

Scientific report on the implementation of the project in the period November 2013-October 2015

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In the period November-December 2013, we started the implementation of the project dealing with the three planned research directions :

- (1) Arithmetic homogeneous spaces ;
- (2) R-equivalence on homogeneous spaces ;
- (3) Application of torsors to infinite dimensional Lie theory.

Documentation has been our main activity. In concrete terms, we started to study several recent papers which are directly related to the first topic. The two members of the project met twice a week as planned.

During the period January 2014-December 2014, the implementation of our project (phase II) continued successfully. On the three research themes, the activity has been intense and fruitful. All objectives were reached and are presented now.

In 2015, the four main new results are the second paper [5] on maximal tori of groups of type G_2 , the preprint [6] on the common slot lemma, the preprint on the classification of torsors over Laurent polynomials [7] and the achievement of the book [8].

Obtained results

Theme 1: Arithmetic homogeneous spaces : In the framework of this main theme of the project, we mention the publication *Sur la classification des schémas en groupes semi-simples* ([2] in the references below). This study provides an extension of certain classical results by Demazure-Grothendieck on reductive group schemes and projective homogeneous spaces G/P on a general base scheme. The most significant result states that parabolic subgroups of a given reductive group scheme G over a ring occur as limit subgroups for cocharacters (theorem 7.3). This fact is well known for algebraic

The web site of the project http://math.univ-lyon1.fr/homes-www/gille/idei_pg.html provides an abstract of activities, link to papers and details on all talks and conferences given.

groups, less known over a local ring. It is remarkable that it globalizes and has applications for example in the third theme.

On the other hand, the preprint [4] (accepted for publication in 2015) investigates maximal tori of semisimple algebraic groups of type G_2 and associated homogeneous spaces. It has been written by the members of the project together with T.Y. Lee (EPFL, Lausanne). Unexpectedly that study gives rise to the construction of such a homogeneous space which a quadratic point and a cubic point but has no rational point. It is the simplest example of homogeneous space satisfying that property.

Note that it cannot occur over a number field; in that case, we have shown that relevant homogeneous spaces satisfy the Hasse principle.

The new paper [5] went further. Its aim is the construction of examples of non-isomorphic algebraic groups of type G_2 having same families of maximal tori. This answers a question raised by Andrei Rapinchuk. Finally, the preprint [6] is a complement on the paper [4] on the special case of characteristic two.

Theme 2: R-equivalence on homogeneous spaces : This theme is the book on preparation [8]. It deals with homogeneous spaces over a field of cohomological dimension ≤ 2 or a number field. The first step is the case of principal homogeneous spaces which is obviously related to Galois cohomology. This part advanced well and the methods are quite different compared with the existing literature. In 2015, our efforts became fruitful. New cases of Serre's conjecture II in Galois cohomology were established. The main outcome is the case of exceptional groups of type E_7 which is fully settled. Given a field k of cohomological dimension 2 and a semisimple simply connected algebraic group G of type E_7 , we have that $H^1(k, G) = 1$, that is each G -torsor is trivial. The book is submitted for publication.

Theme 3: Applications of torsors to infinite dimensional Lie theory : This continues a fruitful collaboration with V. Chernousov and A. Pianzola (Edmonton, Canada).

In 2013, in the third research direction, we obtained results on the generation of certain groups which are important in the Kac-Moody theory. Those are related to Lie algebras over Laurent polynomials. Precisely this is the paper [1] written with V. Chernousov and A. Pianzola. The methods are quite classical and based on a precise study of a group action over a suitable Bruhat-Tits building. This limits the investigations to the nullity one case

but we have some hope for the general case which looks much more complicated.

The publication [3] presents a generalization of the Onsager algebra $\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}, \frac{1}{t-1}]$. It is a kind of Kac-Moody algebra where the ring of Laurent polynomials is replaced by the ring of regular algebraic functions over the Riemann sphere minus three points. The Grothendieck's theory of "Dessins d'enfants" (children drawings) describes the (finite) covers of the Riemann sphere minus three points and contributes to the classification of those Lie algebras. In the case of triality type D_4 , a very special algebra occurs: that algebra admits an infinite family of maximal diagonalizable subalgebras (MAD's) which are not pairwise conjugated.

As planned, that theme was pursued in 2015. Given an algebraic group G defined over a field k , the preprint [7] presents a classification of G -torsors over Laurent polynomials in terms of nice G -torsors (called toral or loop) and unramified Galois cohomology. This applies to the classification of twisted toroidal Lie algebras. The methods are mainly of algebraic geometry.

Scientific production of the members of the team in the project

Publications

- [1] V. Chernousov, P. Gille and A. Pianzola, *Whitehead groups of loop group schemes of nullity one*, Journal of the Ramanujan Mathematical Society 29 (2014), 1-26.
- [2] P. Gille, *Sur la classification des schémas en groupes semi-simples*, to appear in "Autour des Schémas en groupes, III", Panoramas et Synthèses, SMF, 81 pages.
- [3] V. Chernousov, P. Gille and A. Pianzola, *Three-point Lie algebras and Grothendieck's dessins d'enfants*, to appear in Mathematical Research Letters, 16 pages.
- [4] C. Beli, P. Gille and T.-Y. Lee, *Maximal tori of algebraic groups of type G_2* , to appear in Pacific Journal of Mathematics (special issue in the memory of Robert Steinberg), 28 pages.
- [5] C. Beli, P. Gille and T.-Y. Lee, *Examples of algebraic groups of type G_2 having the same maximal tori*, arXiv:1503.07046, to appear in Proceedings of the Steklov Institute of Mathematics (special issue for Vladimir Platonov 75-th birthday), 13 pages.

Preprints

- [6] C. Beli, *Common slot lemmas in characteristic 2*, October 2015, arXiv :1510.08330, submitted.
- [7] V. Chernousov, P. Gille and A. Pianzola, *A classification of torsors over Laurent polynomial rings*, October 2015, hal-01217408, 18 pages, submitted.

Book

- [8] P. Gille, *Groupes algébriques semi-simples sur un corps de dimension cohomologique séparable ≤ 2* , 108 pages, submitted to Lectures Notes of Mathematics. Below is the summary.

1. Generalities
2. Reductive groups
3. Subgroups of algebraic groups, splitting
4. Separable cohomological dimension
5. Algebraic tori, Conjecture I and normgroups
6. Conjecture II, the quasi-split case
7. Classical groups
8. Exceptional groups
9. Applications

Academical exchanges

The financial opening of the project for 2014 has been made only in the end of October and therefore the majority of planned exchanges for the mini-conference of *Strong approximation on algebraic groups* have been cancelled.

The only invited person has been Rony Bitan from Bar-Ilan University (Tel Aviv, Israël) for one week in April. He gave a talk in the number theory seminar. We collaborated together on the main theme of the project namely arithmetic homogeneous spaces.

On the other hand, in March 2014, P. Gille has been invited to Montréal (CRM) in the thematic semester *New directions in Lie theory*. He gave a specialized talk on the third theme and started a collaboration with E. Neher (Ottawa) about arithmetic homogeneous spaces for group schemes of type G_2 . It has some connections with the paper [4].

As planned in the project, P. Gille visited Gopal Prasad (Ann Arbor, University of Michigan) with the scope to work also with B. Conrad on Gabber's compactifications of algebraic groups and homogeneous spaces.

In December 2014, P. Gille has been invited to Edmonton (Canada) for continuing the collaboration with V. Chernousov and A. Pianzola on infinite dimensional Lie theory related with group schemes over Laurent polynomials.

A similar visit was done in February 2015. Also P. Gille was invited to the University of Ottawa in March 2015 as distinguished visiting researcher and visited again Gopal Prasad (Ann Arbor, University Michigan). Three collaborators came to the Simion Stoilow Institute in 2015 for short visits : Mathieu Florence (Paris), Laurent Moret-Bailly (Rennes) and Tamás Szamuely (Budapest). They gave nice talks and we have very interesting discussions.

P. Gille participated to several conferences abroad and both members presented their own results to the congress ICTAMI of Alba Iulia (September 2015).