Scientific report on the implementation of the project in the period November 2013-December 2014

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In the period November-December 2013, we started the implementation of the project dealing with the three planned research directions:

(1) Arithmetic homogeneous spaces;

(2) R-equivalence on homogeneous spaces;

(3) Application of torsors to infinite dimensional Lie theory.

Documentation has been our main activity. In concrete terms, we started to study several recent papers which are directly related to the first topic. The two members of the project met twice a week as planned.

During the period January 2014-December 2014, the implementation of our project (phase II) continuated successfully. On the three research themes, the activity has been intense and fruitful. All objectives were reached and are presented now.

Obtained results

Theme 1: Arithmetic homogeneous spaces : In the framework of this main theme of the project, we mention the publication *Sur la classification des schémas en groupes semi-simples* ([2] in the references below). This study provides an extension of certain classical results by Demazure-Grothendieck on reductive group schemes and projective homogeneous spaces G/P on a general base scheme. The most significant result states that parabolic subgroups of a given reductive group scheme G over a ring occur as limit subgroups for cocharacters (theorem 7.3). This fact is well known for algebraic groups, less known over a local ring. It is remarkable that it globalizes and has applications for example in the third theme.

On the other hand, the preprint [4] investigates maximal tori of semi-simple algebraic groups of type G_2 and associated homogeneous spaces. It

The web site of the project http ://math.univ-lyon1.fr/homes-www/gille/idei_pg.html provides an abstract of activities, link to papers and details on all talks and conferences given.

has been written by the members of the project together with T.Y. Lee (EPFL, Lausanne). Unexpectedly that study gives rise to the construction of such a homogeneous space which a quadratic point and a cubic point but has no rational point. It is the simplest example of homogeneous space satisfying that property.

Note that it cannot occur over a number field ; in that case, we have shown that relevant homogeneous spaces satisfy the Hasse principle.

Theme 2: R-equivalence on homogeneous spaces : This theme is on preparation [5]. It deals with homogeneous spaces over a field of cohomological dimension ≤ 2 or a number field. The first step is the case of principal homogeneous spaces which is obviously related to Galois cohomology. This part advanced well and the methods are quite different compared with the existing literature.

Theme 3: Applications of torsors to infinite dimensional Lie theory : This continues a fruitful collaboration with V. Chernousov and A. Pianzola (Edmonton, Canada).

In 2013, in the third research direction, we obtained results on the generation of certain groups which are important in the Kac-Moody theory. Those are related to Lie algebras over Laurent polynomials. Precisely this is the paper [1] written with V. Cher--nousov and A. Pianzola. The methods are quite clasical and based on a precise study of a group action over a suitable Bruhat-Tits building. This limits the investigations to the nullity one case but we have some hope for the general case which looks much more complicated.

The publication [3] presents a generalization of the Onsager algebra $\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathbb{C}\big[t^{\pm 1}, \frac{1}{t-1}\big]$. It is a kind of Kac-Moody algebra where the ring of Laurent polynomials is replaced by the ring of regular algebraic functions over the Riemann sphere minus three points. The Grothendieck's theory of "Dessins d'enfants" (children drawings) describes the (finite) covers of the Riemann sphere minus three points and contributes to the classification of those Lie algebras. In the case of trialiarian type D_4, a very special algebra occurs: that algebra admits an infinite family of maximal diagonalizable subalgebras (MAD's) which are not pairwise conjugated.

Scientific production of the members of the team in the project

Publications

[1] V. Chernousov, P. Gille and A. Pianzola, *Whitehead groups of loop group schemes of nullity one*, Journal of the Ramanujan Mathematical Society 29 (2014), 1-26.

[2] P. Gille, Sur la classification des schémas en groupes semi-simples, to appear in "Autour des Schémas en groupes, III", Panoramas et Synthèses, SMF, 81 pages.

[3] V. Chernousov, P. Gille and A. Pianzola, *Three-point Lie algebras and Grothendieck's dessins d'enfants*, to appear in Mathematical Research Letters, 16 pages.

Preprints

[4] N. Beli, P. Gille and T.-Y. Lee, *Maximal tori of algebraic groups of type* G_2, submitted in November 2014, 28 pages.

Book in construction

[5] P. Gille, *Groupes algébriques semi-simples sur un corps de dimension cohomologique séparable* ≤ 2, three chapters.

Academical exchanges

The financial opening of the project for 2014 has been made only in the end of October and therefore the majority of planned exchanges for the mini-conference of *Strong approximation on algebraic groups* have been cancelled.

The only invited person has been Rony Bitan from Bar-Ilan University (Tel Aviv, Israël) for one week in April. He gave a talk in the number theory seminar. We collaborated together on the main theme of the project namely arithmetic homogeneous spaces.

On the other hand, in March 2014, P. Gille has been invited in Montréal (CRM) in the thematic semester *New directions in Lie theory*. He gave a specialized talk on the third theme and started a collaboration with E. Neher (Ottawa) about arithmetic homogeneous spaces for group schemes of type G_2. It has some connections with the paper [4].

As planned in the project, P. Gille visited Gopal Prasad (Ann Arbor, University Michigan) with the scope to work also with B. Conrad on Gabber's compactifications of algebraic groups and homogeneous spaces.

In December 2014, P. Gille has been invited in Edmonton (Canada) for continuing the collaboration with V. Chernousov and A. Pianzola on infinite dimensional Lie theory related with group schemes over Laurent polynomials.