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Algebraic groups with few subgroups

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Abstract

Every semisimple linear algebraic group over a field F contains nontrivial connected subgroups, namely, maximal tori. In the early 1990s, J. Tits proved that some groups of type E_8 have no others. We give a simpler proof of his result, prove that some groups of type ${}^3D_4$ and ${}^6D_4$ have no nontrivial connected subgroups, and give partial results for types E_6 and E_7. Our result for ${}^3D_4$ uses a general theorem on the indexes of Tits algebras that is of independent interest.

1. *Introduction*

In [**47**], Tits proved that there exist algebraic groups of type E_8 that have very few subgroups. Specifically, he proved that they have no proper nontrivial closed connected subgroups other than maximal tori. Tits also gave a partial result towards the existence of groups of type E_6 with the same property in [**46**].

We say that a reductive (connected) group G defined over a field F is *almost abelian* (this term was suggested by A. Premet) if G/F has no proper nontrivial reductive F-subgroups other than maximal tori. In characteristic zero, an almost abelian reductive F-group G has no proper nontrivial closed connected subgroups other than maximal tori; see Lemma 2.4. In this language, Tits proved that some algebraic groups of type E_8 over some fields of characteristic zero are almost abelian.

Since his work, new tools have appeared, making the problem of the construction of almost abelian groups easier. We give a simpler proof of his result for groups of type E_8 (see § 9), extend his result for groups of type E_6 (Propositions 11.1 and 12.1), and prove a similar partial result for groups of type E_7 (Proposition 13.1). Roughly speaking, for groups of types ${}^1E_6$, ${}^2E_6$, and E_7, we settle the question of the existence of reductive subgroups up to prime-to-p extensions of the base field for each prime p.

For groups of type D_4, we can do more. We prove the following theorem.

Theorem 1.1. *If G is a superversal group of type ${}^3D_4$ or ${}^6D_4$ over a field of characteristic different from* 3*, then G is almost abelian.*

This answers a question by A. Premet and is proved in § 15. The proof uses a result about maximal indexes of Tits algebras for groups of outer type (Theorem 14.1), which is interesting by itself.

A superversal group is, roughly speaking, the most general group of the given type; see Definition 6.3 below. For global fields, such examples do not occur; see § 5.

We remark that groups of type G_2 or F_4 are never almost abelian because the long roots generate a subgroup of type A_2 or D_4, respectively. Thus we have treated the question of the existence of reductive subgroups for all exceptional groups, except that, for types E_6 and E_7, we have only done so up to prime-to-p extensions of the base field for each prime p.

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The idea behind Tits' proofs and ours is to find groups that require a field extension of large degree to split them. We combine his methods with the recent computations of torsion indexes by Totaro [**48**] and the Rost invariant defined in [**27**].

1.2. *Notation and conventions*

The groups we discuss are affine and algebraic, and, in particular, an F-subgroup $H \subset G$ is always closed [**10**, VI_B.1.4.2]. Mainly, we deal with smooth affine algebraic groups that are exactly the linear algebraic groups in the sense of [**1**]. We write μ_n for the group scheme of nth roots of unity. We define 'reductive group' to include the hypothesis that the group is connected. We write G^0 for the identity component of G. A group is *trivial* if it is 1 and *nontrivial* otherwise.

For a field F, we write $\mathrm{Gal}(F)$ for its absolute Galois group, that is, the Galois group of a separable closure F_{sep} of F over F. If $\mathrm{Gal}(F)$ is a pro-p-group (that is, if every finite separable extension of F is of degree a power of p), then we say that F is *p-special*. The subfield F_p of F_{sep} consisting of elements fixed by a p-Sylow subgroup of $\mathrm{Gal}(F)$ is p-special. We call F_p a *co-p-closure* of F.

For a central simple F-algebra D, we write $\mathrm{SL}_1(D)$ for the algebraic groups whose F-points are the norm 1 elements of D.

We use extensively the notion of Tits' index of a semisimple group as defined in [**42**]. The list of possible indexes is given in that paper and also in [**39**, pp. 320–321]. (There is a typo in index #14 in Springer's table; it needs to have an additional circle darkened. His Proposition 17.8.2 is correct.)

2. *First steps*

Let G be a reductive algebraic group over a field F. By the *type* of G, we mean its Killing–Cartan type over an algebraic closure of F.

PARAGRAPH 2.1. The property of being almost abelian is an invariant of isogeny classes; that is, for every central isogeny $G \to G'$, the group G is almost abelian if and only if G' is.

PARAGRAPH 2.2. Our general plan is the following. For a specific semisimple group G over a field F, we prove two statements.

(i) The group G does not contain any F-defined proper semisimple subgroups of the same absolute rank as G.

(ii) Every nontrivial F-torus in G is maximal.

This shows that every nontrivial proper (connected) reductive F-subgroup of G is a maximal torus. Indeed, let H be a nontrivial proper F-reductive subgroup of G. By (ii), H has the same absolute rank as G. Again, by (ii), the central torus of H is either trivial or all of H, that is, H is semisimple (excluded by (i)) or a maximal torus, as claimed.

In order to prove (i), we note that an F-defined semisimple subgroup H of G of the same absolute rank contains, and so is normalized by, a maximal torus of G, and hence H is generated over an algebraic closure of F by root subgroups of G (see [**1**, 13.20]); that is, the roots of H are a subsystem of the roots of G, and H is of one of the types described in [**2**] or [**11**, Theorem 5.3], even though we do not assume that F has characteristic zero.

Item (ii) is the difficult one to prove, in practice. To prove it, our work is cut in half by the following observation whose proof amounts to the fact that the category of finite-dimensional representations of a finite group over $\mathbb{Q}$ is semisimple.

LEMMA 2.3 (Symmetry lemma). *Let G be a reductive algebraic group of absolute rank n. If G contains an F-torus of rank m, then it contains an F-torus of rank $n-m$.*

Proof. Let S be an F-torus in G of rank m. It is contained in a maximal F-torus T of rank n. We switch to the dual category of finitely generated abelian groups with a continuous action by $\mathrm{Gal}(F)$; by continuity, the action of $\mathrm{Gal}(F)$ on T^* and S^* factors through a finite group. Semisimplicity implies that the natural projection $T^* \to S^*$ splits over $\mathbb{Q}$, that is, there is a map $f\colon S^* \otimes \mathbb{Q} \to T^* \otimes \mathbb{Q}$ that is compatible with the Galois action and such that $f(\beta)|_S = \beta$ for all $\beta \in S^*$. Set S'_0 to be the intersection of $f(S^* \otimes \mathbb{Q})$ with T^* and

$$S' := \left(\bigcap_{\alpha \in S'_0} \ker \alpha \right)^0 .$$

Then S' is an F-subtorus of T such that $T = S.S'$ and $S \cap S'$ is finite; it is the desired torus. ∎

A torus over F is *F-irreducible* if it has no nontrivial proper subtori defined over F.

LEMMA 2.4. *Let G/F be an almost abelian reductive group. Then the following conditions hold.*

(i) *The group G is an F-irreducible torus or G is semisimple.*
(ii) *If G has rank at least 2, then G is anisotropic.*
(iii) *If F is perfect, then G has no proper nontrivial smooth connected F-subgroups other than maximal tori.*
(iv) *If $\mathrm{char}(F) = 0$, then G has no proper nontrivial connected subgroups other than maximal tori.*

Proof. (i) The connected center $Z(G)^0$ is an F-subtorus of G. Hence $Z(G)^0 = G$ or $Z(G)^0 = 1$, and hence G is a torus or G is semisimple. If G is an almost abelian torus, then we have already seen that G is irreducible.

(ii) Assume that G is isotropic, that is, there exists an F-embedding $\mathbb{G}_m \to G$. Then the image is a maximal torus.

(iii) Let H be a smooth connected F-subgroup of G and consider its unipotent radical $U = R_u(H)$ (see [**39**, § 12.1.7]), which is a smooth connected unipotent subgroup of H and defined over F (because F is perfect). From [**3**, 3.7], $U(F_{\mathrm{sep}})$ is a subgroup of the unipotent radical of the F_{sep}-parabolic subgroup $\mathcal{P}(U(F_{\mathrm{sep}}))$ constructed by Borel–Tits. Furthermore, this parabolic F_{sep}-subgroup descends in an F-parabolic subgroup P and U is then an F-subgroup of $R_u(P)$. If U is nontrivial, then P is proper and G is isotropic, which contradicts (i). Hence H is reductive and is a maximal F-torus.

Condition (iv) follows from condition (iii) since all algebraic groups are smooth in characteristic zero (Cartier). ∎

REMARK 2.5. Suppose now that G is semisimple and simply connected, and that F is of positive characteristic p, which is not a torsion prime for G as listed in Table 1. Assume that G/F is almost abelian. Then, from [**44**, 2.6], the argument of Lemma 2.4(iii) still shows that G has no proper connected smooth unipotent F-subgroups. We can then conclude in this case that the connected smooth F-subgroups of G are maximal tori or pseudo-reductives (in the sense of [**7**]). For example, if $A = (L/k, \sigma, a)$ is a cyclic division algebra of degree p, then the group $\mathrm{SL}_1(A)$ is almost abelian and contains the pseudo-reductive commutative k-subgroup $R^1_{k(\sqrt[p]{a}))/k}(\mathbb{G}_m)$.

3. *Scholium: groups of type $A_1 \times A_1 \times A_1$*

We illustrate the material in Section 2 by studying the almost abelian groups of type $A_1 \times A_1 \times A_1$ over a field F of characteristic different from 2. (This material is not used in the rest of the paper.) Every group of this type is isogenous to a transfer $R_{L/F}(\mathrm{SL}_1(Q))$, where L is a cubic étale F-algebra and Q is a quaternion L-algebra. If L is not a field, then such a group is obviously not almost abelian because it contains an F-defined subgroup of type A_1. We have the following proposition.

Proposition 3.1. *Maintain the notation of the previous paragraph and suppose that L is a field. Then the following conditions hold.*

(i) *The group $R_{L/F}(\mathrm{SL}_1(Q))$ is almost abelian if and only if Q does not contain a quadratic étale F-algebra.*

(ii) *If $\mathrm{cor}_{L/F}(Q)$ has index at least 4, then $R_{L/F}(\mathrm{SL}_1(Q))$ is almost abelian.*

(iii) *If $R_{L/F}(\mathrm{SL}_1(Q))$ is almost abelian, then $\mathrm{cor}_{L/F}(Q)$ has index at least 2.*

Proof. Suppose first that Q does not contain a quadratic étale F-algebra. The group has no proper semisimple subgroups of rank 3 over an algebraic closure, and hence it cannot have one over F. If $R_{L/F}(\mathrm{SL}_1(Q))$ contains a rank 1 F-torus, then there is a quadratic extension of F splitting Q, which is impossible by the assumption on Q. By the symmetry lemma, $R_{L/F}(\mathrm{SL}_1(Q))$ also has no rank 2 tori. Considering Remark 2.2, we conclude that $R_{L/F}(\mathrm{SL}_1(Q))$ is almost abelian.

Conversely, suppose that Q contains a quadratic étale F-algebra K. The group $R_{L/F}(\mathrm{SL}_1(Q))$ contains the maximal torus $R_{L/F}(R^1_{(K\otimes L)/L}(\mathbb{G}_m))$, and hence the 1-dimensional torus $R^1_{K/F}(\mathbb{G}_m)$. We have proved (i).

The quaternion L-algebra Q is (a, b) for some $a, b \in L^\times$, that is, it is generated as an L-algebra by elements i and j such that $i^2 = a$, $j^2 = b$, and $ij = -ji$. It contains a quadratic étale F-algebra if and only if b can be chosen to lie in $F^\times$. Claim (ii) follows from (i): if b is in $F^\times$, then $\mathrm{cor}_{L/F}(Q)$ is Brauer-equivalent to $(N_{L/F}(b), a)$.

If $\mathrm{cor}_{L/F}(Q)$ is split, then, from [**23**, 43.9], Q contains a quadratic étale F-algebra and (iii) follows. $\square$

In the case where $\mathrm{cor}_{L/F}(Q)$ has index 2, the group $R_{L/F}(\mathrm{SL}_1(Q))$ can be almost abelian or not. Examples where the group is not almost abelian are easy to create using the projection formula as in the proof of part (ii) of the proposition.

Table 1. *Torsion primes for simple root systems; cf.* [**40**, 1.13].

Type	Torsion primes
A_n $(n \geqslant 1)$, C_n $(n \geqslant 2)$	None
B_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$, G_2	2
F_4, E_6, E_7	2, 3
E_8	2, 3, 5

EXAMPLE 3.2. We sketch an example where the corestriction has index 2 and the group is almost abelian. Section 4 of the paper [**35**] gives an example of a cyclic Galois extension L_0/F_0 of degree 3 and a quaternion algebra Q_0 over L_0 such that $\mathrm{cor}_{L_0/F_0}(Q_0)$ has index 8. Let X be the projective variety of right ideals of dimension 16 in $\mathrm{cor}_{L_0/F_0}(Q_0)$. We write $F := F_0(X)$ and $L := L_0(X)$. As F is a regular extension of F_0, the field L is canonically identified with $L_0 \otimes_{F_0} F$. For $Q := Q_0 \otimes_{L_0} L_0(X)$, we have

$$\mathrm{cor}_{L/F}(Q) = \mathrm{cor}_{L/F}\,\mathrm{res}_{L/L_0}(Q_0) = \mathrm{res}_{F/F_0}\,\mathrm{cor}_{L_0/F_0}(Q_0),$$

and thus $\mathrm{cor}_{L/F}(Q)$ has index 2.

For sake of contradiction, we suppose that $R_{L/F}(\mathrm{SL}_1(Q))$ is not almost abelian, and hence Q can be written as (a, b) for some $b \in F^\times$. Write ρ for a generator of the Galois group of L/F, equivalently, L_0/F_0. Then $Q \otimes {}^\rho Q$ is Brauer-equivalent to $(a\,\rho(a), b)$, and hence has index 2. On the other hand, $Q \otimes {}^\rho Q$ is $(Q_0 \otimes {}^\rho Q_0) \otimes_{L_0} L_0(X)$. However, $Q_0 \otimes {}^\rho Q_0$ has index 4 (because $\mathrm{res}_{L_0/F_0}\,\mathrm{cor}_{L_0/F_0}(Q_0)$ has index 8), and the index reduction formula from [**28**, p. 565] shows that $L_0(X)$ does not lower the index of $Q_0 \otimes {}^\rho Q_0$. This is a contradiction, and so $R_{L/F}(\mathrm{SL}_1(Q))$ is almost abelian.

4. *Splitting fields of tori*

We recall the well-known divisibility bounds on finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Q})$ given in Table 2. (We remark that every finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ is conjugate to a finite subgroup in $\mathrm{GL}_n(\mathbb{Z})$ by [**36**, p. 124], and thus the bounds are the same whether one takes $\mathbb{Z}$ or $\mathbb{Q}$.)

The F-isomorphism class of an n-dimensional torus T can be specified by a continuous homomorphism $\phi\colon \mathrm{Gal}(F) \to \mathrm{GL}_n(\mathbb{Z})$ (representing the natural Galois action on T^*), and T is split over F if and only if ϕ is zero. It follows that every torus of rank n can be split by a field extension of degree $|\operatorname{im}\phi|$, and this order is bounded as in Table 2.

We start with the following elementary fact.

PROPOSITION 4.1 (Type ${}^1A_{p-1}$). *Let D be a central division algebra of prime degree p. Then the group $\mathrm{SL}_1(D)$ is almost abelian.*

Proof. We follow Paragraph 2.2. The group $\mathrm{SL}_1(D)$ is clearly anisotropic, and Remark 2.2(i) holds because there are no proper semisimple subgroups of $\mathrm{SL}_1(D)$ of rank $p-1$ even over an algebraic closure of F.

It remains to show that every nonmaximal torus in $\mathrm{SL}_1(D)$ is trivial. First, we may replace F by a co-p-closure F_p, because D remains division over F_p. Each maximal torus T is then a norm one torus $R^1_{K/F}(\mathbb{G}_m)$ for some cyclic Galois F-algebra K of dimension p. Since T is anisotropic, it follows that K is a cyclic field extension of F. The corresponding representation $\mathrm{Gal}(K/F) \to \mathrm{GL}(\mathbb{Z}^{p-1}) \to \mathrm{GL}(\mathbb{Q}^{p-1})$ is $\mathbb{Q}$-irreducible. Thus T does not admit nontrivial proper subtori. $\square$

TABLE 2. *Divisibility bounds on finite subgroups of* $\mathrm{GL}_n(\mathbb{Z})$ *and* $\mathrm{GL}_n(\mathbb{Q})$.

n	Finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$ or $\mathrm{GL}_n(\mathbb{Q})$ have order dividing	References
1	2	
2	2^3 or $2^2 \cdot 3$	For example, [**25**, 4.1]
3	$2^4 \cdot 3$	[**41**]
4	$2^7 \cdot 3^2$ or $2^3 \cdot 5$ or $2^4 \cdot 3 \cdot 5$	[**9**, **47**, p. 1137]

In contrast, we note that a group of type ${}^1A_{n-1}$ where n has at least two distinct prime factors is never almost abelian because a central simple algebra of degree n is decomposable; see, for example, [**18**, 4.5.16].

Corollary 4.2 (Type ${}^2A_{p-1}$). *Let D be a central division algebra of prime degree p over a quadratic extension L/F. If D has a unitary involution τ, then the group $\mathrm{SU}(D,\tau)$ is almost abelian.*

Proof. Over L, the group $\mathrm{SU}(D,\tau)$ becomes isomorphic to $\mathrm{SL}_1(D)$, which is almost abelian by Proposition 4.1. $\square$

Over special fields, other inner groups of type A are not almost abelian.

Lemma 4.3 (Type ${}^1A_{p^r-1}$). *Assume that F is p-special and let D be an F-central division algebra of degree $p^r > p$. Then the group $\mathrm{SL}_1(D)$ is not almost abelian.*

Proof. Let $L \subset D$ be a separable maximal subfield of D so that $[L:F] = p^r$ (see [**18**, § 4.5]). Since $\mathrm{Gal}(F)$ is a pro-p-group, it follows that there exists an intermediate extension $F \subset K \subset L$ with $[K:F] = p$ (see [**18**, 6.1.8]). Thus the torus $R^1_{K/F}(\mathbb{G}_m)$ is a proper subtorus of the maximal torus $R^1_{L/F}(\mathbb{G}_m)$ of $\mathrm{SL}_1(D)$. $\square$

Example 4.4 (Type ${}^1A_3$). By a result of Albert [**20**, Theorem 2.9.21], we know that a central division algebra D of degree 4 contains a maximal subfield that is Galois of group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence $\mathrm{SL}_1(D)$ contains a 1-dimensional torus $R^1_{K/k}(\mathbb{G}_m)$. Semisimple groups of type ${}^1A_3$ are not almost abelian groups.

We ignore whether there exist almost abelian groups of type ${}^1A_{p^2-1}$ for p odd.

Example 4.5 (Types B, C, and D). Suppose, for convenience, that F has characteristic different from 2.

A group of type B_n for some $n \geqslant 2$ is isogenous to the special orthogonal group of a nondegenerate quadratic form $\sum_{i=1}^{2n+1} a_i x_i^2$ for some $a_i \in F^\times$ and indeterminates x_i (see [**23**, 26.12]). Such a group is not almost abelian because it contains the special orthogonal group of, say, $\sum_{i=1}^{2n} a_i x_i^2$ as a subgroup.

A group G of type C_n for $n \geqslant 3$ or of type ${}^1D_n$ or ${}^2D_n$ for $n \geqslant 4$ is isogenous to the symplectic or the special orthogonal group, respectively, of a central simple algebra A of degree $2n$ with involution σ of the first kind; see [**23**, 26.14, 26.15, 44.7]. If A is not a division algebra, for example, if n is not a power of 2, then G is not almost abelian by the same reasoning as in the preceding paragraph. We ignore the case where A is division and n is a power of 2, except to note that a group of type C_4 is never almost abelian by [**15**, Corollary 6.2].

Finally, we note the following lemma.

Lemma 4.6. *Every 3-dimensional torus over a 3-special field is isotropic.*

Proof. Over a 3-special field F, a torus T corresponds to a homomorphism $\phi\colon \mathrm{Gal}(F) \to \mathrm{GL}_3(\mathbb{Z})$ that necessarily factors through $\mathbb{Z}/3\mathbb{Z}$ by Table 2. However, any representation $\mathbb{Z}/3\mathbb{Z} \to \mathrm{GL}_3(\mathbb{Q})$ admits the trivial representation as a direct summand, and hence T is isotropic. $\square$

5. *Global fields*

We assume in this section that F is a *global* field F. This case is essentially known since the construction of proper subgroups is one of the key ingredients in the proof of the Hasse principle, particularly for exceptional types (Harder's method; see [**19**, **32**, Chapter VI]). We prove the following proposition.

PROPOSITION 5.1. *If an absolutely almost simple group G over a global field F is almost abelian, then G has type A_{p-1} for some prime p.*

Conversely, there exist almost abelian groups of type ${}^1A_{p-1}$ and ${}^2A_{p-1}$ over F with p prime by Proposition 4.1 and Corollary 4.2.

The proof is based on the following consequence of the Hasse Principle.

LEMMA 5.2. *If F is a global field with no real embeddings, then the anisotropic semisimple F-groups are of type $A \times \ldots \times A$.*

For the number field case, see [**32**, Theorem 6.25]. For the function field case, this is a result of Harder [**19**, Satz C, p. 133].

The only tricky case of our proof is that of outer type A. From Prasad–Rapinchuk's study of subalgebras of algebras with involutions [**33**, Appendix A; **34**], we can derive the following fact.

PROPOSITION 5.3. *Let L be a quadratic field extension of F. Let A be a central simple L-algebra of degree n equipped with an involution τ of the second kind such that $L^\tau = F$. If $\sqrt{-1} \in F$ or if n is odd, then there exists a cyclic field extension E/F of degree n such that $E \otimes_F L$ is a field and such that $(E \otimes_F L, \mathrm{Id} \otimes \tau)$ embeds in (A, τ).*

Proof. Let S be a finite set of places such that the group $\mathrm{SU}(A, \tau)$ is quasi-split for places away from S. We assume that S contains a finite place v_0 such that $L \otimes_F F_v$ is not a field; such a place exists by Cebotarev density and, if none are in S, then we add one.

For each place $v \in S$, we pick a cyclic étale F_v-algebra E_v of degree n such that $(E_v, \mathrm{Id}) \otimes_{F_v} (L \otimes F_v, \tau)$ embeds in $(A, \tau) \otimes_F F_v$ as an $(L \otimes_F F_v)$-algebra with involution.

Case 1: Suppose first that $L \otimes_F F_v$ is not a field. Then $A \otimes_L (L \otimes_F F_v) \cong A_v \times A_v$, where A_v is a central F_v-algebra.

(i) If v is finite, then, by local class field theory, A_v is split by the unramified extension of F_v of degree n. We take it to be E_v.

(ii) If v is infinite, then $F_v = \mathbb{C}$ or n is odd. In either case, $A \otimes_L (L \otimes_F F_v)$ is split. Hence $\mathrm{SU}(A, \tau) \times_F F_v$ is split; in this case, we delete v from S.

Case 2: Suppose now that $L \otimes_F F_v$ is a field L_w. We claim that the algebra $A \otimes_L L_w$ is split. It is obvious in the archimedean case; in the finite case, it follows from the bijectivity of the norm map $\mathrm{Br}(L_w) \to \mathrm{Br}(F_v)$. Then $(A, \tau) \otimes_F F_v$ is adjoint to a hermitian form for L_w/F_v. By diagonalizing it, it follows that L_w^n embeds in $(A \otimes_F F_v, \tau)$. We take E_v to be F_v^n.

The map

$$H^1(F, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \prod_{v \in S} H^1(F_v, \mathbb{Z}/n\mathbb{Z})$$

is onto from [**31**, Corollary 9.2.8] (see also Example 1 on p. 535 of that book). Hence there exists a cyclic étale F-algebra E such that $E \otimes F_v$ is isomorphic to E_v for all $v \in S$. Note that E is a field because E_{v_0} is. Further, $L \otimes_F F_{v_0}$ is not a field, but $E \otimes_F F_{v_0}$ is, and hence E does not contain a copy of L. As E is Galois over F, it follows that $E \otimes L$ is a field.

By construction, for $v \in S$, the algebra with involution $(E_v, \mathrm{Id}) \otimes_{F_v} (L \otimes F_v, \tau)$ embeds in $(A, \tau) \otimes_F F_v$. Nevertheless, the same is true for v not in S because the group $\mathrm{SU}(A, \tau) \times F_v$ is quasi-split; cf. [**32**, p. 340]. By [**34**, Theorem 4.1] (using that $E \otimes L$ is a field), $(E, \mathrm{Id}) \otimes (L, \tau)$ embeds in (A, τ) as F-algebras with involution. □

Proof of Proposition 5.1. Let G be an absolutely almost simple group over F, not of type A_{n-1} for n prime. We prove case by case that G is not almost abelian.

Type ${}^1A$: If G has type ${}^1A_{n-1}$, then it is isogenous to $\mathrm{PGL}_1(A)$, where A is a central simple algebra of degree n. By the Brauer–Hasse–Noether theorem, A is a cyclic algebra, and hence contains a central simple proper subalgebra A' if n is not prime. Then $\mathrm{PGL}_1(A')$ is a proper semisimple subgroup of $\mathrm{PGL}_1(A)$, and hence $\mathrm{PGL}_1(A)$ is not almost abelian.

Type ${}^2A$: If G has type ${}^2A_{n-1}$ with n not prime, then it is isogenous to $\mathrm{SU}(B, \tau)$, where B is a central simple algebra of degree $n \geqslant 3$ defined over a quadratic separable extension L/F and equipped with an involution of the second kind. If n is even, then we know that B contains a quadratic subalgebra L'/L that is stable under τ (see [**22**, § 5.7, p. 109]). Thus $\mathrm{SU}(L', \tau)$ is a rank 1 k-torus of $\mathrm{SU}(B, \tau)$, and hence $\mathrm{SU}(B, \tau)$ is not almost abelian. If n is odd, then Proposition 5.3 shows that (B, τ) admits a subalgebra $(E \otimes_F L, \mathrm{Id} \otimes \tau)$, where E is a cyclic Galois F-algebra of degree n. Because n is not prime, there is a field E_0 such that $F \subsetneq E_0 \subsetneq E$, and the special unitary group of $(E_0 \otimes L, \mathrm{Id} \otimes \tau)$ is a nonmaximal nonzero torus in $\mathrm{SU}(B, \tau)$. Again, $\mathrm{SU}(B, \tau)$ is not almost abelian.

Other types: By Lemma 5.2, we can assume that F has real places. Then we write P for the parabolic subgroup of G defined over $F(\sqrt{-1})$ from Lemma 5.2. It follows that G has an F-subgroup L that is conjugate over $F(\sqrt{-1})$ to a Levi subgroup of P (see [**32**, Lemma 6.17′, p. 383], and hence G is not almost abelian. □

6. *Versal groups*

We first recall the definition of a versal torsor.

Definition 6.1 (Serre [**27**, § 5]). Let G be a smooth affine algebraic group defined over a field F_0. A *versal G-torsor* P is a G-torsor defined over a finitely generated extension F of F_0 such that there exists a smooth irreducible variety X over F_0 with function field F and a G-torsor $\mathfrak{Q} \to X$, with base X, with the following two properties.

(i) The fiber of $\mathfrak{Q}$ at the generic point of X is P.

(ii) For every extension E of F_0, with E infinite, every G-torsor T over E, and every nonempty open subvariety U of X, there exists $x \in U(E)$ such that the fiber $\mathfrak{Q}_x$ is isomorphic to T.

Versal torsors exist and one construction is to take a faithful linear representation $G \to \mathrm{GL}_N$ and the torsor $\mathrm{GL}_N \to X = \mathrm{GL}_N / G$. Here we mimic this definition for versal groups, which are the 'most general' groups of a given quasi-split type.

Definition 6.2. Fix a field F_0, a connected Dynkin diagram Δ, and a class $\phi \in H^1(F_0, \mathrm{Aut}(\Delta))$, namely, a continuous homomorphism of the absolute Galois group $\mathrm{Gal}(F_0)$

by graph automorphisms on Δ, up to conjugation by an element of $\mathrm{Aut}(\Delta)$. A versal group of type (Δ, ϕ) is an adjoint semisimple group G defined over a finitely generated extension F of F_0 and of quasi-split type (Δ, ϕ_F) such that there exist a smooth irreducible variety X over F_0 with function field F and an adjoint semisimple group scheme $\mathfrak{G}/X$ with the following two properties.

(i) The fiber of $\mathfrak{G}$ at the generic point of X is G.

(ii) For every extension E of F_0, with E infinite, every adjoint semisimple group H/E of quasi-split type (Δ, ϕ_E), and every nonempty open subvariety U of X, there exists $x \in U(E)$ such that the fiber $\mathfrak{G}_x$ is isomorphic to H.

Versal groups exist. There is a unique quasi-split simple adjoint group G^q of quasi-split type (Δ, ϕ). We embed G^q in some GL_N, and consider $X = \mathrm{GL}_N / G^q$ and $F = F_0(X)$. By étale descent [30, p. 134], we can twist inner automorphisms $G^q \times_{F_0} X$ by the G^q-torsor $\mathrm{GL}_N \to X$; this defines $\mathfrak{G}/X$ and its generic fiber G/F. Since every adjoint semisimple group H/E of quasi-split type (Δ, ϕ_E) is obtained by twisting by a G^q-torsor over E, it follows that G/F is a versal group of quasi-split type.

A *versal group* is a group G (over F) obtained in this manner from some F_0, Δ, and ϕ. As an abbreviation, we say that G is a *versal group of type* ${}^aT_n$, with T_n being the Killing–Cartan type of Δ and $a = |\mathrm{im}\, \phi|$.

There is a further notion of versal groups when we allow the $*$-action to vary inside $H^1(E, \mathrm{Aut}(\Delta))$ with image in the conjugate in a subgroup $\Gamma \subset \mathrm{Aut}(\Delta)$.

Definition 6.3. Fix a field F_0, a connected Dynkin diagram Δ, and a subgroup $\Gamma \subset \mathrm{Aut}(\Delta)$. A superversal group of type (Δ, Γ) is an adjoint semisimple group G defined over a finitely generated extension F of F_0 and of quasi-split type (Δ, ϕ_G) such that

$$\phi_G \in \mathrm{Im}(H^1(F, \Gamma) \longrightarrow H^1(F, \mathrm{Aut}(\Delta)))$$

and such that there exists a smooth irreducible variety X over F_0 with function field F and an adjoint semisimple group scheme $\mathfrak{G}/X$ with the following two properties.

(i) The fiber of $\mathfrak{G}$ at the generic point of X is G.

(ii) For every extension E of F_0, with E infinite, every adjoint semisimple group H/E of quasi-split type (Δ, ϕ_H) such that

$$\phi_H \in \mathrm{Im}(H^1(E, \Gamma) \longrightarrow H^1(E, \mathrm{Aut}(\Delta))), \tag{6.4}$$

and every nonempty open subvariety U of X, there exists $x \in U(E)$ such that the fiber $\mathfrak{G}_x$ is isomorphic to H.

Similarly, superversal groups exist. Let G^d be the split adjoint group of type Δ. Fix a subgroup $\Gamma \subset \mathrm{Aut}(\Delta)$. Write $\mathrm{Aut}_\Gamma(G^d)$ for the pull back by the inclusion $\Gamma \subset \mathrm{Aut}(\Delta)$ of the exact sequence as follows:

$$1 \longrightarrow G^d \longrightarrow \mathrm{Aut}(G^d) \longrightarrow \mathrm{Aut}(\Delta) \longrightarrow 1.$$

We embed $\mathrm{Aut}_\Gamma(G^d)$ in some GL_N, and consider $X = \mathrm{GL}_N / \mathrm{Aut}_\Gamma(G^d)$ and $F = F_0(X)$. We twist $G^q \times_{F_0} X$ by the $\mathrm{Aut}_\Gamma(G^q)$-torsor $\mathrm{GL}_N \to X$: this defines $\mathfrak{G}/X$ and its generic class G/F. We shall see below (Hypothesis 6.5) that every adjoint semisimple group H/E of quasi-split type (Δ, ϕ_H) satisfying the condition (6.4) is obtained by twisting G^q by a $\mathrm{Aut}_\Gamma(G^q)$-torsor over E, and it follows that G/F is a superversal group of type (Δ, Γ).

A *superversal group* is a group G (over F) obtained in this manner from some F_0, Δ, and Γ. As an abbreviation, we say that G is a *superversal group of type* ${}^aT_n$, with T_n being the Killing–Cartan type of Δ and $a = |\Gamma|$.

The case $\Gamma = 1$ is nothing but the versal inner form.

We maintain the notation of Definition 6.3. Suppose that M is an adjoint group with the same Dynkin diagram Δ, defined over an extension E of F_0, and the $*$-action on the Dynkin diagram of M belongs to the image of $H^1(E,\Gamma) \to H^1(E,\mathrm{Aut}(\Delta))$.

LEMMA 6.5. *The isomorphism class of M is in the image of the map $H^1(E,\mathrm{Aut}_\Gamma(G^d)) \to H^1(E,\mathrm{Aut}(G^d))$.*

Proof. We have to check that these kinds of groups occur as twisted forms of G^d by cocycles with value in $\mathrm{Aut}_\Gamma(G^d)$. There exists $[z] \in H^1(E,\mathrm{Aut}(G^d))$ such that $M \cong G^d_z$. We consider the following commutative diagram

$$\begin{array}{ccc} H^1(E,\mathrm{Aut}(G^d)) & \xrightarrow{\pi} & H^1(E,\mathrm{Aut}(\Delta)) \\ \uparrow & & \uparrow \\ H^1(E,\mathrm{Aut}_\Gamma(G^d)) & \longrightarrow & H^1(E,\Gamma). \end{array}$$

Our hypothesis is that $\pi_*[z] \in \mathrm{Im}(H^1(E,\Gamma) \to H^1(E,\mathrm{Aut}(\Delta)))$. According to [**38**, §I.5, Proposition 37], this is equivalent to the fact that ${}_{\pi_*(z)}(\mathrm{Aut}(\Delta)/\Gamma)(E) \neq \emptyset$. This finite set is nothing but ${}_{\pi_*(z)}(\mathrm{Aut}(G^d)/\mathrm{Aut}_\Gamma(G^d))(E)$ and hence the claim. □

We conclude this section by the following observation.

LEMMA 6.6. *Let Ψ be a root datum in the sense of Springer* [**39**, 7.4.1]. *The following conditions are equivalent.*

(i) *The superversal group G/F contains a reductive subgroup of type Ψ.*

(ii) *For every field extension E/F_0 and every adjoint group M/E of type Δ such that $\varphi_M \in \mathrm{Im}(H^1(E,\Gamma) \to H^1(E,\mathrm{Aut}(\Delta)))$, the group M/E has a reductive subgroup of type Ψ.*

Note that there is an analogous statement for versal groups.

We use a root datum instead of a root system so as to also include tori and the kernel of the map from the simply connected cover.

Proof of Lemma 6.6. The statement does not depend on the choice of the superversal group. We can then work with our construction, that is, with an $\mathrm{Aut}_\Gamma(G)$-torsor V/F arising as the generic fiber of a $\mathrm{Aut}_\Gamma(G)$-torsor $\mathfrak{V}/X$, where X is a smooth F_0-variety such that $F = F_0(X)$ and $\mathfrak{G}/X$ is the twisted X-form of G^d by $\mathfrak{V}$. The implication (ii) ⇒ (i) is obvious. Conversely, assume that G/F admits a reductive F-subgroup $H \subset G/F$ of type Ψ. The idea is to extend H locally. The quotient $Q = G/H$ is a G-variety over F. According to [**10**, VI_B.10.16], there exists an open (nonempty) Zariski-connected subset $U \subset X$ and a homogeneous $\mathfrak{G}$-scheme $\mathfrak{Q}/U$ such that $Q = \mathfrak{Q} \times_U F_0(X)$. Shrinking U further, if necessary, we can assume that the origin of Q extends to a point $q \in \mathfrak{Q}(U)$. We define the closed U-subgroup scheme $\mathfrak{H} := \mathrm{Stab}_{\mathfrak{G}}(q)$. Again, shrinking U, if necessary, we can assume that $\mathfrak{H}/U$ is smooth and connected. By [**10**, XIX.2.6], $\mathfrak{H}/U$ is a reductive group scheme. By Demazure's type unicity theorem [**10**, XXII.2.8], $\mathfrak{H}$ is of type Ψ, that is, all geometric fibers are of type Ψ.

We can now proceed to the proof. We are given a field E/F_0 and an adjoint group M/E of type Δ such that its $*$-action $\varphi_M \in H^1(E,\mathrm{Aut}(\Delta))$ comes from $H^1(F,\Gamma)$. By Hypothesis 6.5, there exists $x \in U(E)$ such that M is isomorphic to the fiber $x^*\mathfrak{G}$. Then M/E has the closed subgroup $x^*\mathfrak{H}$ that is reductive of type Ψ. □

COROLLARY 6.7. *Let G be a superversal group as in Definition 6.3. The following conditions are equivalent.*

(i) *The superversal group G/F is not almost abelian.*

(ii) *For every field extension E/F_0 and every adjoint group M/E of type Δ such that $\varphi_M \in \mathrm{Im}(H^1(E,\Gamma) \to H^1(E,\mathrm{Aut}(\Delta)))$, the group M/E is not almost abelian.*

Proof. Again (ii) $\Rightarrow$ (i) is obvious. Conversely, assume that G/F admits a proper nonzero reductive F-subgroup $H \subset G$. Hence Lemma 6.6 applies. $\square$

7. *Rost invariant and adjoint groups*

PARAGRAPH 7.1. Let G^q be a quasi-split simple adjoint algebraic group, and write $\widetilde{G}^q$ for its simply connected cover. The *Rost invariant* of $\widetilde{G}^q$ is a morphism of functors $\mathsf{Fields}_{/F} \to \mathsf{Sets}$ given by

$$r_{\widetilde{G}^q} : H^1(*, \widetilde{G}^q) \longrightarrow H^3(*, \mathbb{Z}/\delta\mathbb{Z}(2)),$$

where δ is the *Dynkin index* of $\widetilde{G}^q$ as defined in [**27**, § 10]. (We remind the reader that, in the case where the characteristic of F divides δ, the group $H^3(F, \mathbb{Z}/\delta\mathbb{Z}(2))$ is defined using logarithmic differential forms as in [**21**]; see [**27**, Appendix A]. In the case where the characteristic of F does not divide δ, it is the Galois cohomology group $H^3(F, \mu_\delta^{\otimes 2})$.) Factor δ as $\delta_0 m$, where m is maximal with the property of being relatively prime to both δ_0 and the exponent of the group $Z^*(F_{\mathrm{sep}})$ of F_{sep}-points of the cocenter of $\widetilde{G}^q$. The number m is 1 exactly for G^q of type ${}^1A_n, B_n$, or C_n for all n, of type ${}^2A_n$ for n odd, or of type ${}^1D_n$ or ${}^2D_n$ for $n \geqslant 4$. The number m is δ when Z is trivial, that is, for G^q of type E_8, F_4, or G_2. All the other cases, that is, where $m \neq 1, \delta$, are given in Table 3.

For each extension K/F, the group $H^3(K, \mathbb{Z}/\delta\mathbb{Z}(2))$ can naturally be written as a direct sum of its m and δ_0 torsion parts, that is, as $H^3(K, \mathbb{Z}/m\mathbb{Z}(2)) \oplus H^3(K, \mathbb{Z}/\delta_0\mathbb{Z}(2))$.

PROPOSITION 7.2. *Suppose that the characteristic of F does not divide m. Then there is a unique invariant*

$$r_{G^q,m} : H^1(*, G^q) \longrightarrow H^3(*, \mathbb{Z}/m\mathbb{Z}(2))$$

such that the diagram

$$\begin{array}{ccc} H^1(*, \widetilde{G}^q) & \xrightarrow{r_{\widetilde{G}^q}} & H^3(*, \mathbb{Z}/\delta\mathbb{Z}(2)) \\ \big\downarrow & & \big\downarrow \\ H^1(*, G^q) & \xrightarrow{r_{G^q,m}} & H^3(*, \mathbb{Z}/m\mathbb{Z}(2)) \end{array}$$

commutes.

TABLE 3. *Factorizations of Dynkin indexes for some quasi-split absolutely simple groups. All other quasi-split absolutely simple groups have $m = 1$ or $m = \delta$.*

Type of G^q	m	Dynkin index δ of $\widetilde{G}^q$	Exponent of $Z^*(F_{\mathrm{sep}})$
${}^2A_n$ (n even)	2	2	$n+1$
${}^3D_4, {}^6D_4$	3	6	2
${}^1E_6$	2	6	3
${}^2E_6$	4	12	3
E_7	3	12	2

Proof. The result is trivial if $\widetilde{G}^q = G^q$ (that is, $m = \delta$) or $m = 1$, and so we may assume that m is given in Table 3. In particular, m is a power of some prime p.

Fix an element $y \in H^1(K, G)$ for some extension K/F; we claim that there is an extension L/K such that $\mathrm{res}_{L/K}(y)$ is in the image of $H^1(L, \widetilde{G}^q) \to H^1(L, G)$ and $[L : K]$ is relatively prime to m. Let K_p denote the co-p-closure of the perfect closure of K. The group $H^2_{\mathrm{fppf}}(K_p, Z)$ is zero because the exponent of Z is not divisible by p. The exactness of the sequence $1 \to Z \to \widetilde{G}^q \to G^q \to 1$ implies that there is a finite extension E/K and an $x \in H^1(E, \widetilde{G}^q)$ that maps to $\mathrm{res}_{E/K}(y) \in H^1(E, G^q)$. We take L to be the separable closure of K in E; the sets $H^1(L, \widetilde{G}^q)$ and $H^1(E, \widetilde{G}^q)$ are identified by the restriction map, and the claim is proved.

Now suppose that $x_1, x_2 \in H^1(L, \widetilde{G}^q)$ both map to y. Then there is a $z \in H^1_{\mathrm{fppf}}(L, Z)$ such that $z \cdot x_1 = x_2$. Applying the Rost invariant, we find that

$$r_{\widetilde{G}^q}(x_1) = r_{\widetilde{G}^q}(z \cdot x_2) = r_{\widetilde{G}^q}(x_2)$$

by [**17**, Lemma 7, p. 76]; cf. [**16**, Remark 2.5(i)]. Combining this with the previous paragraph proves the existence and uniqueness of $r_{G^q,m}$ by [**14**, Proposition 7.1] (which uses the hypothesis on the characteristic of F). □

Definition 7.3. Let G be a simple algebraic group over F, and write $\overline{G}$ for its associated adjoint group. There is a unique class $\eta \in H^1(F, \overline{G})$ such that the group $\overline{G}_\eta$ obtained by twisting $\overline{G}$ by η is quasi-split [**23**, 31.6]. Consider m to be the natural number defined for the quasi-split group $\overline{G}_\eta$ in Remark 7.1, and suppose that the following hypothesis holds.

Hypothesis 7.4. The characteristic of F does not divide m, or G has type F_4, G_2, or E_8.

Write θ_η for the twisting isomorphism $H^1(F, \overline{G}_\eta) \xrightarrow{\sim} H^1(F, \overline{G})$ and define

$$r(G) := r_{\overline{G}_\eta,m}(\theta_\eta^{-1}(0)) \in H^3(F, \mathbb{Z}/m\mathbb{Z}(2)) \subseteq H^3(F, \mathbb{Q}/\mathbb{Z}(2)).$$

This element depends only on the isomorphism class of G. If G is one of the groups listed in Table 3, then m divides 24 and $r(G)$ takes values in $H^3(F, \mathbb{Z}/m\mathbb{Z})$ (see [**23**, § VII, Exercise 11]).

It is easy to see that r is an invariant in the sense of [**27**]; for example, for every extension K/F we have

$$\mathrm{res}_{K/F}(r(G)) = r(G \times_F K).$$

Lemma 7.5. *If G is a versal group and Claim* 7.4 *holds, then $r(G)$ has order m.*

Proof. The claim is equivalent to the following statement in the language of Proposition 7.2. If V is a versal G^q-torsor, then $r_{G^q,m}(V)$ has order m in $H^3(F, \mathbb{Z}/m\mathbb{Z}(2))$. To prove this, by the specialization property of versal torsors [**27**, 12.3], it suffices to produce an element $y \in H^1(E, G^q)$ for some extension E/F such that $r_{G^q,m}(y)$ has order m in $H^3(E, \mathbb{Z}/m\mathbb{Z}(2))$. We take y to be the image of a versal $\widetilde{G}^q$-torsor $\tilde{V}$. From [**27**, 10.8], the element $r_{\widetilde{G}^q}(\tilde{V})$ has order δ, and the claim follows. □

8. *Tits algebras*

Let G be a semisimple algebraic group over F. Tits defined certain invariants $\beta_G(\chi)$ of G in [**43**] with the following properties; cf. [**23**, § 27; **26**, § 4; **28**, § 2]. Write Z for the scheme-theoretic center of a simply connected cover of G. The Cartier dual Z^* of the center, the *cocenter*, is an étale group scheme. For $\chi \in Z^*(F_{\mathrm{sep}})$, we write F_χ for the subfield of F_{sep} fixed by the stabilizer of χ in $\mathrm{Gal}(F)$. The *Tits algebra* $\beta_G(\chi)$ is an element of $H^2(F_\chi, \mathbb{G}_m)$, the Brauer group of F_χ.

Let $t_G \in H^2_{\mathrm{fppf}}(F, Z)$ be the Tits' class of G (see [**23**, § 31]). We note the following properties.

Property 8.1. For all $\chi \in Z^*(F_{\mathrm{sep}})$, the image of t_G under the composite map

$$H^2_{\mathrm{fppf}}(F, Z) \xrightarrow{\mathrm{res}} H^2_{\mathrm{fppf}}(F_\chi, Z) \xrightarrow{\chi_*} H^2(F_\chi, \mathbb{G}_m)$$

is $\beta_G(\chi)$; see [**23**, 31.7].

Property 8.2. For G of inner type, the Galois group acts trivially on $Z^*(F_{\mathrm{sep}})$ and $F_\chi = F$ for all χ.

Property 8.3. The Galois action on Z^* is 'compatible' with the Tits algebras in the following sense. For $\sigma \in \mathrm{Gal}(F)$, clearly $F_{\sigma\chi}$ equals $\sigma(F_\chi)$. Further, the natural map

$$\sigma : H^2(F_\chi, \mathbb{G}_m) \longrightarrow H^2(F_{\sigma\chi}, \mathbb{G}_m)$$

sends $\beta_G(\chi)$ to $\beta_G(\sigma\chi)$. This follows from Property 8.1. In particular, the index of the division algebra representing $\beta_G(\chi)$ is the same as the index of the algebra representing $\beta_G(\sigma\chi)$.

Property 8.4. The Tits algebras are compatible with scalar extension in a natural way. If K/F is an extension contained in F_{sep}, then K_χ is the compositum of F_χ and K, and $\beta_{G\times K}(\chi)$ is the restriction $\mathrm{res}_{K_\chi/F_\chi} \beta_G(\chi)$.

Property 8.5. For every integer k, the element $\mathrm{res}_{F_\chi/F_{k\chi}} \beta_G(k\chi)$ equals $k \cdot \beta_G(\chi)$ in $H^2(F_\chi, \mathbb{G}_m)$, by Property 8.1.

We say simply the *index of* $\beta_G(\chi)$ to mean the index of a division algebra representing $\beta_G(\chi)$, or, equivalently, the dimension of the smallest separable extension of F_χ that kills $\beta_G(\chi)$ (see [**18**, 4.5.9]), or, which is the same, the subgroup of $H^2(F_\chi, \mathbb{G}_m)$ generated by $\beta_G(\chi)$. Property 8.5 gives the following property.

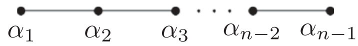
Property 8.6. *If k is relatively prime to the exponent of $Z^*(F_{\mathrm{sep}})$, then $\beta_G(k\chi)$ and $\beta_G(\chi)$ have the same index.* Indeed, χ and $k\chi$ generate the same subgroup of $Z^*(F_{\mathrm{sep}})$, and hence $F_{k\chi}$ equals F_χ.

Define $n_G(\chi)$ to be $\gcd\{\dim \rho\}$, where ρ varies over the representations of G defined over F_{sep} such that $\rho|_Z = \chi$; for example, $n_G(0) = 1$. Note that $n_G(\chi)$ depends only on χ and the Killing–Cartan type of G, and not on the field F. Further, a twisting argument together with Property 8.1 shows that the index of $\beta_G(\chi)$ divides $n_G(\chi)$; see [**28**, Proposition 2.4]. The numbers $n_G(\chi)$ have been computed in [**28**, **29**].

Example 8.7. In the case where $Z^*(F_{\mathrm{sep}})$ is $\mathbb{Z}/p\mathbb{Z}$ for some prime p, we obtain a convenient formula from [**29**, Proposition 6.13(2)]. One finds for nonzero χ that

$$n_G(\chi) = \begin{cases} 27 & \text{for } G \text{ of type } {}^1E_6 \text{ or } {}^2E_6 \text{ from [29, p. 156]}, \\ 8 & \text{for } G \text{ of type } E_7 \text{ from [29, p. 166]}. \end{cases}$$

EXAMPLE 8.8. Label the simple roots of a group of type A_{n-1} as in the following Dynkin diagram.



We write ω_i for the fundamental dominant weight corresponding to the simple root α_i. From [28, pp. 562–564], we can verify from the definition of n_G that $n_G(\omega_i|_Z) = n/\gcd(i, n)$.

Regarding the Tits algebras of versal/superversal groups, our main tool is the following theorem of Merkurjev.

THEOREM 8.9 [26, Theorem 5.2]. *If G is a versal group and χ is fixed by $\mathrm{Gal}(F)$, then the index of the Tits algebra $\beta_G(\chi)$ is $n_G(\chi)$.*

Alternatively, one can find concrete constructions for G of inner type in [28, 29]. We extend this theorem in § 14 below.

9. *Groups of type E_8*

LEMMA 9.1. *If G is a group of type E_8 that is neither split nor anisotropic, then the semisimple anisotropic kernel of G and the order of $r(G)$ are as in one of the rows of Table 4.*

Proof (*sketch*). The list of possibilities for the semisimple anisotropic kernel G^{an} is from [39] or [42]. It remains to verify the claim on $r(G)$ in each case. As G is simply connected with trivial Tits algebras and the $*$-action is trivial, it follows that the same holds for G^{an}. (To see this for the Tits algebras, one applies [43, p. 211].)

By Tits' Witt-type theorem from [39, 16.4.2] or [42, 2.7.1], there is a class $z \in H^1(F, G^{\mathrm{an}})$ such that twisting G^{an} and G by z gives split groups. The definition of $r(G)$ says that it equals $r_{G^{\mathrm{an}}_z}(\theta_z^{-1}(0))$, and hence its order divides the Dynkin index of G^{an}. By hypothesis, G^{an} is anisotropic, and hence z is not zero and the main result of [13] gives that $r(G)$ is not zero. The remaining possibilities for the order of $r(G)$ are those listed in Table 3. □

LEMMA 9.2 (cf. [45, Proposition 5, p. 134]). *Let G be a group of type E_8 over a 5-special field F. If G is not split, then G is anisotropic and every proper nontrivial reductive subgroup of G is a torus of rank 4 or 8 or has semisimple type A_4 or $A_4 \times A_4$.*

We note that all of these possibilities can occur; see [14, 15.7].

TABLE 4. *Possible invariants of a group G of type E_8 that is neither split nor anisotropic.*

Semisimple anisotropic kernel of G	Order of $r(G)$
E_7	$\neq 1$, divides 12
D_7	2
E_6	3 or 6
D_6	2
D_4	2

Proof of Lemma 9.2. As G is not split, if it is isotropic, then $r(G)$ is not zero by Table 4. However, G is split by a separable extension of F, and thus $r(G)$ has order dividing 5. It follows from Table 4 that G is anisotropic. This implies, by Table 2, that G cannot contain tori of ranks 1, 2, or 3. By the previous sentence, every nontrivial reductive subgroup of G has rank 4 or 8. We claim that any semisimple group H of rank 4 over F is isotropic or of type A_4. Indeed, since F has no separable field extension of degree at most 4, it follows that the $*$-action of the absolute Dynkin diagram Δ_H of H is trivial, and thus H is an inner form. If Δ_H contains a summand of either type A_n for $1 \leqslant n \leqslant 3$, B_n for $2 \leqslant n \leqslant 4$, C_3, C_4, or D_4, then H is isotropic. If H is of type F_4, then it contains a subgroup of type D_4, and hence splits as well. Therefore, the only possibility for nonisotropic H is to be of type A_4.

For a subgroup of rank 8, we consult the list of such in [**11**, Table 10]; in addition to $A_4 \times A_4$, we find:

(i) $A_7 \times A_1, A_5 \times A_2 \times A_1, E_6 \times A_2, E_7 \times A_1, D_6 \times A_1^{\times 2}, D_5 \times A_3, A_3^{\times 2} \times A_1^{\times 2}$;
(ii) $A_8, D_8, A_2^{\times 4}, D_4^{\times 2}, D_4 \times A_1^{\times 4}, A_1^{\times 8}$.

Those in list (i) do not occur because they contain tori of ranks neither 4 nor 8. Those in list (ii) do not occur because they are isotropic over a 5-special field. □

LEMMA 9.3. *Let G be a nonsplit group of type E_8 over a* 3*-special field. The following conditions hold.*

(i) *If G is isotropic, then it has rank* 2 *and a semisimple anisotropic kernel of type E_6.*
(ii) *The group G does not contain a semisimple subgroup of type A_4 or $A_4 \times A_4$.*

Proof. Over a 3-special field, every group of type E_7 is isotropic [**14**, 13.1], and hence (i) follows from Table 4. Since the field is 3-special, a semisimple group of type A_4 or $A_4 \times A_4$ is split, contradicting the statement that G has rank at most 2. □

THEOREM 9.4. *A versal group of type E_8 is almost abelian.*

Proof. Every group of type E_8 is simply connected, and thus the versal group G is obtained by twisting the split group G^q of type E_8 by a versal torsor. By [**27**, 16.8], the Rost invariant $r_{G^q}(G)$ has order 60 in $H^3(F, \mathbb{Z}/60\mathbb{Z}(2))$. In particular, this element is not killed by a co-3-closure nor a co-5-closure of F, and so G is not split over such fields. Combining Lemmas 9.2 and 9.3, it suffices to show that G does not contain any rank 4 tori.

To seek a contradiction, we suppose that G has a rank 4 torus S defined over F, and that an extension L/F splits S. Then G has an L-rank of at least 4, and so G is L-split or has a semisimple anisotropic kernel, a strongly inner form of D_4 over L; in either case G is split by an extension of F of degree dividing $2[L : F]$. By [**48**], the torsion index of the compact real Lie group E_8 is 2880; thus every extension that splits G has dimension divisible by 2880, and hence $[L : F]$ is divisible by $1440 = 2^5 \cdot 3^2 \cdot 5$. However, by Table 2, every rank 4 torus over F is split by an extension of degree dividing $2^7 \cdot 3^2$ or $2^3 \cdot 5$ or $2^4 \cdot 3 \cdot 5$. This is a contradiction. □

Applying 2.4 gives the following corollary.

COROLLARY 9.5 (cf. [**47**, Corollary 3, p. 1135]). *Inside a versal group G of type E_8 over a field of characteristic* 0, *the proper nontrivial connected subgroups are the maximal tori.*

In the proof of Theorem 9.4, the second paragraph amounts to a reference to Totaro's general result on the torsion index. Alternatively, one could replace it with a reference to [**47**, Lemma 5], which says that *if a group G of type E_8 contains a regular rank 4 torus, then G is split by an extension of dimension dividing $2^7 \cdot 3^2$ or $2^3 \cdot 5$*. This gives a strengthening of Theorem 9.4 as follows.

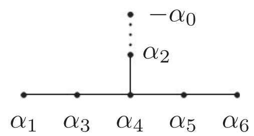
Theorem 9.6. *If G is a group of type E_8 such that $r(G)$ has order 60, then G is almost abelian.*

Proof. An extension of degree dividing $2^7 \cdot 3^2$ or $2^3 \cdot 5$ cannot kill an element of order 60 in $H^3(F, \mathbb{Z}/60\mathbb{Z}(2))$ by a restriction/corestriction argument, and hence G cannot contain a rank 4 torus. Now combine Lemmas 9.2 and 9.3(ii) to complete the proof. □

10. *Weyl group of E_6*

This section prepares the ground for the proof of Proposition 11.1.

Notation 10.1. We label the simple roots in a root system of type E_6 as in the extended Dynkin diagram



where α_0 denotes the highest root. The roots

$$\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_2, -\alpha_0 \tag{1}$$

are a basis for a root subsystem of type $A_2 \times A_2 \times A_2$, and we refer to it simply as $A_2^{\times 3}$. Of course, all subsystems of E_6 of that type are conjugate under the Weyl group $W(E_6)$ (see [**4**, Exercise VI.4.4]).

Lemma 10.2. *There is exactly one conjugacy class of elements of $W(E_6)$ of order 2, determinant 1, and trace -2. Every element of the class normalizes a subroot system of type $A_2^{\times 3}$.*

Here, the determinant and trace are computed relative to the action of $W(E_6)$ on the $\mathbb{Q}$-vector space spanned by the roots.

Proof of Lemma 10.2. The determinant defines an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow W(E_6) \xrightarrow{\det} \pm 1 \longrightarrow 1,$$

where Γ is the finite simple group $U_4(2)$; cf. [**4**, Exercise VI.4.2]. To prove the first claim, it suffices to note that there is exactly one conjugacy class of elements of Γ of order 2 and trace -2, as can be read off of the fourth line of the table from p. 27 of [**8**].

For the second claim, take w to be the composition of the reflections in the (pairwise strongly orthogonal) roots

$$\begin{array}{cc} \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 & \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\ \alpha_5 + \alpha_6 & \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \end{array}$$

Clearly, w has determinant 1. It interchanges α_1 and α_3 with α_2 and $-\alpha_0$, respectively, and sends α_5 and α_6 to $-\alpha_5$ and $-\alpha_6$, respectively. It follows that w has trace -2 and stabilizes the subsystem $A_2^{\times 3}$. □

EXAMPLE 10.3. The roots $\alpha_2, \alpha_3, \alpha_4$, and α_5 span a subroot system of type D_4 in E_6, and this gives an inclusion $W(D_4) \hookrightarrow W(E_6)$. The element -1 of $W(D_4)$ maps to an element $w \in W(E_6)$ of order 2. This element acts as -1 on the 4-dimensional subspace spanned by the roots in D_4 and fixes the (orthogonal) 2-dimensional subspace spanned by the fundamental weights of E_6 dual to α_1 and α_6. It follows that w has determinant 1 and trace -2.

11. *Groups of type* ${}^1E_6$

Let G be a group of type ${}^1E_6$. The cocenter Z^* is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The numbers $n_G(\chi)$ are 27 for nonzero χ, and the Tits algebras $\beta_G(1)$ and $\beta_G(2)$ have the same index by Property 8.6. We prove the following proposition.

PROPOSITION 11.1. *Let G be a group of type ${}^1E_6$ over a field F of characteristic different from 2. The following conditions hold.*

(i) *If F is a p-special field for some prime p, then G contains a semisimple F-subgroup of type $A_2^{\times 3}$.*

(ii) *If $\beta_G(1)$ has index 27, then G has no nontrivial proper reductive F-subgroups other than maximal tori and possibly semisimple subgroups of type $A_2^{\times 3}$.*

Part (ii) of the proposition is known, see [**45**, p. 129].

Versal groups have $\beta_G(1)$ of index 27, and thus (ii) is not empty. (Alternatively, an explicit construction of a group of type ${}^1E_6$ as in (ii) and containing a semisimple subgroup of type $A_2^{\times 3}$ can be found in [**29**, pp. 153–155].)

LEMMA 11.2. *If a group G of type ${}^1E_6$ over a field of characteristic different from 2 is neither split nor anisotropic, then its semisimple anisotropic kernel, the index of $\beta_G(1)$, and $r(G)$ are as in one of the rows of Table 5.*

Proof (*sketch*). As in the proof of the similar result for groups of type E_8 (Lemma 9.1), the list of possible types for the semisimple anisotropic kernel is known. The claim about the index of $\beta_G(1)$ follows from [**43**, p. 211]. As for $r(G)$, it belongs to $H^3(F, \mathbb{Z}/2\mathbb{Z})$, and so we may compute it over a co-2-closure of F, where G has trivial Tits algebras; the claim follows as for E_8. □

REMARK 11.3. We take this opportunity to fill a tiny gap from the classification of possible Tits' indexes from [**39**, p. 311]. That reference omits a proof that a group G of type ${}^1E_6$ over

TABLE 5. *Possible invariants of a group G of type ${}^1E_6$ that is neither split nor anisotropic.*

Semisimple anisotropic kernel of G	Index of $\beta_G(1)$	$r(G)$
$A_2 \times A_2$	3	0
D_4	1	$\neq 0$

a field of characteristic 3 cannot have a semisimple anisotropic kernel of type A_5. To see this, suppose that G has such a semisimple anisotropic kernel; we call it M. It is isomorphic to $\mathrm{SL}_1(D)$ for a division algebra D of index 6. The fundamental weight ω_4 corresponding to the simple root α_4 of G belongs to the root lattice of E_6, and so the Tits algebra $\beta_G(\omega_4)$ is zero. On the other hand, the restriction of ω_4 to M has Tits algebra with Brauer class of $D \otimes D \otimes D$. From [**43**, p. 211], these two Brauer classes are equal; thus $D^{\otimes 3}$ is split and M is isotropic, which is a contradiction.

Proof of Proposition 11.1(ii). It suffices to prove this in the case where F is 3-special. By Table 5, G is anisotropic, and remains anisotropic over every cubic extension of F. It follows from Table 2 that G cannot contain F-tori of dimensions 1, 2, or 3, nor dimensions 4 or 5 by symmetry. Consulting [**11**, Table 10], the only possible types for proper semisimple subgroups of maximal rank are $A_5 \times A_1$ and $A_2^{\times 3}$. However, the first type contains a 1-dimensional torus, and so it cannot be the type of an F-subgroup of G. □

Lemma 11.4. *Let G be a group of type ${}^1E_6$ over a field F of characteristic different from* 2. *If G is split by an extension of F of degree not divisible by* 3, *then G contains a subgroup of type $A_2^{\times 3}$.*

Proof. The hypothesis on G implies that G is split, in which case we are done, or G is isotropic with a semisimple anisotropic kernel of type D_4.

We write E_6, D_4, and G_2 for the split simply connected groups of those types. We view D_4 as a subgroup of E_6 via the inclusion from Example 10.3 and G_2 as the subgroup of D_4 consisting of elements fixed by the outer automorphism ϕ of order 3 that cyclically permutes the simple roots α_2, α_3, and α_5. (Recall that the roots of E_6 are all relative to some fixed split maximal torus T.)

Write n for an element of D_4 normalizing T and representing -1 in the Weyl group of D_4. Replacing n with $n\,\phi(n)\,\phi^2(n)$, we may assume that n belongs to G_2. Write A for the group generated by n and a maximal torus $T_2 := (T \cap G_2)^0$ in G_2. The natural map $H^1(F, A) \to H^1(F, G_2)$ is surjective; cf. [**37**, 1.5.2]. Further, the image of $H^1(F, G_2) \to H^1(F, \mathrm{Aut}(D_4))$ contains the class of the semisimple anisotropic kernel of G because 8-dimensional quadratic forms in I^3 are similar to Pfister forms. Combining these two observations with Tits' Witt-type theorem [**39**, 16.4.2], we deduce that the simply connected cover of G is obtained by twisting E_6 by a 1-cocycle η with values in A. By Lemma 10.2, η stabilizes a subroot system of type $A_2^{\times 3}$, and hence the claim. □

Proof of Proposition 11.1(i). Suppose that $p = 3$; otherwise the claim is Lemma 11.4. The adjoint quotient of G is obtained by twisting the split adjoint E_6 by a 1-cocycle η with values in the normalizer N of a maximal split torus T such that the image of η in N/T is contained in a 3-Sylow subgroup of $W(E_6)$ of our choice. It suffices to note that one can find a 3-Sylow in $W(E_6)$ that normalizes the subsystem $A_2^{\times 3}$ from Remark 10.1; cf. [**5**, Lemma 5.8]. □

12. *Groups of type* ${}^2E_6$

Let G be a group of type ${}^2E_6$ over F; there is a uniquely determined quadratic extension K/F such that G is of type ${}^1E_6$ over K. The cocenter Z^* of G is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ with a twisted Galois action; an element of $\mathrm{Gal}(F)$ acts trivially on Z^* if and only if it restricts to the identity on K. The number $n_\chi(1)$ is 27 and the class $\beta_G(1)$ is represented by a central simple K-algebra of degree 27. We prove in Theorem 14.1(ii) below that there exist such groups G where $\beta_G(1)$ has index 27. Proposition 12.1 is an analog of Proposition 11.1; over a p-special field for $p \neq 2$, every group of type E_6 is inner, and we are back in the situation of § 11.

Proposition 12.1. *Let G be a group of type ${}^2E_6$ over a field F of characteristic different from* 2. *The following conditions hold.*

(i) *If F is a 2-special field, then G contains a semisimple F-subgroup of type D_4.*
(ii) *If $\beta_G(1)$ has index 27, then G has no nontrivial proper closed connected F-subgroups other than maximal tori and possibly semisimple subgroups of type $A_2^{\times 3}$.*

In the case of a 2-special field F, groups of type ${}^2E_6$ can still be quite complicated. For example, they can be anisotropic (like the compact real form of E_6), which does not occur in the inner case, a fact we have exploited in the proof of Proposition 11.1(i).

Proof. (i) We have seen that G is K-isotropic. The field K is 2-special, and hence Table 5 shows that G_K admits a parabolic subgroup P of type $\{1, 6\}$. From [**32**, Lemma 6.17′, p. 383], the group G has a reductive F-subgroup of type D_4, namely, the semisimple part of a Levi subgroup of P.

(ii) Let H be a nontrivial proper closed connected F-subgroup of G. Over K, then H is a maximal torus in G or is semisimple of type $A_2^{\times 3}$ by Proposition 11.1. Hence the same is true over F. □

13. *Groups of type E_7*

In this section, we consider groups G of type E_7. The cocenter of the simply connected cover of G is $\mathbb{Z}/2\mathbb{Z}$, and the Tits algebra $\beta_G(1)$ has index dividing $n_G(1) = 8$. The element $r(G)$ belongs to $H^3(F, \mathbb{Z}/3\mathbb{Z})$. Our goal is to prove the following proposition.

Proposition 13.1. *Let G be a group of type E_7 over a field F of characteristic zero. The following conditions hold.*

(i) *If F is p-special, then G contains semisimple subgroups of type D_4, $A_1^{\times 3}$, and $D_4 \times A_1^{\times 3}$.*
(ii) *If $\beta_G(1)$ has index* 8 *and $r(G)$ is nontrivial, then every proper semisimple subgroup of G* (*if there are any*) *is normalized by a maximal torus of G and has type D_4, $A_1^{\times 3}$, or $D_4 \times A_1^{\times 3}$.*

We remark that groups of type D_4 and $A_1^{\times 3}$ can be almost abelian by Theorem 1.1 and Proposition 3.1. Also, groups G as in (ii) exist; for example, a versal group satisfies the hypothesis by Theorem 8.9 and Lemma 7.5.

Proof of Proposition 13.1(i). If the group of type E_7 is split (for example, if $p \neq 2, 3$), then the result is clear.

If F is 2-special, then we note that the Weyl group of E_7 and of the standard subgroup of type $D_6 \times A_1$ have the same 2-Sylows, and hence every F-group of type E_7 has an F-subgroup of type $D_6 \times A_1$; cf., for example, [**14**, 14.7]. Then it suffices to note that every group of type D_6 contains a subgroup of type $D_4 \times D_2$, that is, $D_4 \times A_1^{\times 2}$.

If F is 3-special, then every simply connected group of type E_7 is obtained by twisting the split group by a 1-cocycle that normalizes the standard D_4 subgroup [**14**, 13.1]. The centralizer of this D_4 subgroup has type $A_1^{\times 3}$, corresponding to the subroot system spanned by

$$\begin{aligned} &\alpha_7, \quad \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \quad \text{and} \\ &2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \end{aligned} \tag{2}$$

and the conclusion follows. □

Besides being split or anisotropic, the possible indexes for G are given in Table 6. The proof is the same as the ones for E_8 and ${}^1E_6$ given in Lemmas 9.1 and 11.2.

LEMMA 13.2. *Suppose that G is as in Proposition 13.1(ii). Then the following conditions hold.*

(i) *The group G is anisotropic and every extension that splits G has dimension divisible by 24.*

(ii) *Let K be a finite extension of F such that G is K-isotropic.*

  (a) *If 3 does not divide $[K : F]$, then G has K-rank 1 and 8 divides $[K : F]$.*

  (b) *If 3 divides $[K : F]$ but 4 does not, then G has K-rank 1 and 6 divides $[K : F]$.*

Proof. Since $\beta_G(1)$ has index 8, G is anisotropic by Table 6. Further, if G is split over an extension L/F, then L must kill both $\beta_G(1)$ and $r(G)$, and hence the dimension $[L : F]$ is divisible by 8 and 3, respectively. This proves (i).

For (ii(a)), the hypothesis on the dimension implies that K does not kill $r(G)$, and hence $G \times K$ must have a semisimple anisotropic kernel E_6 by Table 6.

For (ii(b)), the hypothesis on the dimension implies that $\beta_G(1) \times K$ has index at least 4 by [**18**, 4.5.11(1)], and hence the semisimple anisotropic kernel of $G \times K$ must have type D_6 by Table 6. □

The lower bound 'divisible by 24' in Lemma 13.2(i) is easily obtained. One might hope to prove a stronger bound, but this is impossible. For *any* group G of type E_7, the greatest common divisor of the dimensions $[L : F]$, where L ranges over extensions of F splitting G, divides 24 by [**48**, 6.1].

Proof of Proposition 13.1(ii). Because G is anisotropic and remains so over every quadratic extension of F by Lemma 13.2(ii(a)), it cannot contain a rank 1 torus nor an F-subgroup of type A_1. By symmetry, G contains no rank 6 tori nor any semisimple subgroup of rank 6.

Over a co-3-closure F_3 of F, the group G has rank 1 by Lemma 13.2(ii(a)). Nevertheless, a group of type C_3, G_2, or D_n for $n \neq 4$, of type A_n for $n = 3, 4, 6$, or 7, or of type $A_1^{\times s}$ for $s \neq 3$ has F_3-rank at least 2. Therefore, such a group cannot be an F-subgroup of G.

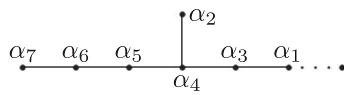
A group of type F_4, A_2, or A_5 has rank at least 2 over an extension of F of degree dividing 6, and hence it cannot be an F-subgroup of G by Lemma 13.2(ii(b)).

TABLE 6. *Possible invariants of a group G of type E_7 that is neither split nor anisotropic.*

Semisimple anisotropic kernel of G	Index of $\beta_G(1)$	$r(G)$
E_6	1	$\neq 0$
D_6	Divides 4	0
$A_1 \times D_5$	2	0
$A_1 \times D_4$	2	0
D_4	1	0
$A_1^{\times 3}$	2	0

Suppose that H is a semisimple F-subgroup of G that is not normalized by a maximal torus of G. From [**11**, Table 34, p. 233] we obtain a list of the possibilities for the type of H, using the assumption that F has characteristic zero. All of them have an isotypic component of type A_1, G_2, F_4, or C_3, and so they cannot be F-subgroups of G by the preceding observations.

Now suppose that H is a semisimple subgroup of G of type $A_2^{\times 2}$. We claim that its centralizer has semisimple type A_2. It suffices to prove this over an algebraic closure of F, where [**11**] says that all subgroups of type $A_2^{\times 2}$ are conjugate. In particular, we may assume that H is generated by the root subgroups corresponding to the highest root, α_1, α_6, and α_7 as in the following extended Dynkin diagram.



(3)

The intersection of the maximal torus T in G with H gives a maximal torus S in H. The centralizer $Z_G(S)$ is reductive with semisimple part generated by the root subgroups for roots α_2 and α_4. Clearly, these root subgroups also centralize H, proving the claim. As G has no F-subgroups of type A_2, this is a contradiction, and so G has no subgroups of type $A_2^{\times 2}$.

To complete the proof of (ii), we consult the list of possible semisimple subalgebras of E_7 in [**11**, Table 11], and note that all the types other than D_4, $A_1^{\times 3}$, and $D_4 \times A_1^{\times 3}$ have rank 6 or have an isotypic component excluded by the previous arguments. □

REMARK 13.3. Suppose now that G is a group of type E_7 over a field F of characteristic different from 2, and that G contains a semisimple subgroup of type $D_4 \times A_1^{\times 3}$. The $A_1^{\times 3}$ component is isogenous to $R_{L/F}(\mathrm{SL}(Q))$ for some cubic étale F-algebra L and quaternion L-algebra Q, and *the Tits algebra of G is Brauer-equivalent to* $\mathrm{cor}_{L/F}[Q]$. To see this, we examine the center of the $A_1^{\times 3}$ component in the case where G is split. Viewing $A_1^{\times 3}$ as generated by the roots from (2), the nonidentity elements of the center of each copy of A_1 are

$$h_{\alpha_7}(-1), \quad h_{\alpha_2}(-1)\, h_{\alpha_3}(-1)\, h_{\alpha_7}(-1), \quad \text{and} \quad h_{\alpha_3}(-1)\, h_{\alpha_5}(-1)\, h_{\alpha_7}(-1),$$

respectively, where $h_{\alpha_i} \colon \mathbb{G}_m \to G$ is the cocharacter corresponding to the simple root α_i. In particular, the fundamental weight ω_7 of G restricts to the product map $\mu_2^{\times 3} \to \mu_2$ on the center of $A_1^{\times 3}$. By [**43**, § 5], this proves the claim.

Remark 13.3 implies that *the Tits algebra of every group of type E_7 over a 2-special field is Brauer-equivalent to a tensor product of three quaternion algebras*. Indeed, it is certainly Brauer-equivalent to $Q_1 \otimes \mathrm{cor}_{K/F}(Q_2)$, where Q_1 and Q_2 are quaternion algebras over F and a quadratic étale F-algebra K, respectively. However, $\mathrm{cor}_{K/F}(Q_2)$ has degree 4 and exponent 2, and thus it is isomorphic to a tensor product of two quaternion algebras.

14. *Maximal indexes of Tits algebras*

The purpose of this section is to extend Theorem 8.9 to also include the most interesting cases where the character χ is not fixed by $\mathrm{Gal}(F)$. Specifically, we prove the following theorem.

THEOREM 14.1. (*i*) *A superversal group of type ${}^2A_{n-1}$ for n odd is isogenous to* $\mathrm{SU}(D, \tau)$ *for D, a division algebra of degree n with unitary involution τ. Further, $\lambda^i D$ has index $n/\gcd(i, n)$ for $1 \leqslant i \leqslant n$.*

(ii) *A superversal group G of type ${}^2E_6$ has Tits algebra $\beta_G(1)$ of index* 27.

(iii) *A superversal group G of type ${}^3D_4$ or ${}^6D_4$ has Tits algebras $\beta_G(\chi)$ of index* 8 *for $\chi \neq 0$.*

To prove Theorem 1.1, we only need the result for type ${}^3D_4$. The result for type ${}^6D_4$ is known (and we provide a different proof here); it is Proposition 8.1 in [**24**].

The theorem follows easily from Lemma 14.3, stated and proved below.

Paragraph 14.2. We now set up the proof of Lemma 14.3. Fix a root system Φ, a set of simple roots Δ, a subgroup Γ of $\mathrm{Aut}(\Delta)$, and a dominant minuscule weight λ. We write G^d for the split simply connected group with root system Φ, and (I_μ, ρ_μ) for the irreducible representation of G^d with highest weight μ. Define a representation (V^d, ρ^d) of G^d by setting

$$V^d := \bigoplus_{\mu \in \Gamma\cdot\lambda} I_\mu \otimes I_\mu^*$$

as a vector space, where G^d acts on I_μ via ρ_μ and *acts trivially on* I_μ^*. As a representation of G^d, we have

$$(V^d, \rho^d) \cong \bigoplus_{\mu\in\Gamma\cdot\lambda} \bigoplus^{\dim I_\mu} (I_\mu, \rho_\mu).$$

We also define a representation $\rho^A \colon \mathrm{Aut}_\Gamma(G^d) \to \mathrm{GL}(V^d)$ for $\mathrm{Aut}_\Gamma(G^d)$ as in Definition 6.3. The identity component of $\mathrm{Aut}_\Gamma(G^d)$ is the adjoint group, and it acts via the natural representation on I_μ and I_μ^*. (The representations ρ_μ and ρ_μ^* of G^d do not factor through the adjoint group, but $\rho_\mu \otimes \rho_\mu^*$ does.) For each $\gamma \in \Gamma$, we have $\rho_\mu(\gamma^{-1}g) = \mathrm{Int}(a)\rho_{\gamma\mu}(g)$ for some matrix a depending on γ and μ. Combining these matrices with the natural permutation representation on the weights in $\Gamma\cdot\lambda$, we find an action of $\mathrm{Aut}_\Gamma(G^d)$ on V^d. Furthermore, for $\alpha \in \mathrm{Aut}_\Gamma(G^d)$ and $g \in G^d$, we have

$$\rho^d(\alpha g) = \mathrm{Int}(\rho^A(\alpha))\rho^d(g).$$

Now fix a 1-cocycle $z \in Z^1(F, \mathrm{Aut}_\Gamma(G^d))$. We twist G^d and ρ^d by z to obtain a group $G = G^d_z$ and a homomorphism $\rho : G \to \mathrm{GL}(V^d)_z$. Hilbert 90 gives an isomorphism between $\mathrm{GL}(V^d)_z$ and $\mathrm{GL}(V^d)$, and so ρ is a representation of G on V^d.

More concretely, G is the same as G^d over F_{sep}, but the Galois group $\mathrm{Gal}(F)$ acts on $G(F_{\mathrm{sep}})$ via the action $\cdot_z$ defined by $\sigma \cdot_z g := z_\sigma \sigma(g)$. The 1-cocycle $\rho^A(z)$ takes values in $\mathrm{GL}(V^d)$, and so is $\sigma \mapsto h^{-1}\,{}^\sigma h$ for some $h \in \mathrm{GL}(V)(F_{\mathrm{sep}})$. We write $\rho := \mathrm{Int}(h)\rho^d$. Then, for $g \in G(F_{\mathrm{sep}})$, we have

$$\sigma\rho(g) = \mathrm{Int}(\sigma h)\rho^d(\sigma g) = \mathrm{Int}(h)\,\mathrm{Int}(\rho^A(z_\sigma))\rho^d(\sigma g) = \mathrm{Int}(h)\rho^d(z_\sigma \sigma g) = \rho(\sigma \cdot_z g),$$

which confirms that ρ is defined over F (see [**1**, AG.14.3]).

Lemma 14.3. *Let λ be a minuscule dominant weight of a root system Φ with a set of simple roots Δ. Fix $\Gamma \subset \mathrm{Aut}(\Delta)$. If $|Z^*(F_{\mathrm{sep}})|$ and $|\Gamma\cdot\lambda|$ are relatively prime, then $\beta_G(\lambda)$ has index $n_G(\lambda)$ for every superversal group G constructed from Δ and Γ.*

We define $\beta_G(\lambda)$ and $n_G(\lambda)$, where the subscript λ is a weight of Φ, to be $\beta_G(\chi)$ and $n_G(\chi)$ (in the notation of § 8), where χ is the restriction of λ to the center of G.

Proof of Lemma 14.3. Let H be a group of type Φ over a field F. The fundamental property we use is that there is a unique irreducible representation $J_{H,\lambda}$ of H over F such that the composition series for $J_{H,\lambda}$ over an algebraic closure of F has I_λ as a quotient; this follows from [**43**, Theorem 7.2]. This representation satisfies

$$\dim J_{H,\lambda} = |\mathrm{Gal}(F) * \lambda| \cdot \mathrm{ind}\,\beta_H(\lambda) \cdot \dim I_\lambda. \tag{4}$$

Twisting the representation ρ^d of G^d defined in Remark 14.2 by a superversal torsor gives a representation W of G, and the simple quotients in the composition series of W are copies of $J_{G,\lambda}$.

We specialize the superversal group G to a versal group G' of inner type defined over a field K'. The representation W of G specializes to a representation W' of G'. As λ is fixed by $\mathrm{Gal}(K')$, the composition series for W' over K' has as irreducible quotients various copies of $J_{G',\mu}$ for $\mu \in \Gamma \cdot \lambda$.

We view the irreducible quotients of the composition series as partitioning the weights of W, or, equivalently, the weights of W'. Because G-invariant subspaces of W specialize to G'-invariant subspaces of W', the partition coming from the composition series under G' is a refinement of the partition of the composition series under G. The dimension of $J_{G',\mu}$ depends only on the $\mathrm{Aut}(\Delta)$-orbit of μ, by (4) and Property 8.3, and hence $\dim J_{G,\lambda}$ is a multiple of $\dim J_{G',\lambda}$; that is,

$$\operatorname{ind}\beta_{G'}(\lambda)\cdot\dim I_\lambda \quad \text{divides} \quad |\Gamma\cdot\lambda|\cdot\operatorname{ind}\beta_G(\lambda)\cdot\dim I_\lambda$$

by (4). By Theorem 8.9, the index of $\beta_{G'}(\lambda)$ is the number $n_{G'}(\lambda)$. As multiplication by $|Z^*(F_{\mathrm{sep}})|$ kills $\beta_{G'}(\lambda)$ (by Property 8.1), every prime dividing $n_{G'}(\lambda)$ also divides $|Z^*(F_{\mathrm{sep}})|$ (see [**18**, 4.5.13(2)]). By the hypothesis on $|Z^*(F_{\mathrm{sep}})|$, we deduce that

$$n_{G'}(\lambda) = \operatorname{ind}\beta_{G'}(\lambda) \mid \operatorname{ind}\beta_G(\lambda) \mid n_G(\lambda).$$

Since the numbers $n_{G'}(\lambda)$ and $n_G(\lambda)$ depend only on the root system, it follows that they are equal, and this proves the lemma. □

Proof of Theorem 14.1. The numbers appearing in Lemma 14.3 are given in Table 7, where the notation for type A follows Example 8.8. As $|Z^*(F_{\mathrm{sep}})|$ and $|\Gamma\cdot\lambda|$ are relatively prime for each of the cases listed in the theorem, the claim follows. □

15. *Groups of type* ${}^3D_4$ *or* ${}^6D_4$

Let G be a simple group of type D_4 over F. The cocenter of its simply connected cover is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ with a possibly nontrivial Galois action, and $n_G(\chi)$ is 8 for $\chi \neq 0$.

LEMMA 15.1. *Suppose that G is a group of type ${}^1D_4$ or ${}^3D_4$ such that $\beta_G(\chi)$ has index* 8 *for $\chi\neq 0$. Then the following conditions hold.*

(i) *The group G is anisotropic.*

(ii) *If K/F is a finite extension of F of dimension not divisible by* 8, *then the K-rank of G is at most* 1.

Proof. Claim (i) is known. For G of type ${}^3D_4$, it is [**12**, 2.5], and for type ${}^1D_4$ the proof is even easier.

TABLE 7. *Data for groups and weights considered in Theorem* 14.1, *in the notation of Lemma* 14.3.

Type of G	$\lvert Z^*(F_{\mathrm{sep}})\rvert$	λ	$\lvert\Gamma\cdot\lambda\rvert$	$n_G(\lambda)$
${}^2A_{n-1}$	n	ω_i	2	$n/\gcd(i,n)$
${}^2E_6$	3	Minuscule	2	27
${}^3D_4$ or ${}^6D_4$	4	Minuscule	3	8

For (ii), first suppose that G has type ${}^1D_4$, that is, that G is isogenous to $\mathrm{SO}(D, \sigma, f)$ for some central division F-algebra of degree 8 with quadratic pair such that the even Clifford algebra $C_0(D, \sigma, f)$ is isomorphic to $A \times B$ for A and B being central division F-algebras of degree 8. By the hypothesis on K, the index of $A \otimes K$, $B \otimes K$, and $D \otimes K$ is in all cases at least 2. It follows that the involution σ is not K-hyperbolic [**23**, 8.31], which is the claim.

For G of type ${}^3D_4$, there is a unique cubic Galois extension L/F over which G is of type ${}^1D_4$. The compositum KL has dimension $[KL : L]$ dividing $[K : F]$, and, in particular, $[KL : L]$ is not divisible by 8. Replacing F and K with L and KL, we are reduced to the type ${}^1D_4$ case. □

LEMMA 15.2. *Let G be a group of type ${}^3D_4$ or ${}^6D_4$ over a field of characteristic different from* 3*. If $r(G) \neq 0$, then G is anisotropic.*

The hypothesis on the characteristic is here so that $r(G)$ is defined; cf. Proposition 7.2. Recall from Table 3 that $r(G)$ is an element of $H^3(F, \mathbb{Z}/3\mathbb{Z})$. If G has type ${}^1D_4$ (or ${}^2D_4$), then $r(G)$ is automatically zero.

Proof of Lemma 15.2. The group G is not quasi-split; suppose that it is isotropic. The paper [**12**] gives a description of G by Galois descent (which also works in characteristic 2 with only cosmetic changes), and, in particular, G is quasi-split by a separable quadratic extension K of the field. As K kills $r(G)$ and the dimension of K is not divisible by 3, it follows that $r(G)$ is zero. □

THEOREM 15.3. *Let G be a group of type ${}^3D_4$ over a field of characteristic different from* 3*. If the Tits algebras $\beta_G(\chi)$ have index* 8 *for nonzero χ and $r(G)$ is not zero, then G is almost abelian.*

Proof. By Lemma 15.2, G is anisotropic and remains anisotropic over a co-3-closure of F. It follows that G does not contain a rank 1 F-torus nor a rank 2 F-torus split by an extension of degree dividing 8. By Lemma 15.1, G contains no rank 2 F-tori split by an extension of degree dividing 12. By symmetry and Table 2, every nontrivial F-torus in G is maximal.

Over an algebraic closure, a proper semisimple subgroup of G of rank 4 has type $A_1^{\times 4}$. An F-group of this type becomes isotropic over a separable extension of F of dimension a power of 2, and hence, by Lemma 15.2, cannot be contained in G. □

The hypothesis on $r(G)$ was used to exclude transfers of groups of type A_1 from a quartic field extension L/F. These are genuine dangers: the paper [**6**] shows that every isotropic group of type ${}^3D_4$ or ${}^6D_4$ contains a subgroup isogenous to $R_{L/F}(\mathrm{SL}_2)$.

We close by proving the theorem stated in the introduction.

Proof of Theorem 1.1. Let G be a superversal group of type ${}^3D_4$ over a field F of characteristic different from 3 (itself an extension of some base field F_0). Theorem 14.1 gives that $\beta_G(\chi)$ has index 8 for nonzero χ. Because a superversal group is, in particular, a versal group, we have $r(G) \neq 0$ by Lemma 7.5. Then G is almost abelian by Theorem 15.3 and Remark 2.5.

As for type ${}^6D_4$, since there are almost abelian groups of type ${}^3D_4$ defined over a suitable extension of F_0, Corollary 6.7 implies that the superversal group G of type ${}^6D_4$ is almost abelian. □

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