak{M}^n}$ is trivial on $\mathfrak{D}(A/B) \times_R R/\mathfrak{M}^n$ for each $n \geq 1$. Therefore $f_{\widehat{R}}$ is trivial on $\mathfrak{D}(A/B)$ (injectivity in Theorem 10.3.1) and f is then trivial on $\mathfrak{D}(A/B)$ because $\widehat{R}$ is faithfully flat over R. Up to mod out by $\mathfrak{D}(A/B)$, we can then assume then f_k is a monomorphism. Then f is a monomorphism according to Theorem 10.1.4. □

17. Groups of multiplicative type

17.1. **Definitions.**

17.1.1. **Definition.** *A finitely presented affine group scheme $\mathfrak{G}/R$ is of multiplicative type is there exists a flat cover $S_1 \times \dots\dots S_l$ of R such that $\mathfrak{G} \times_R S_i$ is a diagonalizable S_i-group scheme.*

If $\mathfrak{G}_{S_i}$ is isomorphic to some $\mathfrak{D}(\mathbb{Z}^{r_i})_{S_i}$ for each i, we say that $\mathfrak{G}$ is a torus.

If R is connected, this is equivalent to ask that $\mathfrak{G} \times_R S$ is diagonalizable for a single flat cover S/R. By descent, the nice properties of diagonalizable groups generalize. More precisely:

(1) The rigidity properties;

(2) Existence of quotients for free actions on affine schemes;

(3) The category of group of multiplicative type admits kernels and cokernels, it is an abelian category. We denote it by $\mathcal{M}/R$.

(4) Each R-group of multiplicative type $\mathfrak{G}$ fits in a canonical exact sequence $1 \to \mathfrak{T} \to \mathfrak{G} \to \mathfrak{G}' \to 1$ where $\mathfrak{T}/R$ is a R-torus and $\mathfrak{G}'/R$ is finite.

17.1.2. **Example.** If S/R is a finite étale cover and A a f.g. abelian group, the Weil restriction $\mathfrak{G} = \prod\limits_{S/R} \mathfrak{D}(A)_S$ is an R–group of multiplicative type[10]. We have a natural map $\mathfrak{D}(A) \to \mathfrak{G}$.

17.1.3. **Definition.** *An R–group of multiplicative type is called*

isotrivial if there exists a finite étale cover S/R such that $\mathfrak{G}_S$ is diagonalizable;

quasi-isotrivial if there exists an étale cover S/R such that $\mathfrak{G}_S$ is diagonalizable.

For example, the Weil restriction example is isotrivial. We consider the case of a connected Galois cover S/R of finite group Γ. By the yoga of forms, for each f.g. abelian group A, we have a correspondence

$$\Big\{ S/R\text{-forms of } \mathfrak{D}(A) \Big\} \quad < - - > \quad H^1(S/R, \mathrm{GL}(A)) = H^1(\Gamma, \mathrm{GL}(A)).$$

The point is that $\mathrm{Aut}(\mathfrak{D}(A))^{op} = (\mathrm{GL}(A))_R$. Also since $H^1(\Gamma, \mathrm{GL}(A)) = \mathrm{Hom}_{gr}(\Gamma, \mathrm{GL}(A))/\mathrm{GL}(A)$, it implies that there is a minimal Galois subextension $S_{\mathfrak{G}}/R$ of S/R which splits $\mathfrak{G}$. This can be pushed further as follows.

17.1.4. **Proposition.** *The subcategory of $\mathcal{M}/R$ consisting of R–groups of multiplicative type split by S/R is full and abelian. It is antiequivalent to the category of f.g. Γ-modules over $\mathbb{Z}$.*

We can pass that to the limit on Galois covers.

17.1.5. **Corollary.** *Suppose that R is connected and let $f : R \to F$ be a base point where f is a separably closed field. The subcategory $\mathcal{M}_R$ consisting of isotrivial R–groups of multiplicative type is a full and abelian. It is antiequivalent to the category of discrete $\pi_1(R, f)$-modules which are finitely generated over $\mathbb{Z}$.*

17.1.6. **Remarks.** (1) For an isotrivial R–group of multiplicative type $\mathfrak{G}$, there is a minimal Galois subextension $R_{\mathfrak{G}}/R$ of R^{sc} which splits $\mathfrak{G}$.
(2) Since a Γ-module M (f.g. over $\mathbb{Z}$) is a quotient of a free module $\mathbb{Z}[\Gamma]^r$, it follows that each isotrivial R–group of multiplicative type embeds in a quasi-trivial torus namely of the shape $\prod_{S/R} \mathbb{G}_{m,S}$ for a suitable finite étale cover of R.

To find the best way to present a given R–group of multiplicative type is then a natural question which is linked with representation theory. We can mention here the theory of flasque resolutions by Colliot-Thélène and Sansuc which deals with isotrivial objects [CTS1]. In the general case, not much in known beyond the following fact.

[10]Up to localize, we can assume that S/R is locally free of rank $d \geq 1$. We prove it by using a finite étale cover T such that $S \otimes_R T \cong T^d$. Then $\mathfrak{G}_T = \prod\limits_{S \otimes_R T/T} \mathfrak{D}(A)_{S \otimes_R T} = \prod\limits_{T^d/T} \mathfrak{D}(A)_{T^d} = \mathfrak{D}(A)_T^d$.

17.1.7. **Proposition.** *(Conrad,* [C, B.3.8]*) Let $\mathfrak{G}$ be an R–group of multiplicative type. Then $\mathfrak{G}$ embeds as closed subgroup scheme in an R-torus.*

17.2. **Splitting results.**

17.2.1. **Lemma.** *Assume that R is connected. Let $\mathfrak{G}/R$ be a finite group of multiplicative type. Then $\mathfrak{G}$ is isotrivial.*

Proof. There exist a flat cover S/R such that $\mathfrak{G} \times_R S \xrightarrow{\sim} \mathfrak{D}(A)_R$ where A is a finite abelian group. In other words, $\mathfrak{G}$ is an R-form of $\mathfrak{D}(A)$. But those forms are classified by the pointed set $H^1(R, \mathrm{GL}_1(A))$ which classifies also Galois R–covers of group $\mathrm{GL}_1(A)$. Therefore $\mathfrak{G}$ defines a class of Galois covers $[S/R]$ which split $\mathfrak{G}$. $\square$

17.2.2. **Proposition.** *Let k be a field. Then the k-groups of multiplicative type are isotrivial.*

Proof. Since it holds in the finite case, we can deal with a torus T/k of rank d. There exists a finite field extension L/k and an isomorphism $\phi : \mathbb{G}_{m,L}^d \xrightarrow{\sim} T_L$. If L is separable, there is nothing to do. If not, k is of characteristic $p > 0$ and there exists a subextension $F \subsetneq L$ such that $L = F(\sqrt[p]{x})$. We claim that ϕ descend to F. We consider the ring $R = L \otimes_F L = L[t]/(t^p - x) \cong L[u]/u^p$, it is an artinian local ring of residue field L. Theorem 10.2.1.(2) shows that

$$\mathrm{Hom}_{R-gp}(\mathbb{G}_{m,R}^d, T_R) \xrightarrow{\sim} \mathrm{Hom}_{L-gp}(\mathbb{G}_{m,L}^d, T_L).$$

It follows that $d_{1,*}(\phi) = d_{2,*}(\phi) : \mathbb{G}_R^d \to T_R$. By faithfully flat descent, ϕ descends then to F. and this is an R–group isomorphism. We can continue this process which stops when reaching the maximal separable subextension of L/k. $\square$

17.2.3. **Corollary.** *Let A be a f.g. abelian group. Then we have*

$$\mathrm{Hom}_{ct}(\mathrm{Gal}(k_s/k), \mathrm{GL}(A))/\mathrm{GL}(A) \xrightarrow{\sim} H^1(k, \mathrm{GL}(A)).$$

17.2.4. **Theorem.** *Let $\mathfrak{G}/R$ be an R–group of multiplicative type. Then $\mathfrak{G}/R$ is quasi-isotrivial.*

In the present proof, we use Artin's approximation theorem which came six years after the SGA3 seminar.

Proof. By the classical limit argument, we can assume that R is of finite type over $\mathbb{Z}$ and in particular that R is noetherian. Up to localize, we can assume that $\mathfrak{G}$ is an R-form of $\mathfrak{D}(A)$. The statement holds in the case A finite (Lemma 17.2.1) and behave well under extensions, so we can assume that $A = \mathbb{Z}^d$. We switch then to the torus notation $\mathfrak{T} = \mathfrak{G}$. We consider the R-functor (which is a flat sheaf)

$$F(S) = \mathrm{Hom}_{S-gr}(\mathbb{G}_{m,S}^d, \mathfrak{T}_S).$$

We have seen that F is formally étale (Theorem 10.2.1.(2)) and we observe that F is locally finitely presented, that is commutes with filtered direct limits of rings.

Let $x \in \mathrm{Spec}(R)$ be a closed point and denote by $\mathfrak{M}_x$ the underlying prime ideal of R and $k(x) = R/\mathfrak{M}_x$. We denote by R_x^{sh} the strict henselization of the local ring R_x, see §21.4. The plan is to construct an element of $F(R_x^{sh})$ which is an isomorphism. We choose firstly a separable field extension $k'/k(x)$ which splits $\mathfrak{T}_{k(x)}$. Up to shrink R, it lifts to a finite étale connected cover R'/R. Without lost of generality, we can assume then that $\mathfrak{T}_{k(x)}$ is split. In other words, there exists a group $k(x)$-isomorphism

$$\phi_0 : \mathbb{G}_{m,k(x)}^d \xrightarrow{\sim} \mathfrak{T}_{k(x)}.$$

We denote by $\widehat{R}_x$ the $\mathfrak{M}_x$-adic completion of R. From 10.3.3.(3), we have see that ϕ_0 lifts uniquely to a morphism

$$\widehat{\phi} : \mathbb{G}_{m,\widehat{R}_x}^d \to \mathfrak{T}_{\widehat{R}_x}$$

and $\widehat{\phi}$ is a monomorphism. The cokernel of $\widehat{\phi}$ is a $\widehat{R}_x$-group of multiplicative type whose special fiber is trivial. It follows that $\widehat{\phi}$ is an isomorphism. We apply now the Artin's approximation theorem 21.5.1 to the locally finitely presented functor $E(S) = \mathrm{Isom}_{S-gr}(\mathbb{G}_{m,S}^d, \mathfrak{T}_S)$. It implies that

$$\mathrm{Im}\Big(E(R_x^h) \to E(k(x))\Big) = \mathrm{Im}\Big(E(\widehat{R}) \to E(k(x))\Big).$$

In our case, it provides an isomophism $\phi : \mathbb{G}_{m,R_x^h}^d \xrightarrow{\sim} \mathfrak{T} \times_R R_x^h$. This isomorphism is defined on some étale neighboorhood R'/R of x. $\square$

17.2.5. **Corollary.** *Let R be strictly henselian ring. Then the R–groups of multiplicative type are split.*

17.2.6. **Theorem.** *Assume that R is normal. Let K be the fraction field of R and let $f : R \to K \to K_s$ be an embedding in a separable closure of K.*

(1) The R–groups of multiplicative type are isotrivial.

(2) The category of R-groups of multiplicative type is equivalent to the category of discrete $\pi_1(R, f)$–modules which are f.g. over $\mathbb{Z}$.

Assertion (2) is a formal consequence of (1). We present an alternate proof based on the following step.

17.2.7. **Lemma.** *Let $\mathfrak{T}/R$ be a torus of dimension d. There is a Galois cover R'/R of group $\mathrm{GL}_d(\mathbb{F}_3)$ such that $\mathfrak{T}_{R' \otimes_R K}$ splits.*

Proof. We have an exact sequence of groups

$$1 \to \Theta \to \mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{F}_3) \to 1$$

and Minkowski's lemma states that Θ is torsion free [N, IX.11]. We consider then the exact commutative diagram of pointed sets

$$1 \longrightarrow H^1(R, \Theta) \longrightarrow H^1(R, \mathrm{GL}_d(\mathbb{Z})) \longrightarrow H^1(R, \mathrm{GL}_d(\mathbb{F}_3)).$$

The R-torus $\mathfrak{T}$ defines a class in $H^1(R, \mathrm{GL}_d(\mathbb{Z}))$ whose image in $H^1(R, \mathrm{GL}_d(\mathbb{F}_3))$ is represented by a Galois cover R'/R of group $\mathrm{GL}_d(\mathbb{F}_3)$. As usual we take the right connected component S/R, it a connected Galois cover of group $\Gamma \subset \mathrm{GL}_d(\mathbb{F}_3)$. We put $L = \mathrm{Frac}(S) = S \otimes_R K$ and look at the commutatuve diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & H^1(R,\Theta) & \longrightarrow & H^1(R,\mathrm{GL}_d(\mathbb{Z})) & \longrightarrow & H^1(R,\mathrm{GL}_d(\mathbb{F}_3)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H^1(S,\Theta) & \longrightarrow & H^1(S,\mathrm{GL}_d(\mathbb{Z})) & \longrightarrow & H^1(S,\mathrm{GL}_d(\mathbb{F}_3)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H^1(L,\Theta) & \longrightarrow & H^1(L,\mathrm{GL}_d(\mathbb{Z})) & \longrightarrow & H^1(L,\mathrm{GL}_d(\mathbb{F}_3))
\end{array}$$

where the vertical maps are base change maps.

17.2.8. **Claim.** $H^1(L, \Theta) = 1$.

Indeed a class of $H^1(L, \Theta)$ is represented in $H^1(L, \mathrm{GL}_d(\mathbb{Z}))$ by a continous map $\phi : \mathrm{Gal}(K_s/L) \to \Theta$, so is trivial. By diagram chase, it follows that $[\mathfrak{T}] \in \ker\Big(H^1(R, \mathrm{GL}_d(\mathbb{Z})) \to H^1(L, \mathrm{GL}_d(\mathbb{Z}))\Big)$. Thus $\mathfrak{T}_L$ is a split torus. □

We can proceed to the proof of Theorem 17.2.6.

Proof. We can assume that R is noetherian. Since the result holds in the finite case (lemma 17.2.1) and behaves well under exact sequences, it is enough to deal with the torus case. Let $\mathfrak{T}/R$ be an R–torus of dimension d. Granting to Lemma 17.2.7, we can assume that $\mathfrak{T}_K$ is split. We have then an isomorphism map $\alpha : \mathbb{G}_{m,K}^d \xrightarrow{\sim} \mathfrak{G}_K$. We want to show that it extends to R.

First case: R is a DVR. We have $R^{sc} = R^{hs}$ (see §21.4), so that $\mathfrak{T} \times_R R^{sc}$ splits according to Corollary 17.2.5. Hence there exists a finite Galois connected cover S/R and an isomorphism $\beta : \mathbb{G}_K^d \xrightarrow{\sim} \mathfrak{T}_K$. We put $L = \mathrm{Frac}(S)$ and observe that $\alpha_L \circ \beta_L \in \mathrm{GL}_d(\mathbb{Z})(L) = \mathrm{GL}_d(\mathbb{Z})(S)$. So we can modify β such that $\beta_L = \alpha_L$. It follows that β is $\mathrm{Gal}(S/R)$-invariant and descends to an isomorphism $\mathbb{G}_R^d \xrightarrow{\sim} \mathfrak{T}$. which extends α.

General case: Given $x \in \mathrm{Spec}(R)^{(1)}$, we know from the first case that there exists an isomorphism $\beta_x : \mathbb{G}_{m,R_x}^d \xrightarrow{\sim} \mathfrak{T}_{R_x}$ which extends α. Hence there exists an open subset $\mathfrak{U}$ of $\mathrm{Spec}(R)$ containing all points of codimension one such that α extends to an isomorphism $\widetilde{\alpha} : \mathfrak{D}(A)_{\mathfrak{U}} \xrightarrow{\sim} \mathfrak{G}_{\mathfrak{U}}$.

The map $\widetilde{\alpha} : \mathbb{G}_m^d \times_{\mathbb{Z}} \mathfrak{U} \to \mathfrak{G}$ is defined everywhere in codimension one on the normal scheme[11] $\mathbb{G}_m^d \times_{\mathbb{Z}} R$ so it extends uniquely to a map $\mathbb{G}_{m,R}^d \to \mathfrak{G}$ ([EGA4, 20.4.6] or [Li, 4.1.14]). This map is a group isomorphism with the same kind of arguments. □

[11] Recall that a smooth affine scheme over a normal noetherian ring is normal [Li, 8.2.25].

17.3. **Back to the recognition statement.** We state once again the important Proposition 13.3.4.

17.3.1. **Proposition.** [C, 4.2.2] *Let $\mathfrak{U}/R$ be an affine smooth group scheme whose geometric fibers are rank one additive groups. We assume that $\mathfrak{U}/R$ is equipped with an action of $\mathfrak{G}_m$ such that the $\mathbb{G}_m$–module $L = \mathrm{Lie}(\mathfrak{U})(R)$ is non trivial everywhere. Then there exists a natural R–group isomorphism $\mathfrak{W}(L) \cong \mathfrak{U}$.*

Proof. We can assume that R is noetherian. The statement is of local nature for the flat topology so we can assume that $R = (R, \mathfrak{M}, k)$ is local. Also up to make an essentially étale extension of R, we can assume that the map $\mathfrak{U}(R) \to \mathfrak{U}(k)$ is not trivial.

In particular $\mathbb{G}_m$ acts on L by a character α_n, $n \geq 1$ (up to take the opposite action). Let $\sigma \in \mathfrak{U}(R)$ be a point whose specialization is not trivial. We consider the orbit map

$$q : \mathfrak{G}_{m,R} \to \mathfrak{U}, \quad t \mapsto t.\sigma \qquad .$$

It extends to an R–map $\widetilde{q} : \mathbb{A}^1_R \to \mathfrak{U}$ which is $\mathfrak{G}_m$–equivariant for the scaling action on $\mathfrak{G}_{a,R}$. The induced map $\widetilde{q}_k : \mathbb{A}^1_k \to \mathfrak{U}_k \xrightarrow{\sim} \mathfrak{G}_{a,k}$ is a non constant endomorphism f of the affine line $\mathbb{A}^1_k$ which satisfies $f(t.x) = t^n\, f(x)$, hence $f(t) = a\, t^n$ for some $a \in k^\times$. In particular, f is $\mu_{n,k}$–invariant.

17.3.2. **Claim.** *q is μ_n-invariant.*

Equivalently we have to show that $\mu_n \to \mathfrak{G}_m \to \mathfrak{U}$ is a constant map (of value σ) or that $R[\mathfrak{U}] \to R[\mu_n]/R$ is trivial. Since q_k is trivial, $R_m[\mathfrak{U}] \to R_m[\mu_n]/R_m = R[\mathbb{Z}/n\mathbb{Z}]/R_m$ is trivial where $R_m = R/\mathfrak{M}^m$. By passing to the limit, we get that $\widehat{R}[\mathfrak{U}] \to \widehat{R}[\mu_n]/\widehat{R} = \widehat{R}[\mathbb{Z}/n\mathbb{Z}]/\widehat{R}$ is trivial. Thus $q_{\widehat{R}}$ is a constant map so q is trival as well. The claim is proved.

By moding out by μ_n, we get a factorization $q' : \mathfrak{G}'_{m,R} \to \mathfrak{U}$ and a map $\widetilde{q}' : (\mathbb{A}^1_{m,R})' \to \mathfrak{U}$ which is $\mathfrak{G}_m$-equivariant where the action on $(\mathbb{A}^1_{m,R})'$ is by α_n. It follows that the map $\widetilde{q}'$ is an R–group monomorphism. Also $\widetilde{q}'$ is étale by the differential criterion so that it is an immersion. But its image contains the closed fiber $\mathfrak{U}_k$ which permits to conclude that $\widetilde{q}'$ is an isomorphism. $\square$

17.3.3. **Remark.** The key thing in the proof is the $\mathfrak{G}_m$-action. In positive characteristic, the additive group $\mathbb{G}_a$ has a large automorphism group [DG, II.1.2.7] which is under control there.

Reductive group schemes and descent techniques

18. Splitting reductive group schemes

18.1. **Local splitting.** The next result generalizes the torus case 17.2.4.

18.1.1. **Theorem.** *Let $\mathfrak{G}/R$ be a reductive group scheme. Then there exists an étale cover $S_1 \times \cdots \times S_l$ such that $\mathfrak{G} \times_R S_i$ is a split reductive S_i–group scheme for $i = 1, ..., r$.*

Proof. The proof goes on the same lines than Theorem 17.2.4. By the classical limit argument, we can assume that R is of finite type over $\mathbb{Z}$ and in particular that R is noetherian. Let $x \in \mathrm{Spec}(R)$ be a closed point and denote by $\mathfrak{M}_x$ the underlying prime ideal of R and $k(x) = R/\mathfrak{M}_x$. The $k(x)$–group $\mathfrak{G} \times_R k(x)$ admits a maximal torus $T/k(x)$. It splits after a separable field extension $k(x)'$. Up to shrink R, the extension $k'(x)/k$ lifts to a finite étale connected cover R'/R. It boils down then to the case when $\mathfrak{G} \times_R k(x)$ admits a maximal torus $\mathbb{G}_m^d/k(x)$. According to Theorem 10.3.3.(1), our given $k(x)$-embedding

$$\phi_0 : \mathbb{G}^d_{m,k(x)} \to \mathfrak{G}_{k(x)}.$$

lifts to a $\widehat{R}_x$–monomorphism

$$\widehat{\phi} : \mathbb{G}^d_{m,\widehat{R}_x} \to \mathfrak{G}_{\widehat{R}_x}.$$

We apply now the Artin's approximation theorem 21.5.1 to the locally finitely presented functor $F(S) = \mathrm{Hom}_{S-gr}(\mathbb{G}^d_{m,S}, \mathfrak{G}_S)$. It implies that

$$\mathrm{Im}\Big(F(R_x^h) \to F(k(x))\Big) = \mathrm{Im}\Big(F(\widehat{R}_x) \to F(k(x))\Big).$$

In our case, it provides an R_x^h–map $\phi : \mathbb{G}^d_{m,R_x^h} \to \mathfrak{G}_{R_x^h}$. This map ϕ is a monomorphism by 10.1.4 and a closed immersion by Corollary 16.3.1. Since the absolute rank is a locally constant function (Cor. 11.0.10), we conclude that ϕ defines a split maximal R_x^h–torus of $\mathfrak{G}_{R_x^h}$. The $\mathfrak{G}_{R_x^h}$-group is then split according to Remark 13.3.6. $\square$

18.1.2. **Corollary.** *Assume that R is a strictly henselian local ring. Then each reductive group scheme $\mathfrak{G}/R$ is split.*

18.1.3. **Definition.** *Let $\mathfrak{H}/R$ be a reductive group scheme. For each $x \in$* $\mathrm{Spec}(R)$*, we define the type of $\mathfrak{H}$ at x as the isomorphism class of the root datum $\Psi(\mathfrak{H}_{\overline{\kappa(x)}})$. It is denoted* $\mathrm{type}_x(\mathfrak{H})$.

The isomorphism classes of root data form a set, we denote it by $\mathcal{T}ype$. We refine then the continuity of the rank by the

18.1.4. **Corollary.** *Let $\mathfrak{H}/R$ be an affine smooth group scheme. The map*

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longrightarrow & \mathcal{T}ype \\ x & \mapsto & \mathrm{type}_x(\mathfrak{H}) \end{array}$$

is continuous.

We get also a local characterization of reductive group schemes.

18.1.5. **Corollary.** *Let $\mathfrak{G}/R$ be an affine group scheme. Then the following are equivalent:*

(i) $\mathfrak{G}$ is a reductive group scheme;

(ii) there exists an étale cover $S_1 \times \cdots \times S_l$ such that $\mathfrak{G} \times_R S_i$ is a split reductive S_i–group scheme for $i = 1, ..., l$;

(iii) there exists an flat cover $S_1 \times \cdots \times S_l$ such that $\mathfrak{G} \times_R S_i$ is a split reductive S_i–group scheme for $i = 1, ..., l$.

Proof. Theorem 18.1.1 is exactly $(i) \Longrightarrow (ii)$. The implication $(ii) \Longrightarrow (iii)$ is obvious. The implication $(iii) \Longrightarrow (i)$ is the easy one. Assuming (iii), faithfully flat descent theory yields that $\mathfrak{G}$ is smooth. Also the geometric fibers of $\mathfrak{G}$ are reductive, so we conclude that $\mathfrak{G}$ is a reductive group scheme. □

18.1.6. **Proposition.** *Let $\mathfrak{G}/R$ be a smooth affine group scheme. Maximal tori exist locally for the étale topology and are locally conjugated.*

Proof. Since split reductive group schemes admit maximal tori, Theorem 18.1.1 yields the existence of maximal tori locally for the étale topology. For conjugacy, we can assume that R is finitely generated over $\mathbb{Z}$. Let $\mathfrak{T}_1$, $\mathfrak{T}_2$ be two maximal R–tori of $\mathfrak{G}$. Let $x \in \mathrm{Spec}(R)$. By Theorem 17.2.4, we can localize for the étale topology in order to split $\mathfrak{T}_1$ and $\mathfrak{T}_2$. We can then assume that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are split and see them as the images of $\phi_i : \mathbb{G}_R^d \to \mathfrak{G}$ for $i = 1, 2$. Up to localize furthermore, we know that $\mathfrak{T}_1 \times_R k(x)$ and $\mathfrak{T}_1 \times_R k(x)$ are conjugated [CGP, A.2.10] by an element $g_x \in G(k(x))$ which lifts in $g \in \mathfrak{G}(R)$. This boils down to the case when $\phi_{1,k(x)} = \phi_{2,k(x)}$. By Corollary 10.3.3.(1), there exist $g \in \mathfrak{G}(\widehat{R}_x)$ such that $\phi_1 =^g \phi_2$. Artin approximation's theorem applied to the transporter functor

$$E(S) = \Big\{ h \in \mathfrak{G}(S) \mid \phi_{1,S} = {}^h\phi_{2,S} \Big\}$$

shows that $E(R_x^h)$ is not empty. □

18.2. **Weyl groups.**

18.2.1. **Proposition.** *Let $\mathfrak{G}/R$ be a reductive group scheme equipped with a maximal R-torus $\mathfrak{T}$. We denote by $N = N_{\mathfrak{G}}(\mathfrak{T})$ the normalizer functor of $\mathfrak{T}$ defined by*

$$N(S) = \Big\{ g \in \mathfrak{G}(S) \mid g\mathfrak{T}(S')g^{-1} = \mathfrak{T}(S') \quad \forall S'/S \Big\}.$$

for each S/R.

(1) The functor N is representable by a closed subgroup scheme $\mathfrak{N} = \mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})$ of $\mathfrak{G}$.

(2) $\mathfrak{N}$ is smooth and the quotient $\mathfrak{N}/\mathfrak{T}$ is locally a twisted finite constant group scheme.

(3) If $\mathfrak{B}$ is a Borel subgroup containing $\mathfrak{T}$, we have $\mathfrak{T} = \mathfrak{N} \times_{\mathfrak{G}} \mathfrak{B}$.

(4) If $\mathfrak{T}$ is split and $\mathfrak{G}$ splits respectively to $\mathfrak{T}$, then $\mathfrak{N}/\mathfrak{T}$ is isomorphic to W_R where W is the Weyl group of the root datum $\Psi(G,T)$. Furthermore, the map $\mathfrak{N}(R) \to W_R(R)$ is surjective.

Proof. To be written. □

It applies to Borel subgroups.

18.2.2. **Theorem.** *Let $\mathfrak{G}/R$ be a reductive group scheme and let $\mathfrak{B}/R$ be a Borel R–subgroup scheme. Then $\mathfrak{B}$ is its own normalizer.*

Proof. We consider the normalizer functor of $\mathfrak{B}$ defined by

$$N(S) = \Big\{ g \in \mathfrak{G}(S) \mid g\mathfrak{B}(S')g^{-1} = \mathfrak{B}(S') \quad \forall S'/S \Big\}$$

for each S/R. We note that N is a flat sheaf so that we can localize for flat topology. From Theorems 18.1.1 and 18.5.3, we can assume that $\mathfrak{G}$ is split and that $\mathfrak{B}$ contains a maximal split R-torus $\mathfrak{T}$. Let us show that $\mathfrak{B}(R) = N(R)$. We are given $g \in N(R)$. Then ${}^g\mathfrak{T}$ is a maximal R–torus of $\mathfrak{B}$, so that up to localize for the étale topology, there exists $g \in \mathfrak{B}(R)$ such that ${}^g\mathfrak{T} =^b \mathfrak{T}$ (Th. 18.1.6). We can then assume that ${}^g\mathfrak{T} = \mathfrak{T}$, that is $g \in \mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$. By Proposition 18.2.1.(3), we get that $g \in \mathfrak{T}(R)$. □

18.2.3. **Remark.** An alternative way is to use Theorem 14.9.1. It implies that the normalizer functor is representable by a closed subgroup scheme $\mathfrak{N}$ of $\mathfrak{G}$. Since $\mathfrak{B}_{k(x)} = \mathfrak{N}_{k(x)}$ for each $x \in \mathrm{Spec}(R)$, the closed immersion $\mathfrak{B} \to \mathfrak{N}$ is surjective, hence an isomomorphism.

18.3. **Center of reductive groups.** We say that a split reductive group is semisimple (resp. adjoint, simply connected) is its root datum is semisimple (resp. adjoint, simply connected). The general definition is then provided by descent.

18.3.1. **Corollary.** *Let $\mathfrak{G}/R$ be a reductive group scheme. Then the center of $\mathfrak{G}$ is representable by a R-group of multiplicative type. Furthermore the quotient $\mathfrak{G}/\mathfrak{Z}(G)$ is an adjoint reductive R–group.*

18.4. **Isotriviality issues.** We shall discuss firstly examples. The (normal) ring $\mathbb{Z}$ is simply connected so that all tori are split. Also $\mathrm{Pic}(\mathbb{Z}) = 0$, hence by Remark 13.3.7, we have

A reductive group scheme $\mathfrak{G}/\mathbb{Z}$ is split if and only if $\mathfrak{G}$ carries a maximal $\mathbb{Z}$-torus.

There are semisimple group schemes over $\mathbb{Z}$, the simplest one being the special orthogonal group of the $\mathbb{Z}$–quadratic form Γ_8, see [CG]. Those groups have no maximal tori and this a somehow the first obstruction for splitting a reductive group scheme. A reasonnable question is the following.

18.4.1. **Question.** Assume that R is normal. Let $\mathfrak{G}/R$ be a reductive group scheme such that $\mathfrak{G} \times_R R^{sc}$ admits a maximal (split) R-torus. Is $\mathfrak{G}/R$ isotrivial namely split by R^{sc}?

We discuss this question by means of the following example. Let M/R be a locally free module of rank $d \geq 1$. Then the R–group $\mathfrak{G} = \mathrm{GL}(M)/R$ is isomorphic locally (for the Zariski topology) to GL_d. This R–group is reductive.

18.4.2. **Lemma.** *(1) $\mathfrak{G} = \mathrm{GL}(M)$ admits a split R–torus of rank d if and only if $M = L_1 \oplus \cdots L_d$ where the L_i's are invertible R–modules.*

(2) $\mathfrak{G} = \mathrm{GL}(M)$ is split if and only if there exists an invertible R–module L such that $M \cong L^d$.

Proof. (1) If M decomposes as a sum of invertible modules, $\mathfrak{G}$ contains $\mathbb{G}_m^d$ as closed R–subgroup scheme. Conversely, assume that there is a closed immersion $i : \mathbb{G}^d \to \mathrm{GL}(M)$. By diagonalization, we get a decomposition $M = M_1 \oplus \cdots \oplus M_d$ and the M_i's are projective locally free of rank one.
(2) If $M = L^d$, we have $\mathrm{GL}(M) = \mathrm{GL}_d$. Conversely, assume that $\mathfrak{G}$ is split. In particular, there is there is a closed immersion $i : \mathbb{G}^d \to \mathrm{GL}(M)$ and we have $M = M_1 \oplus \cdots M_d$. We consider the adjoint action of $\mathbb{G}_{m,R}^d$ on $\mathrm{End}_R(M) = \mathrm{Lie}(\mathfrak{G})(R)$. We have the decomposition

$$\mathrm{End}_R(M) = R^d \oplus \bigoplus_{i \neq j} \mathrm{Hom}_R(M_i, M_j)$$

where $\mathrm{Hom}_R(M_i, M_j)$ is the eigenspace for the root $\alpha_i^{-1}\alpha^j$. Since $\mathfrak{G}$ is split, the eigenspaces are free modules, so we conclude that $M_1 \xrightarrow{\sim} M_i$ for $i = 2, ..., d$. $\square$

18.4.3. **Remark.** Assume that R is a Dedekind ring and let L be an invertible module. Then $L \oplus L^*$ is free so that $\mathrm{GL}_2 \xrightarrow{\sim} \mathrm{GL}(L \oplus L^*)$ contains a maximal split torus of $\mathfrak{T}$ rank two and the root decomposition is $gl_{2,R} = \mathrm{Lie}(\mathfrak{T}) \oplus L^{\otimes 2} \oplus (L^{\otimes 2})^*$. Hence $\mathfrak{T}$ is a splitting tous of GL_2 if and only if $[L] \in {}_2\mathrm{Pic}(R)$

Now let $E/\mathbb{C}$ be a projective elliptic curve and put $\mathrm{Spec}(R) = E \setminus \{0\}$. Then R is a Dedekind ring and we have an exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Pic}(E) \to \mathrm{Pic}(R) \to 0.$$

But $\mathrm{Pic}(E) \xrightarrow{\sim} \mathbb{Z} \oplus E(\mathbb{C})$ so that $E(\mathbb{C}) \xrightarrow{\sim} \mathrm{Pic}(R)$. Since $E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/\mathbb{Z}^2$, it follows that $\mathrm{Pic}(R)$ contains a class $[L]$ which is not torsion. Now we consider the R–group $\mathrm{GL}(R \oplus L)$ and claim it is not isotrivial, namely cannot be split by a finite étale extension of R. We reason by sake of contradiction. Assume there exists a finite étale cover R'/R such that $\mathfrak{G} \times_R R'$ splits. Lemma 18.4.2.(2) implies then that $L \otimes_R R' \xrightarrow{\sim} R'$, i.e.

$$[L] \in \ker\Big(\mathrm{Pic}(R) \to \mathrm{Pic}(R')\Big).$$

By means of the norm map $\mathrm{Pic}(R') \to \mathrm{Pic}(R)$ (see [EGA4, 21.5.5]), this kernel is torsion (killed by the degree of $[R' : R]$). This contradicts our assumption over L.

This example shows that the naive question 18.4.1 has a negative answer. In other words, Theorem 17.2.6 is not true for reductive (and semisimple) R-group schemes even for an R–group carrying a split maximal R–torus.

18.4.4. **Proposition.** *Let $\mathfrak{G}/R$ be a reductive group scheme admitting a maximal R–torus which is locally isotrivial. Then $\mathfrak{G}$ is locally isotrivial, that is there exists a Zariski cover $R_1 \times \cdots R_l$ of R such that $\mathfrak{G} \times_R R_i$ is split for $i = 1, .., l$.*

Note that it applies in particular when R is normal and $\mathfrak{G}$ contains a maximal R-torus by Theorem 17.2.6.

Proof. Let $\mathfrak{T}$ be a maximal R–torus. Up to localize, we can suppose that $\mathfrak{G}$ has constant type and that there exists a finite étale connected cover such that $\mathfrak{T} \times_R S \cong \mathbb{G}_S^d$. For each root α, the weight space $\mathrm{Lie}(\mathfrak{G})(S)_\alpha$ is an invertible S–module. For each $x \in \mathrm{Spec}(R)$, $S \otimes_R R_x$ is a semi-local ring so that $\mathrm{Pic}(S \otimes_R R_x) = 1$. It follows that $\mathfrak{G} \times_R (S \otimes_R R_x)$ splits. By quasi-compacity, we conclude that $\mathfrak{G}$ is locally isotrivial. $\square$

18.4.5. **Remark.** The statement is rather weak and can be strenghtened as follows: A semisimple R–group scheme is locally isotrial, see [SGA3, XXIX.4.1.5].

18.5. **Killing pairs.**

18.5.1. **Definition.** *A Killing couple is a pair $(\mathfrak{B}, \mathfrak{T})$ where $\mathfrak{B}$ is a Borel R–subgroup (see 12.3.1) and $\mathfrak{T}$ is a maximal R–torus of $\mathfrak{B}$.*

18.5.2. **Example.** Let $\mathfrak{G}/R$ be a split reductive group and $\mathfrak{T}/R$ be a maximal split torus of $\mathfrak{G}/R$. We denote by $\Psi = \Psi(\mathfrak{G}, \mathfrak{T})$ the associated root datum. Choose $\lambda \in \widehat{\mathfrak{T}}^0$ such that $\langle \alpha, \lambda \rangle \neq 0$ for each root α. We get then a subset of positive roots $\{\alpha \mid \langle \alpha, \lambda \rangle > 0\}$ and a basis Δ of the root system $\Phi(\mathfrak{G}, \mathfrak{T})$ [Bbk3, VI.1.7, cor. 2]. Then we claim that the limit R–group $\mathfrak{B} = \mathfrak{P}_G(\lambda)$ defined in 12.3.1 is a Borel subgroup of $\mathfrak{G}/R$. It is a closed subgroup scheme which is indeed smooth. Its Lie algebra is

$$\mathrm{Lie}(\mathfrak{B})(R) = \mathrm{Lie}(\mathfrak{T})(R) \oplus \bigoplus_{\alpha > 0} \mathrm{Lie}(\mathfrak{G})_\alpha(R).$$

Also its geometric fibers are parabolic subgroups [Sp, §15.1] whose Lie algebras are Borel subalgebras. Therefore the geometric fibers of $\mathfrak{B}$ are Borel subgroups and we conclude that $\mathfrak{B}$ is a Borel subgroup scheme of $\mathfrak{G}$.

18.5.3. **Theorem.** *Let $\mathfrak{G}/R$ be a reductive group scheme.*

(1) Locally for the étale topology, Killing couples of $\mathfrak{G}$ exist and are conjugated.

(2) Locally for the étale topology, Borel subgroups of $\mathfrak{G}$ exist and are conjugated.

Proof. By Theorem 18.1.1, $\mathfrak{G}$ is locally split for the étale topology. The example above shows that $\mathfrak{G}$ admits a Killing couple locally for the étale topology on $\mathrm{Spec}(R)$. It remains to treat the two conjugacy questions with essentually the same method than for 18.1.6. We can assume that R is finitely generated over $\mathbb{Z}$.

(1) Let $(\mathfrak{B}_1, \mathfrak{T}_1)$, $(\mathfrak{B}_2, \mathfrak{T}_2)$ be two Killing couples of $\mathfrak{G}/R$. Let $x \in \mathrm{Spec}(R)$, we want to show the statement étale-locally at x. Since the result holds for separably closed fields, we can localize for the étale topology so that $(\mathfrak{B}_{1,k}, \mathfrak{T}_{1,k}) = (\mathfrak{B}_{2,k}, \mathfrak{T}_{2,k})$. By Theorem 10.3.3.(3), there exist unique $\lambda_i : \mathbb{G}_{m,\widehat{R}_x} \to \mathfrak{T}_{i,\widehat{R}_x}$ which lifts $\lambda_{i,k}$ for $i = 1, 2$.

Lemma 12.3.3.(1) shows that $\mathfrak{B}_{i,\widehat{R}} = \mathfrak{P}_{\mathfrak{G}}(\lambda_i)$ for $i = 1, 2$. Now we use that λ_1 and λ_2 are $\mathfrak{G}(\widehat{R})$–conjugated according to Corollary 10.3.3.(2), i.e. $\lambda_{1,\widehat{R}} =^g \lambda_{2,\widehat{R}}$ for some $g \in \mathfrak{G}(\widehat{R})$. Since $\mathfrak{T}_{i,\widehat{R}} = \mathfrak{Z}_{\mathfrak{G}_{\widehat{R}}}(\lambda_i)$ for $i = 1, 2$, it follows that $(\mathfrak{B}_1, \mathfrak{T}_1)_{\widehat{R}} = {}^g(\mathfrak{B}_2, \mathfrak{T}_2)_{\widehat{R}}$. Once again the Artin approximation theorem enables to conclude that $(\mathfrak{B}_1, \mathfrak{T}_1)$ and $(\mathfrak{B}_2, \mathfrak{T}_2)$ are locally conjugated for the étale topology.

(2) It is a simplification of the previous argument. □

19. Towards the classification of semisimple group schemes

19.1. **Kernel of the adjoint representation.** Let $\mathfrak{G}/R$ be a reductive group scheme and denote by $\mathfrak{g} = \mathrm{Lie}(\mathfrak{G})(R)$ its Lie algebra. We consider the adjoint representation $\mathrm{Ad} : \mathfrak{G} \to \mathrm{GL}(\mathfrak{g}) = \mathfrak{H}$, it factorizes in the sequence of R–group functors

$$\mathfrak{G} \overset{int}{\to} \mathrm{Aut}(\mathfrak{G}) \overset{L}{\to} \mathrm{GL}(\mathfrak{g}) = \mathfrak{H}$$

where L maps an S-isomorphism $\varphi : \mathfrak{G}_S \to \mathfrak{G}_S$ to its differential $\mathrm{Lie}(\varphi) : \mathfrak{g} \otimes_R S \to \mathfrak{g} \otimes_R S$.

19.1.1. **Proposition.** *(1)* [SGA3, XXII.5.14] *The adjoint representation $\mathfrak{G} \to \mathrm{GL}(\mathfrak{g})$ induces a monomorphism $\mathfrak{G}_{ad} \to \mathrm{GL}(\mathfrak{g})$.*

(2) If $\mathfrak{G}$ is adjoint, $L : \mathrm{Aut}(\mathfrak{G}) \to \mathrm{GL}(\mathfrak{G}) = \mathfrak{H}$ is a monomorphism.

(3) If $\mathfrak{G}$ is adjoint, the morphism of R-functors $\mathrm{Aut}(\mathfrak{G}) \to N_{\mathfrak{H}}(\mathfrak{G})$ is an isomorphism and $\mathrm{Aut}(\mathfrak{G})$ is representable by a closed subgroup scheme of $\mathrm{GL}(\mathfrak{g})$.

Proof. (1) The field case is due to Rosenlicht [Rt, Lemma 1 p. 39]. We can assume that we deal with a base field k which is algebraically closed and reductive group G/k. By the Bruhat decomposition with respect to a Killing couple (B,T) an element of $\ker\Big(G(k) \to \mathrm{GL}(\mathfrak{g})(k)\Big)$ can be written up to conjugacy as $g = n_w u$ with $n \in N_G(T)(k)$. By looking at the action on $\mathfrak{g}$, we see that $w = 1$, that is $g \in B(k)$. We have $Z_G(\mathrm{Lie}(T)) = T$ so that $g \in N_G(T)(k) \cap B(k) = T(k)$. By looking at the roots we conclude that $g \in \mathfrak{Z}(\mathfrak{G})(k)$.

The method for reaching the general case is similar to that for showing Theorem 10.1.4. We denote by $\mathfrak{K}$ the kernel of $G_{ad} \to \mathrm{GL}(\mathfrak{G})$. We shall show that $\mathfrak{K}$ is proper by using the valuative criterion. Let A/R be a valuation ring and denote by F its fraction field. Since $\mathfrak{K}(F) = 1$ from the field case, we have $\mathfrak{K}(A) = \mathfrak{K}(F)$ and the criterion is fullfilled. Hence $\mathfrak{K}$ is proper. Since $\mathfrak{K}$ is affine, $\mathfrak{K}$ is finite over R [Li, 3.17]. It follows that $R[\mathfrak{K}]$ is a finite R–algebra such that $R/\mathfrak{M}_x \xrightarrow{\sim} R[\mathfrak{K}]/\mathfrak{M}_x\, R[\mathfrak{K}]$. The Nakayama lemma [Sta, 18.1.11.(6)] shows that the map $R \to R[\mathfrak{K}]$ is surjective. By using the unit section $1_{\mathfrak{K}}$ we conclude that $R = R[\mathfrak{K}]$.

(2) We assume that $\mathfrak{G}$ is adjoint and consider the group functor $L : \mathrm{Aut}(\mathfrak{G}) \to \mathrm{GL}(\mathfrak{G}) = \mathfrak{H}$. Let $\varphi \in \ker(L)(R)$. Then for each S/R and each $g \in \mathfrak{G}(S)$, we have

$$L(\varphi \circ \mathrm{int}(g) \circ \varphi^{-1}) = L(\mathrm{int}(g)) = \mathrm{Ad}(g).$$

But $\varphi \circ \mathrm{int}(g) \circ \varphi^{-1} = \mathrm{int}(\varphi(g))$ so we have $\mathrm{Ad}(\varphi(g)) = \mathrm{Ad}(g)$. Since Ad is a monomorphism by (1), it follows that $\varphi(g) = g$. This shows that $\varphi = id_{\mathfrak{G}}$. We conclude that L is a monomorphism.

(3) The map $\mathrm{Aut}(\mathfrak{G}) \to N_{\mathfrak{H}}(\mathfrak{G})$ is then a monomorphism. But this map admits a splitting hence is an isomorphism. By Theorem 14.9.1, we have that $\mathrm{Aut}(\mathfrak{G})$ is representable by a closed subgroup of $\mathfrak{H}$. □

19.2. **Pinnings.**

19.2.1. **Definition.** *Let $\mathfrak{G}/R$ be a split reductive group scheme equipped with a maximal torus $\mathfrak{T}$. A pinning of $\mathfrak{G}/R$ is a couple $E = (\Delta, (X_\alpha)_{\alpha\in\Delta})$ where Δ is base of the root datum $\Psi(\mathfrak{G},\mathfrak{T})$ and each X_α is an R–base of the invertible free R–module* $\mathrm{Lie}(\mathfrak{G})(R)_\alpha$.

If R is connected and $g \in \mathfrak{G}(R)$, then we can talk about the conjugated pinning ${}^gE_p$ relative to the R–torus ${}^g\mathfrak{T}$.

19.2.2. **Lemma.** *Assume that R is connected and let $\mathfrak{G}/R$ be an adjoint split group scheme which splits relatively to a split R–torus $\mathfrak{T}$.*

(1) The group $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$ acts simply transitively on the pinnings relatively to $\mathfrak{T}$.

(2) The group $\mathfrak{G}(R)$ acts simply transitively on the couples $(\mathfrak{S}, Ep)$ consisting of a maximal split torus which splits $\mathfrak{G}$ and a pinning.

Proof. Let (Δ, Ep) be a pinning. We prove firstly the freeness of the action. Let $g \in \mathfrak{G}(R)$ such that ${}^g(\mathfrak{T}, \Delta, (X_\alpha)) = (\mathfrak{T}, \Delta, (X_\alpha))$. Then $g \in \mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$ and we denote by w its image in $W = W_R(R)$. Then ${}^w\Delta = \Delta$ so that $w = 1$ and $g \in \mathfrak{T}(R)$. For each $\alpha \in \Delta$, we have $X_\alpha = {}^g X_\alpha = \alpha(g) X_\alpha$ so that $g \in \ker\Big(\mathfrak{T} \to \mathbb{G}_{m,R}^{\Delta}\Big) = 1$ since $\mathfrak{G}$ is adjoint. We prove now the transitivity.

(1) Let $(\Delta', (X'_{\alpha'}))$ be another pinning relative to $\mathfrak{T}$. There exists (a unique) $w \in W$ such that $\Delta = {}^w\Delta'$. By Proposition 18.2.1, w lifts to an element $n_w \in \mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$. We can then assume than $\Delta = \Delta'$. For each $\alpha \in \Delta$, we have $X'_\alpha = c_\alpha X_\alpha$ for some $c_\alpha \in R^\times$. Since $\mathfrak{T} \xrightarrow{\sim} \mathbb{G}_{m,R}^{\Delta}$, there exists then $t \in \mathfrak{T}(R)$ such that ${}^t X'_\alpha = X_\alpha$ for each $\alpha \in \Delta$. Thus $(\Delta', (X'_{\alpha'}))$ is a $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})(R)$-conjugate of Ep.

(2) Let $\mathfrak{T}'$ be another maximal split torus which splits $\mathfrak{T}'$ and let E'_p be pinning. By the unicity, we can reason étale locally so that $\mathfrak{T}'$ is $\mathfrak{G}(R)$-conjugated to $\mathfrak{T}$. This boils down to the case $\mathfrak{T}' = \mathfrak{T}$ where (1) applies. ☐

19.3. **Automorphism group.**

19.3.1. **Theorem.** [SGA3, XXIV.1] *Let $\mathfrak{G}/R$ be an adjoint reductive group scheme and denote by $\mathfrak{g}$ its Lie algebra.*

(1) The functor $\mathrm{Aut}(\mathfrak{G})$ *is representable by a smooth group scheme and the map* $\mathrm{Aut}(\mathfrak{G}) \to \mathrm{GL}(\mathfrak{g})$ *is a closed immersion.*

(2) The quotient sheaf $\mathrm{Out}(\mathfrak{G}) = \mathrm{Aut}(G)/\mathfrak{G}$ *is representable by a finite étale R–group scheme.*

(3) We assume that $\mathfrak{G}$ is split. Let $\mathfrak{T}$ be a maximal split torus of $\mathfrak{G}$ and let Ep be a pinning. Then $\mathrm{Out}(G)$ *is a finite constant group and the exact sequence*

$$1 \to \mathfrak{G} \to \mathrm{Aut}(\mathfrak{G}) \to \mathrm{Out}(\mathfrak{G}) \to 1$$

splits. More precisely, we have an isomorphism

$$\Big\{ f \in \mathrm{Aut}(\mathfrak{G}, \mathfrak{T})(R) \mid {}^f Ep = Ep \Big\} \xrightarrow{\sim} \mathrm{Out}(\mathfrak{G})(R).$$

Proof. to be written.

☐

19.4. **Unicity and existence theorems.** We give a weak version of the unicity theorem.

19.4.1. **Theorem.** *Let $(\mathfrak{G}, \mathfrak{T})$ and $(\mathfrak{G}', \mathfrak{T}')$ be two split reductive R-group schemes equipped with maximal split tori. Then $(\mathfrak{G}, \mathfrak{T})$ and $(\mathfrak{G}', \mathfrak{T}')$ are isomorphic iff the root data $\Psi(\mathfrak{G}, \mathfrak{T})$ and $\Psi(\mathfrak{G}', \mathfrak{T}')$ are isomorphic.*

We come now to the Chevalley's existence theorem.

19.4.2. **Theorem.** *Let $\Psi = (M, \mathcal{R}, M^\vee, \mathcal{R}^\vee)$ be a reduced root datum. Then there exists a split $\mathbb{Z}$–reductive group scheme $\mathfrak{G}$ equipped with a maximal split $\mathbb{Z}$–torus $\mathfrak{T}$ such that $\Psi(\mathfrak{G}, \mathfrak{T}) \cong \Psi$.*

This $\mathbb{Z}$-group scheme is called the Chevalley group associated to Ψ. If R is connected, a given reductive R–group scheme $\mathfrak{G}/R$ is isomorphic étale locally to a unique Chevalley group scheme $\mathfrak{G}_0 \times_{\mathbb{Z}} R$. It defines then a class in the non-abelian cohomology set $H^1(R, \mathrm{Aut}(\mathfrak{G}_0))$. This set encodes then the classification of reductive group schemes over R.

20. Appendix: smoothness

We refer to [BLR, §2.2]. Let $\mathfrak{X}/R$ be an affine scheme and let $x \in \mathfrak{X}$.

We say that $\mathfrak{X}$ is smooth at x of relative dimension d if there exists a neighbourhood $\mathfrak{U}/R$ of x and an R–immersion $j : \mathfrak{U} \to \mathbb{A}^n_R$ such that the following holds:

(1) Locally at $y = j(x)$, the ideal defining $\mathfrak{U}$ is generated by $n-d$ sections $g_1, ..., g_{n-d}$;

(2) The differentials $dg_1(y), \dots, dg_{n-d}(y)$ are linearly independent in the $\kappa(y)$-vector space $\Omega^1_{R[t_1,\dots,t_n]/R} \otimes_{R[t_1,\dots,t_n]} k(y)$.

We say that $\mathfrak{X}/R$ is étale at x if $\mathfrak{X}/R$ is smooth at x of relative dimension zero. Smoothness (resp. étalness) is an open condition on $\mathfrak{X}$ and the relative rank is locally constant.

We say that $\mathfrak{X}/R$ is smooth (resp étale) if it is smooth (resp. étale) everywhere. Smoothness and étalness are stable by composition and base change. Smoothness (resp. étalness) can be characterized by a property of the functor of points.

20.0.3. **Definition.** *We say that an R–functor is formally smooth (resp. formally étale) if for each R–ring C equipped with an ideal J satisfying $J^2 = 0$, the map*
$$F(C) \to F(C/J)$$
is surjective (resp. bijective).

Note that this property implies that for each R–ring C equipped with a nilpotent ideal J then the map $F(C) \to F(C/J)$ is surjective (resp. bijective).

20.0.4. **Theorem.** *Let $\mathfrak{X}/R$ be an affine scheme of finite presentation. Then $\mathfrak{X}/R$ is smooth (resp. étale) if and only if the R–functor $h_{\mathfrak{X}}$ is formally smooth (resp. formally étale).*

Another important result is the following characterisation of open immersions.

20.0.5. **Theorem.** [EGA4, th. 17.9.1] *Let $f : \mathfrak{X}/R \to \mathfrak{Y}/R$ be a morphism of affine R-schemes. The following are equivalent:*

(1) f is an open immersion;

(2) f is a monomorphism flat of finite presentation.

In particular, a smooth (and a fortiori étale) monomorphism is an open immersion.

A useful consequence is the following

20.0.6. **Corollary.** [EGA4, th. 17.9.5] *Let $f : \mathfrak{X}/R \to \mathfrak{Y}/R$ be a morphism of finite presentation between affine R-schemes. Assume that $\mathfrak{X}/R$ is flat. Then the following are equivalent:*

(1) f is an open immersion (resp. an isomorphism);

(2) $f_s : \mathfrak{X} \times_R \kappa(s) \to \mathfrak{Y} \times_R \kappa(s)$ is an open immersion (resp. an isomorphism) for each $s \in \mathrm{Spec}(R)$.

We recall also the following differential criterion.

20.0.7. **Theorem.** [EGA4, §17.11] *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a finitely presented R–morphism between two smooth affine R-schemes $\mathfrak{X}, \mathfrak{Y}$. Let $x \in \mathfrak{X}$ and put $y = f(x)$. Then the following statements are equivalent :*

(1) f is smooth (resp. étale) at x;

(2) The $\kappa(x)$–map $f^* : \Big(\Omega^1_{\mathfrak{Y}/R} \otimes_{R[\mathfrak{Y}]} \kappa(y)\Big) \otimes_{\kappa(y)} \kappa(x) \to \Omega^1_{\mathfrak{X}/R} \otimes_{R[\mathfrak{X}]} \kappa(x)$ is injective (resp. bijective).

If furthermore, $\kappa(x) = \kappa(y)$, it is also equivalent to

(3) The $\kappa(x)$-map on tangent spaces $T(f) : T_{\mathfrak{X},x} \to T_{\mathfrak{Y},y}$ is surjective (resp. bijective).

20.0.8. **Corollary.** *Let I be an ideal included in $\mathrm{rad}(R)$ (e.g. R is local and I is its maximal ideal). Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a finitely presented R–morphism between two smooth affine R-schemes $\mathfrak{X}, \mathfrak{Y}$. Then the following are equivalent:*

(1) f is smooth (resp. étale);

(2) $f_{R/I} : \mathfrak{X} \times_R R/I \to \mathfrak{Y} \times_R R/I$ is smooth (resp. étale).

Proof. The direct way follows of the base change property of smooth morphisms. Conversely, we assume that $f_{R/I}$ is smooth (resp. étale) and we consider the smooth (resp. étale) locus $\mathfrak{U}$ of f. It is an open subscheme of $\mathfrak{X}$. According to the differential criterion, $\mathfrak{U}$ contains then $\mathfrak{X} \times_R R/I$. We claim that $\mathfrak{U} = \mathfrak{X}$. Assume it is not the case. Since $\mathfrak{X}$ is quasi-compact (see [EGA1, §2.1.3]), $\mathfrak{X} \setminus \mathfrak{U}$ contains a closed point x. It maps to a closed point of $\mathrm{Spec}(R)$, contradiction. □

20.0.9. **Corollary.** *Let $f : \mathfrak{Y} \to \mathrm{Spec}(R)$ be an étale morphism where $\mathfrak{Y}$ is an affine scheme. Let s be a section of f. Then f is an clopen immersion.*

In particular, the diagonal map $\mathfrak{Y} \xrightarrow{\Delta} \mathfrak{Y} \times_R \mathfrak{Y}$ is a clopen immersion.

The general underlying statement is [EGA4, cor. 17.9.4].

Proof. We put $\mathfrak{X} = \mathrm{Spec}(R)$ and see the section s as an R-map $\mathfrak{X} \to \mathfrak{Y}$ between two smooth R–schemes. By considering the carthesian square

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\Delta_{\mathfrak{Y}}} & \mathfrak{Y} \times_R \mathfrak{Y} \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle id \otimes s} \\ \mathfrak{X} & \xrightarrow{s \otimes id} & \mathfrak{Y} \times_R \mathfrak{X} \end{array}$$

we see that s is a closed immersion. It is then enough to show that s is étale. Let $x \in \mathfrak{X}$ with image $y = s(x)$. We have $\kappa(x) = \kappa(y)$ and the $\kappa(x)$-map on tangent spaces $T(s)_x : T_{\mathfrak{X},x} \to T_{\mathfrak{Y},y}$ is a section of the bijective map $T(f)_y$, so is bijective. Therefore s is étale and is an open immersion. □

21. Appendix: universal cover of a connected ring

The interested reader could skip that appendix and read directly at the source [SGA1] or Szamuely's book [Sz]. There is no noetherianity assumption but the reader can assume for simplicity that R is noetherian.

A finite étale cover S/R is flat cover which is finite and étale. Since it is flat, it is projective and then locally free. We state the following nice characterization (not used in the sequel).

21.0.10. **Proposition.** [EGA4, 18.3.1] *Let S/R be a finite ring. Then S/R is étale if and only if S is a projective R-module and S is a projective $S \otimes_R S$-module.*

A special case is that of a Galois cover with respect to a finite abstract group Γ. Recall that our convention (and that of [P]) is that a Galois cover is a étale Γ-algebra for a finite group Γ[12].

If Γ is a finite group, the pointed set $H^1(R, \Gamma)$ classifies Γ-torsors and then Galois covers of R with group Γ. If $u : \Gamma \to \Gamma'$ is a group homomorphism of finite groups, we have an induced map $u_* : H^1(R, \Gamma) \to H^1(R, \Gamma')$. Of course, this map has a description in terms of algebras and by composition, it is enough to deal with the case u surjective and u injective.

If u is surjective, we associate to a Γ-algebra S/R the Γ'-algebra $S^{\ker(u)}$.

If u is injective, we associate to a Γ-algebra S/R its induction

$$\mathrm{Ind}_{\Gamma}^{\Gamma'}(S)$$

which is defined exactly as in the field case [KMRT, 18.17]. In terms of R-modules, $\mathrm{Ind}_{\Gamma}^{\Gamma'}(S) \xrightarrow{\sim} S^{[\Gamma':\Gamma]}$ and

$$\mathrm{Ind}_{\Gamma}^{\Gamma'}(S) = \Big\{ \alpha \in Map(\Gamma', S) \mid \alpha(\gamma\gamma') = \gamma \circ \alpha(\gamma') \ \forall \gamma, \gamma' \Big\}.$$

21.1. **Embedding étale covers into Galois covers.** Our goal is to make the construction of the universal cover of the base ring R with respect to a F-point $f : R \to F$ where F is a separably closed field.

21.1.1. **Lemma.** *Let S/R be a finite étale cover of R. Assume that R is connected and denote by d the rank of S/R.*

(1) Let $s : S \to R$ be a (ring) section of $R \to S$. Then $S = R \times S'$ where S'/R is a finite étale cover and s is the first projection.

(2) S has at least d connected components and those are finite étale covers.

(3) If S/R is Galois, so are its connected components.

[12] Warning: in [SGA1] and [Sz], a Galois cover is by definition connected.

Proof. (1) This follows of Corollary 20.0.9.

(2) We reason by induction on the rank d of S/R, the rank 0 case being trivial. If S is connected, there is nothing to do. Else there exists an idempotent $e \in S \setminus \{0,1\}$ such that $S = Se \times S(1-e)$. Those are direct summands of S, so are f.g. projective or respective rank $r \geq 1$, $d - r \geq 1$. Then $S.e$ and $S(1-e)$ are finite over R and are étale since $\mathrm{Spec}(S.e)$ (resp. $\mathrm{Spec}(S.(1-e))$) is open in $\mathrm{Spec}(S)$. By induction $S.e$ and $S(1-e)$ have at least r and $d-r$ connected components. Thus S has at least d connected components.

Assume firstly that S is noetherian. It has finitely many connected components, hence reads $S = S_1 \times \cdots \times S_l$. Each S_i is locally free over R of rank d_i. Also S_i is étale over R, so $S_i \to R$ is finite étale and is surjective since R is connected.

The general case is a limit argument. We can write $R = \varinjlim_{\alpha \in I} R_\alpha$ and $S = S_{\alpha_0} \otimes_{R_{\alpha_0}} S$ with the R_α noetherians [Sta, 135.3]. We can assume furthermore that the R_α's are connected, then each S_α has at least d connected components. It follows that the number of connected components of S is finite and the components are defined at finite stage.

(3) We assume furthermore than S/R is Galois for a group Γ. Then Γ permutes the connected components so that $S = S_1^{(\Gamma/\Gamma_1)}$ where Γ_1 is the stabilizer of S_1. Then Γ_1 acts freely on S_1 and $R = S_1^{(\Gamma_1)}$. We conclude that S_1 is a Galois cover of group Γ_1. □

21.1.2. **Proposition.** *Assume that R is connected. Let S/R be a ring extension and d be a non-negative integer. Then the following are equivalent:*

(i) S/R is finite étale of degree d;

(ii) There exists a finite étale connected cover T/R such that $S \otimes_R T \xrightarrow{\sim} T \times \cdots T$ (d times);

(iii) There exists a flat cover R'/R such that $S \otimes_R R' \xrightarrow{\sim} R' \times \cdots R'$ (d times).

Proof. The implication $(ii) \Longrightarrow (iii)$ is obvious and the implication $(iii) \Longrightarrow (i)$ is faithfully flat descent. We assume then (i) namely S/R is finite étale of degree d. If $d = 0$, there is nothing to do. We assume then $d \geq 1$ and put $T_1 = S$. According to Lemma 21.1.1.(1), the codiagonal map $S \otimes_R T_1 \to S$ defines a decomposition

$$S \otimes_R T_1 \quad = \quad T_1 \times S_1 \tag{21.1.3}$$

where S_1/T_1 is a finite étale cover of degree $d-1$. By induction, there exists a finite étale cover T_d/T_1 such that

$$S_1 \otimes_R T_d \xrightarrow{\sim} T_d \times \dots T_d \qquad ((d-1) \text{ times}).$$

Then T_d is finite étale over R and by reporting in the identity (21.1.3), we get $S \otimes_R T_d = T_d \times \cdots \times T_d$ (d times). We now take a connected component of T_d and we have $S \otimes_R T = T \times \cdots \times T$ (d times). $\square$

21.1.4. **Remark.** The proof of $(i) \Longrightarrow (ii)$ is a ring version of the construction of a splitting field for a separable polynomial.

21.1.5. **Lemma.** *We consider the flat sheaf in algebras $S \to S^d = F_d(S)$. Then the natural map $S_{d,R} \to \mathrm{Aut}(F_d)$ of R–functors is an isomorphism.*

In other words, the flat sheaf R– $\mathrm{Aut}(F_d)$ is representable by the constant group scheme S_d.

Proof. We can assume that R is noetherian and also connected. We have $S_d \subset \mathrm{Aut}_R(R^d)$. An R-automorphism σ of R^d permutes the idempotent elements, so is a given by a permutation. $\square$

According to Lemma 21.1.1.(3), the yoga of forms 15.2.1 yields

21.1.6. **Corollary.** *Let d be a positive integer. Then there is a one to one correspondence*

$$\Big\{ \textit{Étale algebras of degree } d \Big\} \quad < - - > \quad H^1(R, S_d).$$

In other words, if S is a finite R–étale algebra of degree d, we attach to it the S_d–torsor

$$T \mapsto E_S(T) = \mathrm{Isom}_{T-alg}(T^d, S \otimes_R T).$$

S defines a flat sheaf in algebras $\mathfrak{W}(S)$ and one has a canonical isomorphism ${}^{E_S}(F_d) \xrightarrow{\sim} \mathfrak{W}(S)$.

21.1.7. **Proposition.** *(Serre,* [Se1, §1.5]*) Let S/R be a finite étale cover.*

(1) There exists an étale cover $\widetilde{S}/S$ such that $\widetilde{S}/R$ is a Galois cover.

(2) If S is connected, $\widetilde{S}$ can be chosen to be connected.

Proof. (1) Up to localize, we can assume that S/R is locally free of constant rank, say $d \geq 1$. The idea of the proof is to do it in the split case R^d in a S_d-equivariant way and to twist that construction. The group S_d acts by permutation on the finite set $\Omega_d = \{1, \dots d\} = S_d/S_{d-1}$. We have $R^d = R[S_d/S_{d-1}]$ and it embeds in the group algebra $R[S_d]$ by the norm map $N = N_{S_d/S_{d-1}} : R[S_d/S_{d-1}] \to R[S_d] \cong R^{(S_d)}$.

Since $R[S_d]$ is the split Galois R-algebra of group S_d, that makes the case of R^d. For the general case of S/R, we can twist this construction by the S_d–torsor above, this provides an embedding of S in a Galois S_d–algebra $\widetilde{S}$. By descent, $\widetilde{S}$ is finite étale of degree $(d-1)!$ over S.

(2) This follows of Lemma 21.1.1.(3). $\square$

21.2. **Construction of the universal cover.** We assume that R is connected and equipped with a point $f : R \to F$ where F is a separably closed field. We consider the category $\mathcal{C}$ of pointed connected Galois covers over R. The objects are Galois covers S/R equipped with a map $f_S : S \to F$ extending f. The morphisms are R–morphisms commuting with the maps f_S. More precisely, a map $h : S_1 \to S_2$ is an R–ring map such that $f_{S_1} = f_{S_2} \circ h$.

21.2.1. **Lemma.** *Let $(S_1, f_1), (S_2, f_2) \in \mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{C}}\Big((S_1, f_1), (S_2, f_2)\Big)$ is empty or consists in one map which is a finite étale cover.*

Proof. Let $h : S_1 \to S_2$ be such a map. Proposition 21.1.2 provides a finite connected étale cover T which splits S_1 and S_2. Hence $S_1 \otimes_R T \xrightarrow{\sim} T^{(d_1)}$ and $S_2 \otimes_R T \xrightarrow{\sim} T^{(d_2)}$ so that h_T is finite étale. By descent, h is finite étale. Since S_2 is connected, h is a finite étale cover.

Now, let h, h' be two such maps. We want to show that $h = h'$. The T-maps $h_T, h'_T : T^{(d_1)} \to T^{(d_2)}$ are given by matrices with entries 0 or 1. Since the two maps agree after tensoring by $T \otimes_R F$, we get that $h_T = h'_T$. By descent, we conclude that have $h' = h$. □

This permits to equip that category with the preorder $(S_2, f_2) \geq (S_1, f_1)$ if $\mathrm{Hom}_{\mathcal{C}}((S_1, f_1), (S_2, f_2)) \neq \emptyset$. For each relation $(S_2, f_2) \geq (S_1, f_1)$, there is a unique map $h_{1,2} : S_1 \to S_2$.

This category is directed by the following construction. Given objects $(S_1, f_1), \dots, (S_n, f_n)$ of $\mathcal{C}$, the tensor product $T = S_1 \otimes_R S_2 \cdots \otimes_R S_n$ is a Galois R-algebra equipped with the codiagonal map $f_T : T \to F$. Then T splits as $T \xrightarrow{\sim} T_1 \times \dots T_r$ where the T_i are connected Galois R-algebras such that $f_T(T_j) = 0$ for $j = 2, .., r$. It follows that $(T_1, f_{T_1}) \geq (S_i, f_i)$ for $i = 1, .., n$ and this is the upper bound.

We can define then the simply connected étale cover of (R, f) by

$$R^{sc} \;=\; \varinjlim_{(S,f)} S. \tag{21.2.2}$$

It comes equipped with a map $f^{sc} : R^{sc} \to F$ and we define the fundamental group

$$\pi_1(R, f) = \varprojlim_{(S,f)} \mathrm{Gal}(S/R).$$

It is a profinite group, it acts continously on R^{sc}. For each object (S, f_S) we have then a natural map $S \to R^{sc}$ which is $\pi_1(R, f)$-equivariant. Such a map is unique.

21.2.3. **Remark.** Since the transition maps $\mathrm{Gal}(S_2/R) \to \mathrm{Gal}(S_1/R)$ are onto, it follows that each map $\pi_1(R, f) \to \mathrm{Gal}(S/R)$ is onto.

We record the following property of R^{sc}.

21.2.4. **Proposition.** *R^{sc} is connected and the finite étale covers of R^{sc} are split so that $\pi_1(R^{sc}, f^{sc}) = 1$.*

Proof. The fact R^{sc} is connected follows from the argument as in Lemma 21.1.1.(2). Let L/R^{sc} be a finite étale connected cover. According to [Sta, 135.3], it is defined at finite stage, that is there exists $S \in \mathcal{C}$ and a finite étale cover T/S such that $T \otimes_R R^{sc} = L$. Also T is connected. By Proposition 21.1.7, L has a Galois closure $\widetilde{T}$ which is connected and it enough to show that $\widetilde{T} \otimes_R R^{sc}$ splits. We put $\Gamma = \mathrm{Gal}(\widetilde{T}/R)$. Since $\widetilde{T} \otimes_S F \cong F[\Gamma]$, f extends to a map $\widetilde{f} : \widetilde{T} \to F$. It follows that that $\widetilde{T}$ embeds in R^{sc}. Thus L/R^{sc} splits. □

One can also give a universal property for connected étale covers, not only the Galois ones, see [Sz, th. 5.4.2]. For Galois covers with a given fixed group, we get

21.2.5. **Theorem.** *Let Γ be a finite abstract group. There is a natural bijection*

$$\mathrm{Hom}_{cont}\big(\pi_1(R,f),\Gamma\big)/\Gamma \xrightarrow{\sim} H^1(R,\Gamma).$$

Proof. The pointed set $H^1(R,\Gamma)$ classifies Γ-torsors and equivalently Galois R-algebras of group Γ. Given such a Γ-étale algebra S, we decompose it as $A = S_1 \times \cdots \times S_r$ in connected components which are Galois over R. The subgroups $\Gamma_i = \mathrm{Gal}(S_i/R)$'s are conjugated in Γ. Choose an extension $f_1 : S_1 \to F$ of f, the choice is up to Γ_1–conjugacy. Then by construction, we have a surjective continuous map

$$\phi_1 : \pi_1(R,f) \to \Gamma_1.$$

Up to Γ-conjugacy, the composite map $\phi_1 : \pi_1(R,f) \to \Gamma_1 \subset \Gamma$ does not depend of the choices made. We have then defined a map

$$H^1(R,\Gamma) \to \mathrm{Hom}_{cont}\big(\pi_1(R,f),\Gamma\big)/\Gamma.$$

Let us define the converse map. Let $\phi : \pi_1(R,f) \to \Gamma$ be a continuous homomorphism. Then it factorizes at finite level, that is there exists a Galois connected cover $S \subset R^{sc}$ and ϕ_0 such that

$$\phi : \pi_1(R,f) \to \mathrm{Gal}(S/R) \xrightarrow{\phi_0} \Gamma.$$

Then we can attach to ϕ_0 the class of the Γ-cover $u_*(S)$. We let the reader to check that the two maps are inverse of each other. □

We discuss quickly other functoriality properties of this construction. If $h : R_0 \to R$ is a morphism of rings, we put $f_0 = f \circ h : R_0 \to F$. Given a connected Galois cover S_0/R, we denote by S/R the component of $S_0 \otimes_{R_0} R$ on which $f_{S_0} \otimes f$ is not trivial. By passing to the limit, it yields a natural map

$$R_0^{sc} \to R^{sc}$$

and a continous map $\pi_1(R,f) \to \pi_1(R,f_0)$. This last base change map is onto if and only if $S_0 \otimes_{R_0} R$ is connected for each connected Galois cover

S_0/R_0. One important special case is when R/R_0 is a Galois cover of group Γ_0. In that case, one has the fundamental exact sequence of Galois theory

$$(*) \qquad 1 \to \pi_1(R, f) \to \pi_1(R_0, f_0) \to \Gamma_0 \to 1.$$

Also, in some sense, the group $\pi_1(R, f)$ does not depend of the choice of the base point f, but in a non canonical way see [Sz, prop. 5.5.1].

21.2.6. **Remark.** One needs to be careful with continuity issues. For example, we consider the profinite group $\mathcal{G} = \varprojlim_{n\geq 1} (\mathbb{F}_p)^n$ where the transition maps are the projections on the last coordinates. It is a $\mathbb{F}_p$-vector space. Since there are $\mathfrak{F}_p$ linear forms mapping the vector $(\cdots, 1, 1, \cdots, 1, 1)$ to 1, $\mathcal{G}$ has plenty of no continous group homomorphisms onto $\mathbb{F}_p$.

21.3. **Examples.**

21.3.1. *Case of a normal ring R.* We assume that R is normal with fraction field K. In this case, it is convenient to take F as a separable closure of K. Each point $x \in \mathrm{Spec}(R)$ of codimension 1 defines a discrete valuation v_x on K. We denote by $\widehat{K}_x$ its completion.

Let L/K be a Galois subextension of F and put $\Gamma = \mathrm{Gal}(L/K)$. For each x point of $\mathrm{Spec}(R)$ of codimension 1, we have a decomposition

$$L \otimes_K \widehat{K}_x \xrightarrow{\sim} (\widehat{L}_x)^{(\Gamma/\Gamma_x)}$$

where $\widehat{L}_x/\widehat{K}_x$ is a Galois extension of local fields for a subgroup $\Gamma_x \subset \Gamma$. We say that L/K is unramified at x if the extension $\widehat{L}_x/\widehat{K}_x$ is unramified (that is an uniformizing parameter of $\widehat{K}_x$ is an uniformizing parameter of $\widehat{L}_x$).

We say that L/K is unramified over R if it is unramified at each point of $\mathrm{Spec}(R)^{(1)}$. In this case, it can be shown [Sz, 5.4.9], that the ring of integers R_L of R in L is a finite Galois extension of group Γ. Also R^{sc} is the inductive limit of those R_L, so that $\pi_1(R, f)$ is the maximal unramified quotient of $\pi_1(R, f)$ with respect to $\mathrm{Spec}(R)$.

In particular if R' is a localisation of R, then $\pi_1(R', f)$ maps onto $\pi_1(R, f)$. This applies to the so-called Kummer covers.

21.3.1. **Proposition.** *Let $n \geq 1$ be an integer such that K contains a primitive root of unity.*

(1) Let $a \in K^\times$ and assume that $K_a = K[T]/(T^n - a)$ is a field extension of K. Then K_a/K is unramified over R if and only if if and only if $\mathrm{div}(u) \in n\mathrm{Div}(R)$.

(2) The construction above induces a natural group isomorphism

$$\ker\Big(K^\times/(K^\times)^n \overset{\mathrm{div}}{\to} \mathbb{Z}/n\mathbb{Z}\Big) \xrightarrow{\sim} H^1(R, \mathbb{Z}/n\mathbb{Z}).$$

Proof. (1) The field extension K_a/K is a cyclic extension of degree n. For each $x \in \mathrm{Spec}(R)^{(1)}$, we can write $a = a_x^{m_x}$ with $m_x \mid n$ such that

$$K_a \otimes_R \widetilde{K}_x \xrightarrow{\sim} \widetilde{K}_x(\sqrt[n_x]{a_x}) \times \cdots \times \widetilde{K}_x(\sqrt[n_x]{a_x}) \quad (m_x \text{ times})$$

where $n_x = \frac{n}{m_x}$. If $\mathrm{div}(a) = \big(v_x(a)\big) \in n\mathrm{Div}(R)$, then we can replace write $a = b_x^n\,(u_x)^{m_x}$ with $u_x \in \widehat{R}_x^\times$, so that $\widetilde{K}_x(\sqrt[n_x]{a_x})/\widehat{K}_x = \widetilde{K}_x(\sqrt[n_x]{u_x})$ is an unramified extension. In this case, K_a/K is unramified over R.

Conversely, we assume that K_a/K is unramified over R. Then $a_x \in \widehat{R}_x^\times.(\widehat{K}_x)^{n_x}$ so that $a = a_x^{m_x} \in \widehat{R}_x^\times.(\widehat{K}_x)^n$.

(2) To be written.

□

21.3.2. *Affine line and affine spaces.* Let k be a field of characteristic zero and let k_s be a Galois closure. Consider the point $0_s : \mathrm{Spec}(k_s) \to \mathrm{Spec}(k) \xrightarrow{s_0} \mathbb{A}^1_k$.

21.3.2. **Proposition.** *The map $s_{0,*} : \pi_1(\mathbb{A}^1_k, 0_s) \to \mathrm{Gal}(k_s/k)$ is an isomorphism. In particular, if k is algebraically closed, $\mathbb{A}^1_k$ is simply connected.*

Proof. From the fundamental (split) sequence of Galois theory $(*)$ above

$$1 \to \pi_1(\mathbb{A}^1_{k_s}, 0_s) \to \pi_1(\mathbb{A}^1_k, 0_s) \to \mathrm{Gal}(k_s/k) \to 1$$

we can assume that k is algebraically closed. Let $f : X \to \mathbb{A}^1_k$ be a finite étale cover. By normalization, it extends to a map $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1_k$ where $\widetilde{X}$ is a smooth projective curve. Put $d = [k(X) : k(t)]$ and denote by g the genus of $\widetilde{X}$. Let $p_1, .., p_r$ be the points of $\widetilde{X}(k)$ mapping to ∞ and denote by $e_1, ..., e_r$ their respective multiplicities. Then we have the formula $d = e_1 + e_2 + \cdots + e_r$, in particular $r \leq d$. In the other hand, we have the Hurwitz formula [H, IV.2.4]

$$2g - 2 = d(-2) + (e_1 - 1) + \cdots + (e_r - 1).$$

Hence

$$2g - 2 = -d - r < 0$$

so that $g = 0$ and $d = 1$. We conclude that f is an isomorphism. □

21.3.3. **Remark.** The characteristic zero assumption is essential here (this was used in Hurwitz formula). If k is of characteristic $p > 0$, the Artin-Schreier map

$$f : \mathbb{A}^1_k \to \mathbb{A}^1_k, \;\; t \mapsto t^p - t$$

is a connected Galois cover of group $\mathbb{Z}/p\mathbb{Z}$.

By using §21.3.1, one can derive that $\pi_1(X \times \mathbb{A}^1_k, 0_s \times x_s) \to \pi_1(X, x_s)$ for each geometrically normal variety X/k. In particular, it follows by induction on n that $\pi_1(\mathbb{A}^n_k, 0_s) \xrightarrow{\sim} \mathrm{Gal}(k_s/k)$ for all $n \geq 1$.

21.3.3. *Split tori.* For a split torus $\mathbb{G}_m^n$, one has a complete description of the covers.

21.3.4. **Proposition.** *(1) $\pi_1(\mathbb{G}_m^n) \xrightarrow{\sim} \Big(\varprojlim_m \mu_m(k_s)\Big)^n \rtimes \mathrm{Gal}(k_s/k)$.*

(2) Let S be a connected finite étale cover of $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Let $L \subset S$ be the integral closure of k in S. Then there exists $g \in \mathrm{GL}_n(\mathbb{Z})$, $a_1, ..., a_n \in L^\times$ and positive integers $d_1, ..., d_n$ such that $d_1 \mid d_2 \cdots \mid d_n$ and

$$S \otimes_{R_n} R_n^g \simeq_{R_n\text{-}alg} (R_n \otimes_k L)\Big[\sqrt[d_1]{a_1 t_1}, \cdots , \sqrt[d_n]{a_n t_n} \Big].$$

In particular, S is k–isomorphic to $R_n \otimes_k L$ and $\mathrm{Pic}(S) = 0$.

By R_n^g/R_n we mean the map $g_* : R_n \to R_n$ arising from the left action of $\mathrm{GL}_n(\mathbb{Z})$ on R_n. For (1) (resp. (2)), see [GP2, 2.10] (resp. [GP3, §2.8]).

For $k = k_s$, the simply connected cover is then $\varinjlim_m k\Big[t_1^{\pm \frac{1}{m}}, \cdots t_n^{\pm \frac{1}{m}}\Big]$.

21.3.4. *Semisimple algebraic groups.* We assume that k is algebraically closed of characteristic zero. Let G'/k be a semisimple algebraic group. Denote by $f : G \to G'$ its simply connected cover defined by the theory of algebraic groups [Sp, 10.1.4]; for example SL_n is the simply connected cover of PGL_n). Recall that we require that if T is a maximal torus of G, then the cocharacter group $\widehat{T}^0$ is generated by the coroots of (G, T) (or equivalently that $\widehat{T}$ is the weight lattice). The kernel $\mu = \ker(f)$ is a finite diagonalizable group.

21.3.5. **Proposition.** *(1) $\pi_1(G, 1) = 1$, that is G is simply connected in the sense of* [SGA1].

(2) $f : G \to G'$ is the universal cover of G, so that $\mu(k) \xrightarrow{\sim} \pi_1(G, 1)$.

Proof. (1) Let B and B^- be a pair of opposite Borel subgroups of G such that $B \cap B^- = T$. We denote by U and U^- their respective unipotent radicals. The idea is to use the big cell $V = Un_0UT$ of G. This an open subvariety of G so that the map

$$\pi_1(Un_0UT, n_0) \to \pi_1(G, n_0)$$

is onto. But $V \xrightarrow{\sim} U \times U \times T$ and U is an affine space. From §21.3.1, it follows that $\pi_1(V, n_0) \cong \pi_1(T, 1)$. We have then shown that the map

$$\widehat{T}^0 \otimes_{\mathbb{Z}} \varprojlim_n \mu_n(k) \xrightarrow{\sim} \pi_1(T, 1) \to \pi_1(G, 1)$$

is surjective. In particular, $\pi_1(G, 1)$ is abelian so that all finite étale covers are Galois.

The SL_2*-case:* In this case, $\pi_1(\mathrm{SL}_2)$ is procyclic. Let $p : X \to \mathrm{SL}_2$ be a finite Galois cover. Then p is Galois of group $\mathbb{Z}/n\mathbb{Z}$ and its restriction V is a connected Galois of group $\mathbb{Z}/n\mathbb{Z}$. With coordinates $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the big cell is $V = \{c \neq 0\} \cong \mathbb{A}^2 \times \mathbb{G}_m$. It follows that $k(X) = k(\mathrm{SL}_2)(\sqrt[n]{c})$. We consider the divisor of the rational function c on SL_2.

21.3.6. **Claim.** *(special elementary case of* [BD, lemme 2.4]*) We have* $\mathrm{div}(c) = 1\,[B] \in \mathrm{Div}(\mathrm{SL}_2)$.

By Proposition 21.3.1.(1), the fact that the extension $k(\mathrm{SL}_2)(\sqrt[n]{c})/k$ is unramified over SL_2 implies that $n=1$.

The general case. Since the coroots of (G,T) generate the cocharacter group $\widehat{T}^0$, it is enough to show the triviality of the map $(\alpha^\vee)_* : \varprojlim_m \mu_m(k) \to \pi_1(G,1)$ for each root $\alpha \in \Phi(G,T)$. But $\alpha^\vee : \mathfrak{G}_m \to G$ factorizes by $\alpha^\vee : \mathrm{SL}_2 \to G$, so $(\alpha^\vee)_* : \varprojlim_m \mu_m(k) \to \pi_1(G,1)$ factorizes by $\pi_1(\mathrm{SL}_2,1) = 1$. We conclude that $\pi_1(G,1)=1$.

(2) to be written.

□

21.3.7. **Remark.** This statement is wrong in characteristic $p>0$. We consider the Frobenius map $F : \mathrm{SL}_2 \to \mathrm{SL}_2$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^p & b^p \\ c^p & d^p \end{bmatrix}$. Generalizing the Artin-Schreier cover, the Lang isogeny

$$h : \mathrm{SL}_2 \to \mathrm{SL}_2, \ \ g \mapsto g^{-1}\, F(g)$$

is a finite (connected) Galois cover of group $\mathrm{SL}_2(\mathbb{F}_p)$ (e.g. [G, §1]).

21.4. **Local rings and henselizations.**

21.4.1. **Definition.** *A ring extension S/R is said standard étale if there are two polynomials $f, g \in R[X]$ such that*

(i) f is monic and its derived polynomial f' is invertible in $R[X]_g/f(X)$;

(ii) $S \xrightarrow{\sim} R[X]_g/f(X)$.

The map $R \to S$ decomposes then as $R \to R[X]_g \to R[X]_g/f(X)$ and is then étale by definition. Locally, étale maps are standard [SGA1, §7].

21.4.2. **Proposition.** [Sta, 135.16] *Let $R \to S$ be an étale extension and let $\mathfrak{Q}$ be prime ideal of S. Then there exists $g \notin \mathfrak{Q}$ such that S_g/R is standard étale.*

21.4.3. **Definition.** *Let R be a local ring with maximal ideal $\mathfrak{M}$ and residue field k. Then R is henselian if for each monic polynomial $P(X)$, each coprime factorization $\overline{P}(X) = \overline{P}_1(X)\dots\overline{P}_r(X) \in k[X]$ can be lifted in a factorization $P(X) = P_1(X)\dots P_r(X)$.*

The ring R is strictly henselian if it is henselian and its residue field is separably closed.

There are several characterisations of Henselian rings.

21.4.4. **Theorem.** [Mi, §I.4] *Let $(R, \mathfrak{M}, k)$ a local ring. Then the following are equivalent:*

(1) R is henselian;

(2) Any finite R–algebra is a product of local rings;

(3) For each étale map $f : \mathfrak{Y} \to \mathrm{Spec}(R)$ and each k–point $y \in \mathfrak{Y}$ then f admits a section mapping the closed point x of $\mathrm{Spec}(R)$ *to y.*

(4) For each smooth map $f : \mathfrak{Y} \to \mathrm{Spec}(R)$ and each k–point $y \in \mathfrak{Y}$ then f admits a section mapping the closed point x of $\mathrm{Spec}(R)$ *to y.*

The struct henselization R^{sh} of R is defined by taking a limit. The objects in the category $\mathcal{C}^h$ are the essentially étale rings S/R. It means that there exists an étale ring $S_\sharp/R$ and an ideal $\mathfrak{Q}$ of $S_\sharp$ such that $S = (S_\sharp)_{\mathfrak{Q}}$ and such that the morphism $R \to S$ is local. Given two such objects S_1, S_2, the morphisms are local R–ring morphisms $S_1 \to S_2$ and there exists at least one object. This defines a preorder and this category is directed [EGA4, 18.6.3]. This permits to take the inductive limit.

21.4.5. **Proposition.** *R^{sh} is a strictly henselian local ring.*

Proof. The ring R^{sh} is local by construction and the construction allows to lift separable field algebras $k[t]/f(t)$ so that the residue field of R^{sh} is a separable closure of k. We need to verify that R^{sh} satisfies the third property of the characterization 21.4.4. Consider an étale map $f : \mathfrak{Y} \to \mathrm{Spec}(R^{sh})$ and a k_s–point $y \in \mathfrak{Y}$ mapping to the closed point x of $\mathrm{Spec}(R^{sh})$. Let S be an affine neighboorhood of y and denote by $\mathfrak{M}_y$ the maximal ideal of regular functions vanishing at y. Then $R^{sh} \to S_{\mathfrak{M}_y}$ is a essentially étale morphism. Since this morphism is defined at finite level in the tower, it splits. $\square$

This construction satisfies an universal property, see [R, VIII]. Note also that there is natural map $R^{sc} \to R^{sh}$.

The construction of the henselization is similar, we require furthermore that k is the residue field of each S/R.

21.4.6. **Theorem.** [BLR, §2.4, cor. 9] *R^h and R^{sh} are faithfully flat extensions of R.*

We focus on the case of a normal ring.

21.4.7. **Theorem.** *(Raynaud* [R, XI.1] *or* [BLR, §2.3, prop. 11]*) Let $(R, \mathfrak{M}, k)$ be a normal local ring. We denote by K its fraction field, by K_s/K a separable closure and by $\mathcal{G} = \mathrm{Gal}(K_s/K)$. Denote by C the integral closure of R in K_s and let $\mathfrak{Q}$ be a maximal ideal of C. We consider the two following subgroups of $\mathcal{G}$*

Decomposition group $\mathcal{D} = \Big\{ \sigma \in \mathcal{G} \ \sigma(\mathfrak{Q}) = \mathfrak{Q} \Big\}$; $B = C^{\mathcal{D}}$.

Inertia group $\mathcal{D} = \Big\{ \sigma \in \mathcal{D} \mid \sigma_{\kappa(\mathfrak{Q})} = id_{\kappa(\mathfrak{Q})} \Big\}$; $B' = C^{\mathcal{I}}$.

Then $B_{\mathfrak{Q} \cap B}$ (resp. $B'_{\mathfrak{Q} \cap B'}$) is the henselization (resp. the strict henselization) of R.

Furthermore $B_{\mathfrak{Q} \cap B}$ (resp. $B'_{\mathfrak{Q} \cap B'}$) is a normal ring [BLR, §2.3, prop. 10]. If R is a DVR, the construction shows that $B_{\mathfrak{Q} \cap B}$ (resp. $B'_{\mathfrak{Q} \cap B'}$) is a limit of DVR's. So in the DVR case, we have $R^{sc} = R^{sh}$.

21.4.8. **Remark.** Assume that the R is an excellent DVR which is equivalent to require that the completed fraction field $\widehat{K}$ is separable over K [EGA4, $\mathrm{Err}_I V.27$]. Then R^h consists of the elements of $\widehat{R}$ which are algebraic over R. There exist non excellent DVR, see [Ku, 11.40].

21.5. Artin's approximation theorem.

21.5.1. **Theorem.** [A] *(see also* [BLR, §3.16]*) Let R be a ring finitely generated over $\mathbb{Z}$ or over a field. Let F be an R–functor in sets such that F is locally of finite presentation, that is commutes with filtered direct limits of R–rings. Let $x \in \mathrm{Spec}(R)$ be a point, and denote by $\mathfrak{P}_x$ the associated prime ideal. Let R_x^h the henselization of the local ring at R_x and let $\widehat{R}_x$ be the completion of R_x. Then for each $n \geq 1$, we have*

$$\mathrm{Im}\Big(F(R_x^h) \to F(R_x^h/\mathfrak{P}_x^n R_x^h)\Big) = \mathrm{Im}\Big(F(R_x^h) \to F(\widehat{R}_x/\mathfrak{P}_x^n \widehat{R}_x)\Big).$$

21.5.2. **Remark.** If $\mathfrak{X}/R$ is an affine scheme, the functor $h_{\mathfrak{X}}$ is of locally of finite presentation iff $\mathfrak{X}$ is of finite presentation over R. In this case, the statement is that $\mathfrak{X}(R_x^h)$ is dense in $\mathfrak{X}(\widehat{R}_x)$. This special case is actually everything since the general case follows from a formal argument.

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