Serre's conjecture II: a survey

Philippe Gille

Dedicated to Parimala

Abstract We provide a survey of Serre's conjecture II (1962) on the vanishing of Galois cohomology for simply connected semisimple groups defined over a field of cohomological dimension ≤ 2.

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1 Introduction

Serre's original conjecture II (1962) states that the Galois cohomology set $H^1(k,G)$ vanishes for a semisimple simply connected algebraic group G defined over a perfect field of cohomological dimension ≤ 2 [53, §4.1] [54, II.3.1]. This means that G-torsors (or principal homogeneous spaces) over $\mathrm{Spec}(k)$ are trivial.

For example, if A is a central simple algebra defined over a field k and $c \in k^\times$, the subvariety

$$X_c := \{\mathrm{nrd}(y) = c\} \subset \mathbf{GL}_1(A)$$

of elements of reduced norm c is a torsor under the special linear group $G = \mathbf{SL}_1(A)$ which is semisimple and simply connected. If $\mathrm{cd}(k) \leq 2$, we thus expect that this G-torsor is trivial, i.e., $X_c(k) \neq \emptyset$. Applying this to all $c \in k^\times$, we thus expect that the reduced norm map $A^\times \to k^\times$ is surjective.

For imaginary number fields, the surjectivity of reduced norms goes back to Eichler in 1938 (see [40, §5.4]). For function fields of complex surfaces, this follows from the Tsen-Lang theorem because the reduced norm is a homogeneous form of degree $\deg(A)$ in $\deg(A)^2$-indeterminates [54, II.4.5]. In the general case, the surjec-

Philippe Gille
DMA, UMR 8553 du CNRS, Ecole normale supérieure, 45 rue d'Ulm, F-75005 Paris, France
e-mail: gille@ens.fr

tivity of reduced norms is due to Merkurjev-Suslin in 1981 [60, Th. 24.8], and this fact essentially characterizes fields of cohomological dimension ≤ 2 (see below).

One can discuss the conjecture with respect to the group classification or with respect to fields. With respect to groups, the main evidence for the conjecture is provided by classical groups, for which the result was established by Bayer-Parimala (1995, [4]). With respect to fields, evidence for the conjecture was provided by imaginary number fields (Kneser [40], Harder [36], Chernousov [9], see [51, §6]) and more recently by function fields of complex surfaces. Over the latter fields, the result for exceptional groups with no factors of type E_8 was pointed out in 2002 in the paper [15] by Colliot-Thélène-Gille-Parimala. It was completed in 2008 for all types by He-de Jong-Starr [38], who used deformation methods. This result has a clear geometric meaning: If $G/\mathbf{C}$ is a semisimple simply connected group and X a smooth complex surface, then a G-torsor over X (or a G-bundle) is locally trivial with respect to the Zariski topology (see §6.6).

There are previous surveys on Galois cohomology discussing this topic. Tits' lectures at Collège de France in 1990-91 discuss the Hasse principle and group classification [64]. Serre's Bourbaki seminar [55] deals among other things with progress on the conjecture up to 1994, see also Bayer's survey on classical groups [2]. For function fields of surfaces, see the surveys by Starr [59] and by Lieblich [42].

For exceptional groups (trialitarian, type E_6, E_7 and E_8), the general conjecture is still open despite some progress [11][15][17][29].

We take this opportunity to point out that Serre's conjecture has some analogy with topology. Indeed, if $\mathbf{G}$ is a semisimple simply connected complex group, we know that $\pi_1(\mathbf{G}) = \pi_2(\mathbf{G}) = 0$, hence $\mathbf{G}$ is 2-connected. Then for every CW-complex of dimension ≤ 2, the $\mathbf{G}$-bundles over X are trivial (cf. [61, Th. 11.34]).

2 Fields of cohomological dimension ≤ 2

Let k be a field and l be a prime. Recall that k is of l-cohomological dimension $\mathrm{cd}_l(k) \leq d$ if $H^i(k,A) = 0$ for every finite l-primary Galois module A and for all $i \geq d+1$. We know that it is equivalent to the vanishing of $H^{d+1}(L,\mathbf{Z}/l\mathbf{Z})$ for any finite separable extension L/k.

Examples 2.1. Recall the following examples of fields of cohomological dimension 2.

(1) Imaginary number fields;
(2) Function fields of complex surfaces;
(3) Merkurjev's tower of fields F_∞, namely an extension of $\mathbf{C}(X_1,\dots,X_{2n})$ such that the u-invariant is $u(F_\infty) = 2n$. Every $(2n+1)$-dimensional quadratic form over F_∞ is isotropic, but the form $\langle X_1, X_2, \cdots X_{2n}\rangle$ remains anisotropic over F_∞. Furthermore the tensor product of the quaternion algebras (X_{2i-1}, X_{2i}) for $i = 1,\dots,n$ is a division algebra over F_∞ [45], [46, Th. 3].

The third example shows that central simple algebras and quadratic forms are not in general low dimensional objects. We already mentioned the following characterization which uses Merkurjev-Suslin's theorem [48].

Theorem 2.2. *[60, Th. 24.8] Let l be an invertible prime in k. The following are equivalent:*

1. $\mathrm{cd}_l(k) \leq 2$.
2. *For any finite separable extension L/k and any l-primary central simple L-algebra A/L, the reduced norm* $\mathrm{nrd} : A^\times \to L^\times$ *is surjective.*
3. *For any finite extension L/k and any l-primary central simple L-algebra A/L, the reduced norm* $\mathrm{nrd} : A^\times \to L^\times$ *is surjective.*

We added here the easy implication $2) \Longrightarrow 3)$ which follows from the usual transfer argument. We say that k is of cohomological dimension $\leq d$ if k is of l-cohomological dimension $\mathrm{cd}_l(k) \leq d$ for all primes l.

If k is of positive characteristic p, we always have $\mathrm{cd}_p(k) \leq 1$; this explains the necessary change in the following analogous statement.

Theorem 2.3. *[28, Th. 7] Assume that* $\mathrm{char}(k) = p > 0$*. The following are equivalent:*

1. $H^3_p(L) = 0$ *for any finite separable extension L/k;*
2. *For any finite separable extension L/k and any l-primary central simple L-algebra A/L, the reduced norm* $\mathrm{nrd} : A^\times \to L^\times$ *is surjective.*

Here $H^3_p(k)$ is Kato's cohomology group, defined by means of logarithmic differential forms [39], see [33, §9]. We shall say that k is of separable p-dimension $\leq d$ if $H^{d+1}_p(L) = 0$ for all finite separable extension L/k, this defines the separable dimension $\mathrm{sd}_p(k)$ of k. For $l \neq p$, we let[1] $\mathrm{sd}_l(k) = \mathrm{cd}_l(k)$. If k is perfect, then $H^i_p(L) = 0$ for every finite extension L/k and for every $i \geq 2$. Hence if k is perfect and of cohomological dimension ≤ 2, k is of separable dimension ≤ 2.

Examples 2.4.

(1) The function field of a curve over a finite field is of separable dimension 2.
(2) The function field $k_0(S)$ of a surface over an algebraically closed field k_0 of characteristic $p \geq 0$ is of separable dimension 2.
(3) Given an arbitrary field F, Theorems 2.2 and 2.3 provide a way to construct a "generic" field extension E/F of separable dimension 2, see Ducros [19].

We can now state the strong form of Serre's conjecture II. For each simply connected group G, Serre defined the set $S(G)$ of primes in terms of the Cartan-Killing type of G, cf. [55, §2.2]. For absolutely almost simple groups, the primes are listed in Table 1.

[1] Kato defined the p-dimension $\dim_p(k)$ as follows [39]. If $[k : k^p] = \infty$, define $\dim_p(k) = \infty$. If $[k : k^p] = p^r < \infty$, $\dim_p(k) = r$ if $H^{r+1}_p(L) = 0$ for any finite extension L/k, and $\dim_p(k) = r+1$ otherwise.

type	$S(G)$
A_n ($n \geq 1$)	2 and the prime divisors of $n+1$
B_n ($n \geq 3$), C_n ($n \geq 2$), D_n (non-trialitarian for $n=4$)	2
G_2	2
trialitarian D_4, F_4, E_6, E_7	2, 3
E_8	2, 3, 5

Table 1 $S(G)$ for absolutely almost simple groups

Conjecture 2.5. *Let G be a semisimple and simply connected algebraic group. If $\mathrm{sd}_l(k) \leq 2$ for every prime $l \in S(G)$, then $H^1(k,G) = 0$.*

In the original conjecture, k was assumed perfect and of cohomological dimension ≤ 2. In characteristic $p > 0$, Serre's strengthened question furthermore assumed that $[k : k^p] \leq p^2$ if p belongs to $S(G)$ [55, §5.5]. Known results do not seem to indicate that this restriction is necessary.

So Conjecture 2.5 is indeed stronger than the original one. Theorems 2.2 and 2.3 show that the conjecture holds for groups of inner type A and that the hypothesis on k is sharp.

3 Link between the conjecture and the classification of groups

The classification of semisimple groups essentially reduces to that of semisimple simply connected groups G which are absolutely almost simple [41, §31.5][62]. This means that $G \times_k k_s$ is isomorphic to $\mathbf{SL}_{n,k_s}$, $\mathbf{Spin}_{2n+1,k_s}$, $\mathbf{Sp}_{2n,k_s}$, $\mathbf{Spin}_{2n,k_s}$,...

Let G/k be such a k-group and let $G \to G_{ad}$ be the adjoint quotient of G. Denote by G^q its quasi-split form and by G^q_{ad} its adjoint quotient. Then G is an inner twist of G^q, i.e., there exists a cocycle $z \in Z^1(k_s/k, G^q_{ad}(k_s))$ such that $G \cong {}_zG^q$. We then identify G and ${}_zG^q$.

The other way around, we know that there exists a unique class $\nu_G = [a] \in H^1(k, G_{ad})$ such that $G^q \cong {}_aG$ [41, 31.6]. We denote by $z^{op} \in Z^1(k, {}_zG^q_{ad})$ the opposite cocycle of z, it is defined by $\sigma \mapsto z_\sigma^{-1} \in {}_zG(k_s)$.

We have $G^q \cong_{z^{op}} \big({}_zG^q\big)$. Hence the image of ν_G under $H^1(k, G_{ad}) \xrightarrow{\sim} H^1(k, {}_zG^q{}_{ad})$ is nothing but $[z^{op}]$. We have an exact sequence

$$1 \to Z(G) \to G \to G_{ad} \to 1$$

of k-algebraic groups with respect to the $fppf$-topology (faithfully flat of finite presentation, see [18, III] or [57]). This gives rise to an exact sequence of pointed sets [5, app. B]

$$\begin{aligned} 1 \to Z(G)(k) \to G(k) \to G_{ad}(k) \xrightarrow{\varphi_G} H^1_{fppf}(k, Z(G)) \to \\ \to H^1_{fppf}(k, G) \to H^1_{fppf}(k, G_{ad}) \xrightarrow{\delta_G} H^2_{fppf}(k, Z(G)). \end{aligned} \quad (3.1)$$

The homomorphism φ_G is called the characteristic map and the mapping δ_G is the boundary. Since G (resp. G_{ad}) is smooth, the $fppf$-cohomology of G (resp. G_{ad}) coincides with Galois cohomology [56, XXIV.8], i.e., we have a bijection $H^1(k,G) \xrightarrow{\sim} H^1_{fppf}(k,G)$. Following [41, 31.6], one defines the Tits class of G by the following formula

$$t_G = -\delta_G\big(\nu_G\big) \in H^2_{fppf}(k, Z(G)).$$

By the compatibility property[2] under the torsion bijection τ_z [34, IV.4.2]

$$\begin{array}{ccc} H^1(k,G_{ad}) & \xrightarrow{\delta_G} & H^2_{fppf}(k,Z(G)) \\ \tau_z \big\downarrow \wr & & ?+\delta_{G^q}([z]) \big\downarrow \wr \\ H^1(k,G^q_{ad}) & \xrightarrow{\delta_{G_{ad}}} & H^2_{fppf}(k,Z(G^q)), \end{array}$$

we see that $t_G = \delta_{G^q}([z])$ which is indeed Tits' definition [64, §1].

Proposition 3.2. *Assume that $H^1(k,G) = 1$.*

(1) The boundary map $H^1(k,G_{ad}) \to H^2_{fppf}(k,Z(G))$ has trivial kernel.
(2) Let G' be an inner k-form of G^q. Then G and G' are isomorphic if and only if $t_G = t_{G'}$.

Proof. (1) follows from the exact sequence (3.1). For (2), let $z' \in Z^1(k, G^q_{ad})$ be a cocycle such that $G' \cong {}_{z'}G$. We assume that $t_G = t_{G'}$. Hence $\delta_{G^q}([z]) = \delta_{G^q}([z']) \in H^2_{fppf}(k, Z(G^q))$. The compatibility above shows that

$$\tau_z^{-1}([z']) \in \ker\big(H^1(k,G_{ad})\big) \to H^2_{fppf}(k,Z(G))\big).$$

By 1), we have $\tau_z^{-1}([z']) = [1] \in H^1(k,G_{ad})$, hence $[z] = [z'] \in H^1(k,G_{ad})$. Thus G and G' are k-isomorphic. □

In conclusion, Serre's conjecture II implies that semisimple k-groups are classified by their quasi-split forms and their Tits classes. For more precise results for classical groups, see Tignol-Lewis [43]. The classification is of special importance in view of the rationality question for groups (Chernousov-Platonov [13], see also Merkurjev [47]) and then also for the Kneser-Tits problem (Gille [32]).

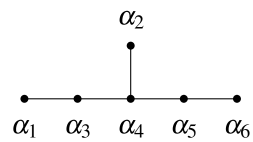
4 Approaches to the conjecture

We would like to describe a few ways to attack the conjecture, and their limitations. This is somehow artificial because in practice we work with all tools.

[2] Note that $Z(G) = Z(G^q)$ since G^q_{ad} acts trivially on $Z(G^q)$.

4.1 Subgroup trick

Let us explain it within the following example due to Tits [65]. Let G/k be the split semisimple simply connected group of type

E_6


Assume that k is infinite. Let $z \in Z^1(k_s/k, G)$ and consider the twisted group $G' = {}_zG$. Since $t_{G'} = 0$, the 27-dimensional standard representation of G of highest weight $\overline{\omega}_6$ descends to G' by [63]. We then have a representation $\rho' : G' \to \mathbf{GL}(V)$. The point is that G' has a dense orbit in the projective space $X = \mathbf{P}(V)$, so there exists a k-rational point $[x]$ in that orbit. The connected stabilizer $(G'_x)^0$ is then semisimple of type F_4 [22, 9.12]. Assuming that Conjecture 2.5 holds for groups of type F_4, it follows that $(G'_x)^0$ is split. Hence G' has relative rank ≥ 4 and a glance at Tits' tables [62] tells us that G' is split. It is then easy to conclude that $[z] = 1 \in H^1(k, G)$.

The subgroup trick (and variants) was fully investigated by Garibaldi in his Lens lectures [22]. The underlying topic is that of prehomogeneous spaces, namely projective G-varieties with a dense orbit.

Unfortunately, this trick works only in few cases. Tits has shown that the general form of type E_8 is "almost abelian" namely has no non trivial other reductive subgroups than maximal tori [65]. Together with Garibaldi, we have shown that the general trialitarian group is almost abelian [23].

4.2 Rost invariant

In this case, the idea is to derive Serre's conjecture II from a more general setting. The Rost invariant [25] generalizes the Arason invariant for 3-fold Pfister form which (in characteristic $\neq 2$) attaches to a Pfister form $\phi = \langle\langle a, b, c \rangle\rangle$ the cup-product $e_3(\phi) = (a) \cup (b) \cup (c) \in H^3(k, \mathbf{Z}/2\mathbf{Z})$. We now see it as the cohomological invariant $H^1(k, \mathbf{Spin}_8) \to H^3(k, \mathbf{Z}/2\mathbf{Z}(2))$. More generally, for G/k simply connected and absolutely almost simple, there is a cohomological invariant

$$r_k : H^1(k, G) \to H^3(k, \mathbf{Q}/\mathbf{Z}Z(2))$$

where the p-primary part has to be understood in Kato's setting [25]. If this invariant has trivial kernel, then $H^1(k, G) = 1$ for G/k satisfying the hypothesis of Conjecture 2.5. This is the case for $\mathbf{Spin}_8$ by Arason's theorem, namely the invariant $e_3(\phi)$ determines ϕ.

4.3 Serre's injectivity question

A special case of a question raised by Serre in 1962 ([53], see also [55, §2.4]), is the following.

Question 4.1. *Let G/k be a connected linear algebraic group. Let $(k_i)_{i=1,..,r}$ be a family of finite field extensions of k such that $g.c.d.([k_i : k]) = 1$. Is the kernel of the map*

$$H^1(k,G) \to \prod_{i=1,..,r} H^1(k_i,G)$$

trivial ?

Remarks.

(1) The hypothesis of connectedness is necessary since there are counterexamples with finite constant groups [35][50].
(2) The question was generalized by Totaro [66, question 0.2], see also [24].
(3) If k is of positive characteristic p, there exists a complete DVR R with residue field k and field of fractions $K = Frac(R)$ of characteristic zero, and an R-group scheme $\mathbf{G}$ with special fiber G. An answer for $\mathbf{G}_K$ to Serre's question yields an answer for G. A fortiori and without lost of generality we can assume that the extensions k_i/k are separable.

We shall rephrase the question in terms of "special fields".

Definition 4.2. Let l be a prime. We say that a field k is l-special if every finite separable extension of k is of degree a power of l.

The subfield k_l of k consisting of elements fixed by a p-Sylow subgroup of $\mathcal{G}al(k_s/k)$ is l-special. We call k_l a *co-l-closure* of k. If we restrict Serre's question to finite separable extensions k_i/k and consider all cases, it can be rephrased by asking whether the map

$$H^1(k,G) \to \prod_l H^1(k_l,G)$$

has trivial kernel for l running over all primes. If the answer to this question is yes, then Conjecture II becomes a question for l-special fields for primes l in $S(G)$.

Here are the few cases where a positive answer to Serre's question is known: unitary groups (Bayer-Lenstra [3]), groups of type G_2, quasi-split groups of type D_4, F_4, E_6, E_7 [27] [11] [21].

If we know that the Rost invariant has zero kernel, then we easily deduce that the answer to Question 4.1 is yes. Thus we can answer Serre's question for groups of type G_2, and quasi-split semisimple simply connected groups of type D_4, F_4, E_6 and E_7.

5 Known cases in terms of groups

5.1 Classical groups

Recall that a semisimple simply connected group is called classical if its factors are of type A, B, C or D, and there is no triality involved.

Theorem 5.1. *Let G be an absolutely almost simple and simply connected classical group over a field k as in Conjecture 2.5. Then* $H^1(k,G) = 1$.

If k is perfect or $\mathrm{char}(k) \neq 2$, this is the original Serre's conjecture II proven by Bayer-Parimala [4]. The general case is a recent work by Berhuy-Frings-Tignol [5]. Its proof is based on Weil's presentation of classical groups in terms of unitary groups of algebras with involutions [67]. This proof is characteristic free; in particular, it provides a quite different proof of Bayer-Parimala's theorem.

Possibly the trickiest case is that of outer groups of type A, namely unitary groups of central simple algebras equipped with an involution of the second kind. The proof in the number field case (which uses Landherr's Theorem) is already difficult, see [40, §5.5].

5.2 Quasi-split exceptional groups

For such groups, the best approach is by investigating the Rost invariant.

Theorem 5.2. *Let G/k be a quasi-split semisimple simply connected group of Cartan-Killing type G_2, F_4, D_4, E_6 or E_7. Then the Rost invariant* $H^1(k,G) \to H^3(k,\mathbf{Q}/\mathbf{Z}(2))$ *has trivial kernel.*

Here the field k is arbitrary, but proving the theorem boils down to the characteristic zero case by a lifting argument [28]. For the cases G_2, F_4, see [4] or [55]. As pointed out by Garibaldi, the D_4 case is done in *The Book of Involutions* but not stated in that shape. We need to know that a trialitarian algebra whose underlying algebra is split arises as the endomorphism of a twisted composition [41, 44.16] and to use results on degree 3 invariants of twisted compositions (ibid., 40.16). For type E_6 and E_7, this is due independently to Chernousov [11] and Garibaldi [21].

Thus Conjecture 2.5 holds for quasi-split groups of all types except E_8; we have an independent proof which is quite different since it is based on Bruhat-Tits theory [29]. For the split group of type E_8 denoted by E_8, the Rost invariant in general has a nontrivial kernel [31, appendix]. In characteristic 0, Semenov recently constructed a higher invariant

$$\ker\big[H^1(k,E_8) \to H^3(k,\mathbf{Q}/\mathbf{Z}(2))\big] \to H^5(k,\mathbf{Z}/2\mathbf{Z})$$

which is nontrivial since it is so already over the reals [58, §8]. Semenov's invariant has trivial kernel for 2-special fields.

By means of norm group of varieties of Borel subgroups, the case of quasi-split groups is the input for proving the following.

Theorem 5.3. *[29, Th. 6] Let G/k be a semisimple simply connected group which satisfies the hypothesis of Conjecture 2.5. Let $\mu \subset G$ be a finite central subgroup of G. Then the characteristic map*

$$(G/\mu)(k) \to H^1_{fppf}(k,\mu)$$

is surjective.

Flat cohomology (see [57], [5, app. B] or [34]) is the right object if the order of μ is not invertible in k, it coincides with Galois cohomology if this order is invertible. By continuing the exact sequence of pointed sets

$$1 \to \mu(k) \to G(k) \to (G/\mu)(k) \to H^1_{fppf}(k,\mu) \to H^1_{fppf}(k,G),$$

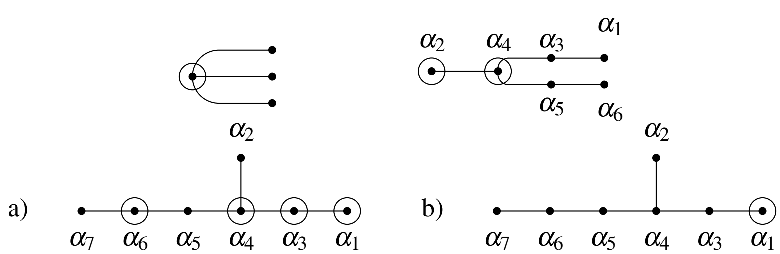
we see that $H^1_{fppf}(k,\mu) \to H^1_{fppf}(k,G)$ is the trivial map. In other words, the center of G does not contribute to $H^1(k,G)$. [The reason why we can avoid the type E_8 is that such groups have trivial center.]

5.3 Other exceptional groups

Theorem 5.4. *[11] [29] Let G/k be a semisimple group satisfying the hypothesis of Conjecture 2.5. Then $H^1(k,G) = 1$ in the following cases:*

1. *G is trialitarian and its Allen algebra is of index ≤ 2.*
2. *G is of quasi-split type ${}^1E_6$ or ${}^2E_6$ and its Tits algebra is of index ≤ 3.*
3. *G is of type E_7 and its Tits algebra is of index ≤ 4.*

Furthermore those groups are quasi-split or isotropic respectively of Tits indexes



where case *a*) (resp. *b*)) is that of Tits algebra of index 2 (resp. 4). One more reason why other exceptional groups are not easy to work with is because they are anisotropic.

Corollary 5.5. *[15] Let G/k as in Theorem 5.4. For every separable finite field extension L/k, assume that every central simple L-algebras of period* 2 *(resp.* 3*) is of index* ≤ 2 *(resp.* 3*). If G is trialitarian or of type E_7 (resp. E_6), then* $H^1(k,G) = 1$.

This is the case for function fields of surfaces as pointed out by Artin [1], thus Corollary 5.5 holds for these fields. In the paper [15] with Colliot-Thélène and Parimala, we exploited Serre's conjecture II for the study of arithmetic properties in this framework by proceeding with analogies with Sansuc's paper [52] in the number field case. On this topic, see also the paper by Borovoi and Kunyavskiĭ [7].

6 Known cases in terms of fields

6.1 *l*-special fields

(a) If $l = 2,3,5$ and k is an l-special field of separable dimension ≤ 2, Conjecture 2.5 holds for the split group of type E_8, see [10] for $l = 5$ and [29, §III.2].

(b) If $l = 3$ and k is an l-special field of characteristic $\neq 2$ and separable dimension ≤ 2, then Conjecture 2.5 holds for trialitarian groups. For $l = 3$, this follows from Theorem 5.2.

In both cases, a positive answer to Serre's injectivity question would provide Conjecture 2.5 for those groups.

6.2 *Complete valued fields*

Let K be a henselian valued field for a discrete valuation with perfect residue field κ. A consequence of the Bruhat-Tits decomposition for Galois cohomology over complete fields is the following.

Theorem 6.1. *(Bruhat-Tits [8, cor. 3.15]) Assume that κ is of cohomological dimension ≤ 1. Let G/K be a simply connected semisimple group. Then* $H^1(K,G) = 1$.

Note that the hypotheses imply that K is of separable dimension ≤ 2. Serre asked whether it can be generalized when assuming $[\kappa : \kappa^p] \leq p$ [55, 5.1]. The hypothesis $[\kappa : \kappa^p] \leq p$ alone is not enough here because $K = \mathbf{F}_p((x))((y))$ is of separable dimension 3 and is complete with residue field $\mathbf{F}_p((x))$.

But if κ is separably closed and $[\kappa : \kappa^p] \leq p$, then K is of separable dimension 1 and enough cases of the vanishing of $H^1(\kappa((x)),G)$ have been established in view of the proof of Tits conjectures on unipotent subgroups [30]. The general case is still open.

Note also that the conjecture is proven for fraction fields of henselian two dimensional local rings (e.g. $\mathbf{C}[[x,y]]$) with algebraically closed residue field of characteristic zero [15]. For the E_8 case, a key point is that the derived group of the absolute Galois group is of cohomological dimension 1 [17, Th. 2.2].

6.3 Global fields

The number field case is due to Kneser for classical groups [40], Harder for exceptional groups except the type E_8 [36, I, II], and Chernousov for the type E_8 [9], see [51]. The function field case is due to Harder [36, III].

6.4 Function fields

He, de Jong and Starr have proven Conjecture 2.5 for split groups over function fields in a uniform way and in arbitrary characteristic.

Theorem 6.2. *[38, cor. 1.5] Let k be an algebraically closed field and let K be the function field of a quasi-projective smooth surface S. Let G be a split semisimple simply connected group over k. Then $H^1(K,G)=1$.*

For cases other than E_8, the conjecture had been establised by case by case considerations [15]. Hence Conjecture 2.5 is fully proven for function fields of surfaces. The proof of Theorem 6.2 is based on the existence of sections for fibrations in rationally simply connected varieties.

Theorem 6.3. *[38, Th. 1.4] Let S/k as in Theorem 6.2. Let X/S be a projective morphism whose geometric generic fiber is a twisted flag variety. Assume that $\mathrm{Pic}(X) \to \mathrm{Pic}(X \times_K \overline{K})$ is surjective. Then $X \to S$ has a rational section.*

The assumption on the Picard group means that there is no "Brauer obstruction". By application to higher Severi-Brauer schemes, this statement yields as corollary de Jong's theorem "period=index" [37] for central simple algebras over such fields; see also [14].

It is the first classification-free item in this survey.

6.5 Why Theorem 6.3 implies Theorem 6.2

We take the opportunity to reproduce here our argument.

Lemma 6.4. *Let G/F be a semisimple simply connected group over a field F. Let E/F be a G-torsor.*

(1) $\mathrm{Pic}(E) = 0$ *and we have an exact sequence*

$$0 \to \mathrm{Br}(F) \to \mathrm{Br}(E) \to \mathrm{Br}(E \times_F F_s).$$

(2) Let P be an F-parabolic subgroup of G and let E/P be the variety of parabolic subgroups of the twisted F-group $E(G)$ of the same type as P. Then we have an exact sequence

$$0 \to \mathrm{Br}(F) \to \mathrm{Br}(E/P) \to \mathrm{Br}(E/P \times_F F_s)$$

and an isomorphism $\mathrm{Pic}(E/P) \xrightarrow{\sim} \mathrm{Pic}(E/P \times_F F_s)^{\mathscr{G}al(F_s/F)}$.

Proof. (1): We have $H^1(F,(F_s)^\times) = 0$ and $\mathrm{Pic}(E \times_F F_s) \cong \mathrm{Pic}(G \times_F F_s) = 0$ since G is simply connected [20]. The first terms of the Hochschild-Serre spectral sequence $H^p(\mathscr{G}al(F_s/F), H^q(E \times_F F_s, \mathbf{G}_m)) \Longrightarrow H^{p+q}(E, \mathbf{G}_m)$ show that $\mathrm{Pic}(E) = 0$ and that the sequence $0 \to \mathrm{Br}(F) \to \mathrm{Br}(E) \to \mathrm{Br}(E \times_F F_s)$ is exact.

(2): The morphism $E \to E/P$ (i.e., the twist of $G \to G/P$ by the torsor E) gives rise to a map $\mathrm{Br}(E/P) \to \mathrm{Br}(E)$. We claim that this map is injective. Since the generic fiber of $E \to E/P$ is isomorphic to $P \times_F F(E/P)$, the map $\mathrm{Br}(F(E/P)) \to \mathrm{Br}(F(E))$ is an injection by the specialization trick [33, lem. 5.4.6]. But $\mathrm{Br}(E/P)$ injects in $\mathrm{Br}(F(E/P))$, hence the claim. We look at the commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \mathrm{Br}(E/P) & \longrightarrow & \mathrm{Br}(E/P \times_F F_s) \\
 & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \mathrm{Br}(E) & \longrightarrow & \mathrm{Br}(E \times_F F_s).
\end{array}$$

Since the bottom sequence is exact, we get by diagram chasing that the upper horizontal sequence is exact as well. The second isomorphism $\mathrm{Pic}(E/P) \xrightarrow{\sim} \mathrm{Pic}(E/P \times_F F_s)^{\mathscr{G}al(F_s/F)}$ comes from the Hochschild-Serre spectral sequence. □

For complete results on Picard and Brauer groups of twisted flag varieties, see Merkurjev-Tignol [49, §2].

Proposition 6.5. *[38, Th. 1.4] Let S/k be as in Theorem 6.2. Let G/K be a semisimple simply connected K-group which is an inner form and let P be a K-parabolic subgroup of G. Then the map $H^1(K,P) \to H^1(K,G)$ is bijective.*

Proposition 6.5 implies Theorem 6.2 by taking a Borel subgroup of G because $H^1(K,B) = 1$.

Proof. Injectivity is a general fact due to Borel-Tits ([6], théorème 4.13.a). Let E/K be a G-torsor of class $[E] \in H^1(K,G)$. After shrinking S, we can assume that G/K extends to a semisimple group scheme $\mathbf{G}/S$, P/K extends to an S-parabolic subgroup scheme $\mathbf{P}/S$ and that E/K extends to a $\mathbf{G}$-torsor $\mathbf{E}/S$ [44]. By étale descent, we can twist the S-group scheme $\mathbf{G}/S$ by inner automorphisms, namely define the S-group scheme $\mathbf{E}(\mathbf{G})/S$. We then define $\mathbf{V}/S := \mathbf{E}/\mathbf{P}$, i.e., the scheme of parabolic subgroup schemes of $\mathbf{E}(\mathbf{G})/S$ ([56], exp. XXVI) of the same type as P. The morphism $\pi\colon \mathbf{V} \to X$ is projective, smooth and has geometrically integral fibers. Set $V = \mathbf{V} \times_S K$; this is a generalized twisted flag variety. Since G is assumed to be an inner form, $\mathrm{Pic}(V \times_K K_s)$ is a trivial $\mathcal{G}al(K_s/K)$-module. By Lemma 6.4.2, the map

$$\mathrm{Pic}(V) \to \mathrm{Pic}(V \times_K K_s)$$

is onto. Thus the composite map $\mathrm{Pic}(\mathbf{V}) \to \mathrm{Pic}(V) \to \mathrm{Pic}(V \times_K K_s)$ is onto. Theorem 6.2 applies and shows that $V(K) \neq \emptyset$. Thus the torsor E admits a reduction to P ([54], §I.5, proposition 37), that is $[E] \in \mathrm{im}\big(H^1(K,P) \to H^1(K,G)\big)$. We conclude that the mapping $H^1(K,P) \to H^1(K,G)$ is surjective. □

The Grothendieck-Serre conjecture on rationally trivial torsors was proven by Colliot-Thélène and Ojanguren for torsors over a semisimple group defined over an algebraic closed field [16]. Thus He-de Jong-Starr's theorem has the following geometric application.

Corollary 6.6. *Let S/k be a smooth quasi-projective surface. Let G/k be a (split) semisimple simply connected group. Let E/S be a G-torsor. Then E is locally trivial for the Zariski topology.*

7 Remaining cases and open questions

Here are some of the remaining cases and open questions.

• Provide a classification free proof for the case of totally imaginary number fields, at least in the quasi-split case.

• The first remaining cases of Conjecture 2.5 are those of trialitarian groups, groups of type E_6 over a 3-special field, groups of type E_7 over a 2-special field and groups of type E_8.

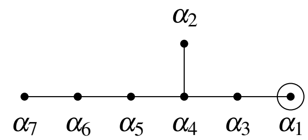
• What about higher mod 3 cohomological invariants of E_8?

• Let K be the function field of a surface over an algebraically closed field. Are K-division algebras cyclic? Is it true that $\mathrm{cd}(K_{ab}) = 1$ where K_{ab} stands for the abelian closure of K?

In the global field case, class field theory answers both questions positively. This question on K_{ab} is due to Bogomolov and makes sense for arbitrary fields. As noticed by Chernousov, Reichstein and the reviewer, a positive answer would provide a positive answer to Serre's conjecture II for groups of type E_8 [12].

• For the Kneser-Tits conjecture for perfect fields of cohomological dimension ≤ 2, there remains only the case of a group with the following Tits index, see [32, §8.2]:

$E_{7,1}^{66}$



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