Remarks on Margaux-Soulé's theorem

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1 Introduction

Let k be a field and let $\mathbf{G}$ be a semisimple simply connected absolutely almost simple k–group. The Margaux-Soulé's theorem provides a presentation of the group $\Gamma = \mathbf{G}(k[t])$ by studying its action on the Bruhat-Tits building of $\mathbf{G}_{k((\frac{1}{t}))}$.

For $\mathbf{G}$ split, this is Soulé's original paper [6] which was generalized by Margaux [5]. Our main remark is to show that Soulé's original theorem has a good behaviour under Galois descent yielding then its generalization.

2 Action of the group $\mathbf{G}(k[t])$ on the Bruhat-Tits building, the statement

Throughout k and $\mathbf{G}$ will be as above. We adopt the same notations as Margaux paper.

2.1 Notations

Let $\mathbf{S}$ be a maximal k–split torus of $\mathbf{G}$, and $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ containing $\mathbf{S}$. Recall that $\mathbf{S}_K$ is a maximal K-split torus of $\mathbf{G}_K$. Let $\widetilde{k}/k$ a finite Galois extension which splits $\mathbf{T}$ (hence also $\mathbf{G}$). Set $\mathcal{G} = \mathrm{Gal}(\widetilde{k}/k)$ and $\widetilde{\mathbf{T}} = \mathbf{T} \times_k \widetilde{k}$.

Let $\widetilde{\mathbf{G}} = \mathbf{G} \times_k \widetilde{k}$ and $\widetilde{\mathbf{S}} = \mathbf{S} \times_k \widetilde{k}$. We choose compatible orderings on the root systems $\Phi = \Phi(\mathbf{G}, \mathbf{S})$ and $\widetilde{\Phi} = \Phi(\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}})$ (see [1]). We then have a set Δ of relative simple roots and a set $\widetilde{\Delta}$ of absolute simple roots.

- $A = k[t]$, $K = k((\frac{1}{t}))$;

• ω the valuation defined on K the valuation on K at ∞, that is, the valuation on K having $\mathcal{O} = k[[\frac{1}{t}]]$ as its ring of integers.

We also have the analogous to the above objects for $\widetilde{k}$, namely

• $\widetilde{A} = \widetilde{k}[t]$, $\widetilde{K} = \widetilde{k}((\frac{1}{t}))$, $\widetilde{\Gamma} = \mathbf{G}(\widetilde{A})$, and $\widetilde{\mathcal{O}} = \widetilde{k}[[\frac{1}{t}]]$.

At the level of buildings :

• $\mathcal{T}$ the (affine) Bruhat-Tits building of the K–group $\mathbf{G}_K := \mathbf{G} \times_k K$ and $\widetilde{\mathcal{T}}$ the Bruhat-Tits building of the $\tilde{K}$–group $\mathbf{G}_{\tilde{K}} := \mathbf{G} \times_k \widetilde{K}$ [3, §4.2].

The building $\mathcal{T}$ is equipped with an action of $\mathbf{G}(K)$ and that $\widetilde{\mathcal{T}}$ is equipped with an action of $\mathbf{G}(\widetilde{K}) \rtimes \mathcal{G}$. There is an isometric embedding $j : \mathcal{T} \to \widetilde{\mathcal{T}}$ which identifies $\mathcal{T}$ with $\widetilde{\mathcal{T}}^{\mathcal{G}}$. The hyperspecial group $\mathbf{G}(\widetilde{O})$ of $\mathbf{G}(\widetilde{K})$ fixes a unique point $\widetilde{\phi}$ of $\widetilde{\mathcal{T}}$ [2, §9.1.9.c]. This point descends to a point ϕ of $\mathcal{T}$.

We denote by $\mathcal{A}$ the standard apartment of $\mathcal{T}$ associated to $\mathbf{S}$ (this is a real affine space) and similarly by $\widetilde{\mathcal{A}}$ the standard apartment associated to $\widetilde{\mathbf{T}}$. The point $\widetilde{\phi}$ belongs to $\widetilde{\mathcal{A}}$ (*ibid.*). Since $\mathrm{Hom}_{k-gr}(\mathbf{G}_m, \mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{k-gr}(\mathbf{G}_m, \mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \Big(\mathrm{Hom}_{\widetilde{k}-gr}(\mathbf{G}_{m,\widetilde{k}}, \widetilde{\mathbf{T}}) \otimes_{\mathbb{Z}} \mathbb{R}\Big)^{\mathcal{G}}$ [3, §4.2], we have $j(\mathcal{A}) = \widetilde{\mathcal{A}}^{\mathcal{G}}$ so ϕ belongs to $\mathcal{A}$ and

$$\mathcal{A} = \phi \,+\, \mathrm{Hom}_{k-gr}(\mathbf{G}_m, \mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

By means of the canonical pairing $\langle\, , \,\rangle : \mathrm{Hom}_{k-gr}(\mathbf{S}, \mathbf{G}_m) \times \mathrm{Hom}_{k-gr}(\mathbf{S}, \mathbf{G}_m) \to \mathbb{Z}$ we can then define the *sector* (quartier)

$$\mathcal{Q} := \phi + D \ \text{ where } \ D := \big\{ v \in \mathrm{Hom}_{k-gr}(\mathbf{S}, \mathbf{G}_m) \otimes_{\mathbb{Z}} \mathbb{R} \ \mid \ \langle b, \lambda \rangle \geq 0 \ \ \forall \ \ b \in \Delta \big\}.$$

We discuss then the

Theorem 2.1. *The set $\mathcal{Q}$ is a simplicial fundamental domain for the action of $\mathbf{G}(k[t])$ on $\mathcal{T}$. In other words, any simplex of $\mathcal{T}$ is equivalent under the action of $\mathbf{G}(k[t])$ to a unique simplex of $\mathcal{Q}$.*

3 Descent argument

We shall explain why Soulé's theorem (i.e. the split case) implies the general one. The only thing to show is that $\Gamma.\mathcal{Q} = \mathcal{T}$. Let $y \in \mathcal{T}$. By Soulé's theorem applied to $\mathbf{G} \times_k \widetilde{k}$, there exist $x \in \tilde{\mathcal{Q}}$ and $h \in \widetilde{\Gamma}$ such that

$$y = h.x \ .$$

Since such a x is unique, it is invariant under the Galois group $\mathcal{G}$, hence $x \in \mathcal{Q} = \tilde{\mathcal{Q}}^{\mathcal{G}}$. Denote by $\widetilde{\Gamma}_x \subset \tilde{\Gamma}$ the isotropy group of x. The function $\mathcal{G} \to \widetilde{\Gamma}_x$, $s \to z_s = h^{-1}\, s(h)$ is a 1–cocycle. The Galois cohomology class

$$[z_s] \in H^1(\mathcal{G}, \widetilde{\Gamma}_x)$$

is the obstruction to the existence of $h_0 \in \Gamma$ such that $y = h_0.x$. Furthermore, since y and x have same type, they are conjugate under $\mathbf{G}(K)$ and $[z_s]$ maps to the trivial class in $H^1(\mathcal{G}, \mathbf{G}(\widetilde{K}))$. Therefore, it is enough to prove the following

Lemma 3.1. *The map*

$$H^1(\mathcal{G}, \widetilde{\Gamma}_x) \to H^1(\mathcal{G}, \mathbf{G}(\widetilde{K}))$$

is injective.

Proof. We need here the precise shape[1] of the isotropy group $\tilde{\Gamma}_x$. Define the subset of roots

$$I_x = \left\{ b \in \Delta \;\mid\; b(x - \phi) = 0 \right\}.$$

and by $\mathbf{P}_I = \mathbf{U}_I \rtimes \mathbf{L}_I$ the standard k–parabolic subgroup of $\mathbf{G}$. By [5, prop. 2.5], we have

$$\widetilde{\Gamma}_x = V \rtimes \mathbf{L}_I(\widetilde{k})$$

where V is a certain subgroup of $\mathbf{U}_I(\widetilde{k}[t])$. Since V admits a composition sequence whose quotients are $\widetilde{k}$–vector spaces, we have an isomorphism

$$H^1(\mathcal{G}, \widetilde{\Gamma}_x) \xrightarrow{\sim} H^1(\mathcal{G}, \mathbf{L}_I(\widetilde{k})).$$

But the map $H^1(\mathcal{G}, \mathbf{L}_I(\widetilde{k})) \to H^1(\mathcal{G}, \mathbf{G}(\widetilde{K}))$ decomposes as

$$H^1(\mathcal{G}, \mathbf{L}_I(\widetilde{k})) \to H^1(\mathcal{G}, \mathbf{L}_I(\widetilde{K})) \cong H^1(\mathcal{G}, \mathbf{P}_I(\widetilde{K})) \to H^1(\mathcal{G}, \mathbf{G}(\widetilde{K})).$$

The last map is injective by Borel-Tits [1, th. 4.13], the first one is injective by Bruhat-Tits [4, cor. 3.15]. Thus $H^1(\mathcal{G}, \mathbf{L}_I(\widetilde{k}))$ injects in $H^1(\mathcal{G}, \mathbf{G}(\widetilde{K}))$ as desired. □

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[1]We could appeal also to [6].

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