TORSORS OVER LOCAL FIELDS

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Abstract

Let G be an affine algebraic k-group defined over a local field of characteristic zero. Borel and Serre [BS] have shown in 1964 that there are finitely many isomorphism classes of G-torsors. Also if $f : X \to Y$ is a G-torsor, then the image of the map $X(k) \to Y(k)$ is locally closed. The starting point of the lecture is the investigation of the same issue for local fields of positive characteristic. It turns out that the two preceding results are false. The main result (obtained with O. Gabber and L. Moret-Bailly in 2014) will be that the image of the map $X(k) \to Y(k)$ is locally closed [G-G-MB]. It has consequences of the topology of the set of isomorphism classes of G-torsors. Our setting is actually wider, it involves Henselian valued fields and algebraic spaces.

The goal of the lecture is to cover the proof of the above statement. On the way, we shall revisit actions of algebraic groups on homogeneous spaces, Galois cohomology, topological properties for algebraic varieties defined over a local field, and Gabber's compactifications of algebraic groups.

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1. Lecture 1: overview

Our purpose is to discuss topological properties of group actions arising from algebraic shape. We start with topological groups.

1.1. **Topological groups.** Let $f : G \to H$ be a continuous homomorphism between two Hausdorff topological groups. The kernel $N = \mathrm{Ker}(f)$ is a closed subgroup of G and we get a continuous homomorphism $\underline{f} : G/N \to H$ where G/N is equipped with the quotient topology. On the other hand, the image $I = \mathrm{Im}(f)$ is a subgroup of the topological group H. We get then a continuous group isomorphism $f^\flat : G/N \to I$ between two Hausdorff topological groups. The natural question to address is the following : is $f^\flat$ a homeomorphism?

When it is the case, we say that f is *strict* [B:TG1, §III.8]. We have to pay attention that this notion does not behave well by composition. In presence of compactness assumptions, we can conclude easily. More precisely, when G/N is compact (e.g. if G is compact), then f is strict[1].

1.1.1. **Example.** Let $a \in \mathbb{R}^\times$ and consider the homomorphism

$$f_a : \mathbb{R} \to (\mathbb{R}/\mathbb{Z})^2,\, t \mapsto \big([t], [at]\big).$$

We consider its kernel $N_a = \mathbb{Z} \cap a^{-1}\mathbb{Z}$. If $a = p/q \in \mathbb{Q}$ (p, q coprime), we have $N_a = p\mathbb{Z} \subset \mathbb{R}$ (with the discrete topology) so that $\underline{f}_a : \mathbb{R}/\mathbb{Z} \cong \mathbb{R}/p\mathbb{Z} \to (\mathbb{R}/\mathbb{Z})^2$, $t' \mapsto (pt', qt')$. By using a Bezout relation $\alpha p + \beta q = 1$ with $\alpha, \beta \in \mathbb{Z}$, we see that $f_a^\flat$ is essentially the identity of $\mathbb{R}/\mathbb{Z}$, so is strict.

If $a \not\in \mathbb{Q}$, we have $N_a = 0$ and we claim that f_a is not strict. We know that $\mathbb{Z} + a^{-1}\mathbb{Z}$ is dense in $\mathbb{R}$, so that $a^{-1}\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}$. If follows that $\mathrm{Im}(f_a)$ is dense in $(\mathbb{R}/\mathbb{Z})^2$. There exists then a sequence r_n of real numbers going to ∞ such that $f_a(r_n)$ converges to $[(0,0)]$. Thus f_a is not strict.

There is an important case when we can conclude that f is strict.

1.1.2. **Lemma.** *Assume that I is closed in G, that G is σ-locally compact and that H is locally compact. Then f is open on its image and is strict.*

This is a corollary of a slightly general statement, see [B:TG2, §IX.3, prop. 6]. Note that the conditions on G, H are satisfied for example for Lie groups.

Proof. Our assumption means that G is locally compact and that $G = \bigcup_{n \geq 0} U_n$ where U_n is a sequence of open relatively compact subsets of G such that $\overline{U_n} \subset U_{n+1}$.

Let us prove the openess fact. Without loss of generality, we can assume that $I = H$ and we shall use that H is a Baire space. It is enough to show that $f(V)$ is open in H for all neighborhoods V of 1. Let W be a symmetric compact neighborhood of $1 \in G$ such that $W.W \subset V$.

[1]Observe that the map $G/N \xrightarrow{\sim} I$ is closed hence open.

Since each $\overline{U_n}$ is covered by finitely many translates of $\overset{\circ}{W}$, it follows that $G = \bigcup_{n\geq 0} g_n \overset{\circ}{W} = \bigcup_{n\geq 0} g_n W$ for a sequence (g_n) of G. We have $H = f(G) = \bigcup_{n\geq 0} f(g_n) f(W)$. Since W is compact, $f(W)$ is compact as well and a fortiori closed in H. Baire's property for H shows that $f(\overset{\circ}{W}) \neq \emptyset$. We pick a point $f(w) \in f(\overset{\circ}{W})$. Thus $f(w^{-1}W)$ is a neighborhood of 1 in H and a fortiori, $f(V)$ is a neighborhood of 1 in H.

We come back to the general setting. Since f is open on its image, so is $\underline{f} : G/N \to H$. The induced continuous bijection $G/N \xrightarrow{\sim} I$ is then open. Thus $G/N \to I$ is a homeomorphism and f is strict. □

1.1.3. **Remarks.** (a) The last step is independent of compactness assumptions.

(b) A stronger property than open on the image is that $f : G \to I$ has locally continuous sections. This is the viewpoint of principal fibrations.

1.2. Real Algebraic Groups.

We come to the following statement due to Borel and Harish-Chandra [B-HC, prop. 2.3].

1.2.1. **Proposition.** *Let G, H be real affine algebraic groups. Let $f : G \to H$ be a homomorphism of real algebraic groups. Then $f(G(\mathbb{R})_{\mathrm{Top}})$ is a closed subgroup of $H(\mathbb{R})_{\mathrm{Top}}$ and $f_{\mathrm{Top}} : G(\mathbb{R})_{\mathrm{Top}} \to H(\mathbb{R})_{\mathrm{Top}}$ is strict.*

It means that G is given as a subgroup of some GL_N by algebraic equations; basic examples are orthogonal groups, groups of triangular matrices,... The notation $G(\mathbb{R})_{\mathrm{Top}}$ stands for the underlying topological space[2] seen as subspace of $M_N(\mathbb{R})$.

There is no contradiction between this statement and Example 1.1.1 above. We have $\mathbb{R}/\mathbb{Z} \cong S^1 = \{x^2 + y^2 = 1\}$ so the groups are of algebraic shape but not the morphism f_a. More precisely the exponential $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ has kernel $\mathbb{Z}$ which cannot be defined in $\mathbb{R}$ by algebraic equations.

Sketch of proof. Up to replace H by the Zariski closure of $f(G(\mathbb{C}))$ we can assume that $f : G \to H$ is dominant as morphism of algebraic varieties. Since algebraic groups in characteristic zero are smooth, Chevalley's generic flatness theorem [DG, I.3.3.7] provides a dense open subset V of H such that $f_{|f^{-1}(V)} : f^{-1}(V) \to V$ is faithfully flat. On the other hand, the geometric fibers are smooth of f are smooth so that $f_{|f^{-1}(V)} : f^{-1}(V) \to V$ is smooth by using the fiberwise smoothness criterion [EGA4, 17.8.2]. Then the map $\big(f^{-1}(V)\big)(\mathbb{R})_{\mathrm{Top}} \to V(\mathbb{R})_{\mathrm{Top}}$ is smooth in the differential geometry sense and is open. Thus f_{Top} is open and is strict. Since an open subgroup of a topological group is clopen, we conclude $f(G(\mathbb{R}))_{\mathrm{Top}}$ is closed in $H(\mathbb{R})_{\mathrm{Top}}$.

The original proof goes by the analytic viewpoint; for an extended version, see [Se2, Part II, ch. IV].

[2] We should check later that this topology is intrinsecal.

1.3. **The local non-archimedean field case.** Such a field is a finite extension k of a p-adic field $\mathbb{Q}_p$ or is isomorphic to $F((t))$ where F is a finite field and the topology is induced by the t-adic valuation. The topology of the field induces a locally compact topology on $G(k)$ for G an affine algebraic k-group.

The characteristic zero case is quite similar with the real case so that an algebraic morphism $f : G \to H$ induces a strict morphism $f_{\mathrm{Top}} : G(k)_{\mathrm{Top}} \to H(k)_{\mathrm{Top}}$ having closed image.

In the positive characteristic case, it is much more complicated to prove that $f(G(k))$ is closed in $H(k)_{\mathrm{Top}}$, so that f_{Top} is strict. This has been established by Bernstein-Zelevinskiĭ [B-Z] and we provided an extension of that result.

1.3.1. **Theorem.** [G-G-MB, 1.2, 1.4] *Let K be a henselian valued field such that its completion $\widehat{K}$ is separable over K. Let $f : G \to H$ be a morphism of algebraic K-groups. Then $f(G(K))$ is closed in $H(K)_{\mathrm{Top}}$ and f_{Top} is strict.*

It applies to a much wider class of fields than local fields; it holds in particular for fields like $E((t))$ where E is an arbitrary field and also the separable closure of $E(t)$ in $E((t))$. We cannot use compactness techniques beyond the locally compact case, so other techniques are used and will be presented in this lecture.

1.4. **Group actions.** Group homomorphisms are a special case of group actions. Let G be an affine k-group acting on an affine k-variety X. Let $x \in X(k)$ and consider the orbit map $f : G \to X$, $g \mapsto g.x$. If the orbit of x is closed, Bremigan [Bn, prop. 5.3] has shown that the topological orbit $\big(G(k).x\big)_{\mathrm{Top}}$ is closed for k local field of characteristic zero (so that f_{Top} is strict).

This is not true in the positive characteristic case. We take $k = \mathbb{F}_p((t))$ and consider the action of the group of "affine transformations" $G = \mathbb{G}_a \rtimes_k \mathbb{G}_m$ on the affine line $\mathbf{A}^1_k$ by

$$(a,b).x = a^p + b^p\, x.$$

We consider the orbit map $f_t : G \to \mathbf{A}^1_k$ for the point $t \in k$. We have

$$I = G(k).t = \Big\{ a^p + b^p\, t \ \mid \ a \in k, b \in k^\times \Big\}.$$

Since $t \in k \setminus k^p$, 0 does not belong to $G(k).t$. On the other hand the sequence $t^{-pn}.t$ converges to zero so that $0 \in \overline{G(k).t}$. Thus $f_t(G(k))$ is not closed in k. More precisely we have $\overline{I} = k^p \oplus k^p t$ so that I is open in its closure. In other words, I is locally closed.

The map $(f_t)_{\mathrm{Top}}$ is injective and induces a homeomorphism $G(k)_{\mathrm{Top}} \xrightarrow{\sim} (G(k).t)_{\mathrm{Top}}$. Thus f_t is strict. This example is significant for one of our main result.

1.4.1. **Theorem.** [G-G-MB, 1.5] *Let K be a henselian valued field such that its completion $\widehat{K}$ is separable over K. Let G be an affine algebraic K–group acting on an affine K-variety X. Let $x \in X(K)$ be a rational point. Then the topological orbit $G(K).x$ is locally closed in $X(K)$ and the morphism $f_x : G(K)_{\mathrm{Top}} \to X(K)_{\mathrm{Top}}$ is strict.*

Note that we do not require the orbit $G.x$ to be closed.

2. Lecture 2: Affine group schemes

We are interested mainly in affine algebraic groups over a field k. It is convenient to work in a wider setting of affine group schemes defined over a ring. Basic references are the books of Demazure-Gabriel [DG], Milne [Mi2] and Waterhouse [Wa1].

Convention. If k is a field, a k-variety means a separated k–scheme of finite type. If $f : X \to Y$ is a morphism of affine algebraic k-varieties, the schematic image Y_0 of f is the closed k-subvariety of Y defined by $k[Y_0] = k[Y]/\ker\bigl(k[Y] \xrightarrow{f^\sharp} k[X]\bigr)$.

2.1. **Definition.** Let R be a ring (commutative, unital). An affine R–group scheme $\mathfrak{G}$ is a group object in the category of affine R-schemes. It means that $\mathfrak{G}/R$ is an affine scheme equipped with a section $\epsilon : \mathrm{Spec}(R) \to \mathfrak{G}$, an inverse $\sigma : \mathfrak{G} \to \mathfrak{G}$ and a multiplication $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ such that the three following diagrams commute:

Associativity:

$$\begin{tikzcd}
(\mathfrak{G} \times_R \mathfrak{G}) \times_R \mathfrak{G} \arrow[r, "m\times id"] \arrow[d, "can"', "\cong"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[r, "m"] & \mathfrak{G} \\
\mathfrak{G} \times_R (\mathfrak{G} \times_R \mathfrak{G}) \arrow[r, "id\times m"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[ur, "m"'] &
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathrm{Spec}(R) \arrow[r, "id\times\epsilon"] \arrow[dr, "p_2"'] & \mathfrak{G} \times_R \mathfrak{G} \arrow[d, "m"] & \mathrm{Spec}(R) \times \mathfrak{G} \arrow[l, "\epsilon\times id"'] \arrow[dl, "p_1"] \\
& \mathfrak{G} &
\end{tikzcd}$$

Symetry:

$$\begin{tikzcd}
\mathfrak{G} \arrow[r, "id\times\sigma"] \arrow[d, "s_{\mathfrak{G}}"'] & \mathfrak{G} \times_R \mathfrak{G}. \arrow[d, "m"] \\
\mathrm{Spec}(R) \arrow[r, "\epsilon"] & \mathfrak{G}
\end{tikzcd}$$

We say that $\mathfrak{G}$ is commutative if furthermore the following diagram commutes

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathrm{Spec}(R) \arrow[r, "switch"] \arrow[d, "m"'] & \mathfrak{G} \times_R \mathfrak{G} \arrow[d, "m"] \\
\mathfrak{G} \arrow[r, phantom, "="] & \mathfrak{G}.
\end{tikzcd}$$

Let $R[\mathfrak{G}]$ be the coordinate ring of $\mathfrak{G}$. We call $\epsilon^* : R[\mathfrak{G}] \to \mathfrak{G}$ the counit (augmentation), $\sigma^* : R[\mathfrak{G}] \to R[G]$ the coinverse (antipode), and denote by $\Delta = m^* : R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ the comultiplication. They satisfies the following rules:

Co-associativity:

$$\begin{tikzcd}
R[\mathfrak{G}] \arrow[r, "m^*"] \arrow[dr, "m^*"'] & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[r, "m^*\otimes id"] & (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]) \otimes_R R[\mathfrak{G}] \\
 & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[r, "id\otimes m^*"] & R[\mathfrak{G}] \otimes_R (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]). \arrow[u, "can", "\cong"']
\end{tikzcd}$$

Counit:

$$\begin{tikzcd}
R[\mathfrak{G}] \arrow[r, "id\otimes\epsilon^*"] & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] & & R[\mathfrak{G}] \arrow[ll, "\epsilon^*\times id"'] \\
 & R[\mathfrak{G}] \arrow[ul, "id"] \arrow[u, "m^*"] \arrow[urr, "id"'] & &
\end{tikzcd}$$

Cosymmetry:

$$\begin{tikzcd}
R[\mathfrak{G}] \otimes R[\mathfrak{G}] \arrow[r, "\sigma^*\otimes id"] & R[\mathfrak{G}]. \\
R[\mathfrak{G}] \arrow[u, "m^*"] \arrow[r, "\epsilon^*"] & R. \arrow[u, "s^*_{\mathfrak{G}}"]
\end{tikzcd}$$

In other words, $(R[\mathfrak{G}], m^*, \sigma^*, \epsilon^*)$ is a commutative Hopf R–algebra[3]. Given an affine R–scheme $\mathfrak{X}$, there is then a one to one correspondence between group structures on $\mathfrak{X}$ and Hopf R–algebra structures on $R[\mathfrak{X}]$.

The notion of homomorphisms of group schemes is clear.

2.2. **Examples.**

2.2.1. *Additive and multiplicative groups.* The additive R–group scheme is $\mathbb{G}_a = \mathbf{A}^1_R$ with the group law $\Delta : R[X] \to R[Y] \otimes_R R[Z]$ defined by $X \to Y \otimes 1 + 1 \otimes Z$.

The multiplicative R-group scheme is $\mathbb{G}_{m,R} = \mathrm{Spec}\big(R[T,T^{-1}]\big)$ with group law $\Delta : R[T,T^{-1}] \to R[U,U^{-1}] \otimes_R R[V,V^{-1}]$ defined by $T \to U \otimes V$.

2.2.2. *Constant group schemes.* Let Γ be an abstract group. We consider the R–scheme $\mathfrak{G} = \bigsqcup_{\gamma\in\Gamma} \mathrm{Spec}(R)$. Then the group structure on Γ induces a group scheme structure on $\mathfrak{G}$ with multiplication

$$\mathfrak{G} \times_R \mathfrak{G} = \bigsqcup_{(\gamma,\gamma')\in\Gamma^2} \mathrm{Spec}(R) \to \mathfrak{G} = \bigsqcup_{\gamma\in\Gamma} \mathrm{Spec}(R)$$

applying the component (γ,γ') to $\gamma\gamma'$; This group scheme is affine iff Γ is finite.

There usual notation for such an object is Γ_R. This group scheme occurs as solution of the following universal problem.

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[3]This is Waterhouse definition [Wa1, §I.4], other people talk about cocommutative coassociative Hopf algebra.

2.2.3. *Vector groups.* Let N be an R–module. We consider the commutative group functors

$$V_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto \mathrm{Hom}_S(N \otimes_R S, S) = (N \otimes_R S)^\vee,$$

$$W_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto N \otimes_R S.$$

2.2.4. **Lemma.** *The R–group functor V_N is representable by the affine R–scheme $\mathfrak{V}(N) = \mathrm{Spec}(S^*(N))$ which is then a commutative R–group scheme. Furthermore N is of finite presentation if and only if $\mathfrak{V}(N)$ is of finite presentation.*

Proof. It follows readily of the universal property of the symmetric algebra

$$\mathrm{Hom}_{R'-mod}(N \otimes_R R', R') \xleftarrow{\sim} \mathrm{Hom}_{R-mod}(N, R') \xrightarrow{\sim} \mathrm{Hom}_{R-alg}(S^*(N), R')$$

for each R-algebra R'. □

The commutative group scheme $\mathfrak{V}(N)$ is called the vector group-scheme associated to N. We note that $N = \mathfrak{V}(N)(R)$.

Its group law on the R–group scheme $\mathfrak{V}(N)$ is given by $m^* : S^*(N) \to S^*(N) \otimes_R S^*(N)$, applying each $X \in N$ to $X \otimes 1 + 1 \otimes X$. The counit is $\sigma^* : S^*(N) \to S^*(N)$, $X \mapsto -X$.

2.2.5. **Remarks.** (1) If $N = R$, we get the affine line over R. Given a map $f : N \to N'$ of R–modules, there is a natural map $f^* : \mathfrak{V}(N') \to \mathfrak{V}(N)$.

(2) If N is projective and finitely generated, we have $W(N) = V(N^\vee)$ so that $\mathfrak{W}(N)$ is representable by an affine group scheme.

(3) If R is noetherian, Nitsure showed the converse holds [Ni04]. If N is finitely generated projective, then $\mathfrak{W}(N)$ is representable iff N is locally free.

2.2.6. **Lemma.** *The construction of (1) provides an antiequivalence of categories between the category of R-modules and that of vector group R-schemes.*

2.2.7. *Group of invertible elements, linear groups.* Let A/R be an algebra (unital, associative). We consider the R-functor

$$S \mapsto \mathrm{GL}_1(A)(S) = (A \otimes_R S)^\times.$$

2.2.8. **Lemma.** *If A/R is finitely generated projective, then* $\mathrm{GL}_1(A)$ *is representable by an affine group scheme. Furthermore,* $\mathrm{GL}_1(A)$ *is of finite presentation.*

Proof. We shall use the norm map $N : A \to R$ defined by $a \mapsto \det(L_a)$ constructed by glueing. We have $A^\times = N^{-1}(R^\times)$ since the inverse of L_a can be written L_b by using the characteristic polynomial of L_a. The same is true after tensoring by S, so that

$$\mathrm{GL}_1(A)(S) = \Big\{ a \in (A \otimes_R S) = \mathfrak{W}(A)(S) \;\; \mid \;\; N(a) \in R^\times \Big\}.$$

We conclude that $\mathrm{GL}_1(A)$ is representable by the fibered product

$$\begin{array}{ccc} \mathfrak{G} & \longrightarrow & \mathfrak{W}(A) \\ \downarrow & & \downarrow N \\ \mathfrak{G}_{m,R} & \longrightarrow & \mathfrak{W}(R). \end{array}$$

□

Given an R–module N, we consider the R–group functor

$$S \mapsto \mathrm{GL}_1(N)(S) = \mathrm{Aut}_{S-mod}(N \otimes_R S).$$

So if N is finitely generated projective. then $\mathrm{GL}_1(N)$ is representable by an affine R–group scheme. Furthermore $\mathrm{GL}_1(N)$ is of finite presentation.

2.2.9. **Remark.** If R is noetherian, Nitsure has proven that $\mathrm{GL}_1(N)$ is representable if and only if N is projective [Ni04].

2.2.10. *Diagonalizable group schemes.* Let A be a commutative abelian (abstract) group. We denote by $R[A]$ the group R–algebra of A. As R-module, we have

$$R[A] = \bigoplus_{a \in A} R\, e_a$$

and the multiplication is given by $e_a\, e_b = e_{a+b}$ for all $a, b \in A$.

For $A = \mathbb{Z}$, $R[\mathbb{Z}] = R[T, T^{-1}]$ is the Laurent polynomial ring over R. We have an isomorphism $R[A] \otimes_R R[B] \xrightarrow{\sim} R[A \times B]$. The R-algebra $R[A]$ carries the following Hopf algebra structure:

Comultiplication: $\Delta : R[A] \to R[A] \otimes R[A]$, $\Delta(e_a) = e_a \otimes e_a$,

Antipode: $\sigma^* : R[A] \to R[A]$, $\sigma^*(e_a) = e_{-a}$;

Augmentation: $\epsilon^* : R[A] \to R$, $\epsilon(e_a) = 1$.

2.2.11. **Definition.** We denote by $\mathfrak{D}(A)/R$ (or $\widehat{A}$) the affine commutative group scheme $\mathrm{Spec}(R[A])$. It is called the diagonalizable R–group scheme of base A. An affine R–group scheme is diagonalizable if it is isomorphic to some $\mathfrak{D}(B)$.

We denote by $\mathfrak{G}_m = \mathfrak{D}(\mathbb{Z}) = \mathrm{Spec}(R[T, T^{-1}])$, it is called the multiplicative group scheme. We note also that there is a natural group scheme isomorphism $\mathfrak{D}(A \oplus B) \xrightarrow{\sim} \mathfrak{D}(A) \times_R \mathfrak{D}(B)$. We let in exercise the following fact.

2.2.12. **Lemma.** *The following are equivalent:*

(i) A is finitely generated;

(ii) $\mathfrak{D}(A)/R$ is of finite presentation;

(iii) $\mathfrak{D}(A)/R$ is of finite type.

If $f : B \to A$ is a morphism of abelian groups, it induces a group homomorphism $f^* : \mathfrak{D}(A) \to \mathfrak{D}(B)$. In particular, when taking $B = \mathbb{Z}$, we have a natural mapping

$$\eta_A : A \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{G}_m).$$

2.2.13. **Lemma.** *If R is connected, η_A is bijective.*

Proof. Let $f : \mathfrak{D}(A) \to \mathfrak{G}_m$ be a group R–morphisms. Equivalently it is given by the map $f^* : R[T, T^{-1}] \to R[A]$ of Hopf algebra. In other words, it is determined by the function $X = f(T) \in R[A]^\times$ satisfying $\Delta(X) = X \otimes X$. Writing $X = \sum_{a \in A} r_a e_a$, the relation reads as follows $r_a r_b = 0$ if $a \neq b$ and $r_a \, r_a = r_a$. Since the ring is connected, 0 and 1 are the only idempotents so that $r_a = 0$ or $r_a = 1$. Then there exists a unique a such that $r_a = 1$ and $r_b = 0$ for $b \neq a$. This shows that the map η_A is surjective. It is obviously injective so we conclude that η_A is bijective. □

2.2.14. **Proposition.** *Assume that R is connected. The above construction induces an anti-equivalence of categories between the category of abelian groups and that of diagonalizable R–group schemes.*

Proof. It is enough to contruct the inverse map $\mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{D}(B)) \to \mathrm{Hom}(A, B)$ for abelian groups A, B. We are given a group homomorphism $f : \mathfrak{D}(A) \to \mathfrak{D}(B)$. It induces a map

$$f^* : \mathrm{Hom}_{R-gp}(\mathfrak{D}(B), \mathfrak{G}_m) \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{G}_m),$$

hence a map $B \to A$. □

If $\mathfrak{G}/R$ is an affine R–group scheme, then for each R–algebra S the abtract group $\mathfrak{G}(S)$ is equipped with a natural group structure. The multiplication is $m(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \to \mathfrak{G}(S)$, the unit element is $1_S = (\epsilon \times_R S) \in \mathfrak{G}(S)$ and the inverse is $\sigma(S) : \mathfrak{G}(S) \to \mathfrak{G}(S)$.

2.3. **Largest smooth subgroup.** An affine k-variety X is *absolutely reduced* if the ring $k[X] \otimes_k K$ is reduced for all field extensions K/k; it is enough to check it on the algebraic closure $\overline{k}$ of k [GW, prop. 5.49]. Another terminology is to say that X is *separable*; the geometric meaning of this notion is that X is generically smooth over k (*loc. cit.*, 6.20.(ii)).

2.3.1. **Lemma.** [CGP, C.11] *There is a unique geometrically reduced closed subscheme $X^\dagger \subseteq X$ such that $X^\dagger(F) = X(F)$ for all separable extension fields F/k. The formation of $X^\dagger$ is functorial in X and commutes with the formation of products over k and separable extensions of the ground field.*

2.3.2. **Remarks.** (a) In particular we have $X^\dagger(k) = X(k)$. It means we can often replace X by $X^\sharp$ when we study rational points.

(b) If k is separably closed, then $X^\dagger$ is the Zariski closure of $X(k)$ in X. What is not obvious is that the construction behave well under separable extensions.

(c) If k is algebraically closed, then $X^\dagger$ is X_{red}, that is the reduced k–subscheme of X.

(d) We have to pay attention that the functorial behaviour fails for inseparable extensions. If k is imperfect of characteristic $p > 0$, we consider the variety $x^p - ay^p = 0$ for $a \in k \setminus k^p$. Then $X(k_s) = \emptyset$ so that $X^\dagger = \emptyset$. Since $X\big(k(\sqrt[p]{a})\big) \neq \emptyset$, we have $\big(X_{k(\sqrt[p]{a})}\big)^\dagger \neq \emptyset$.

If G is an affine algebraic k-group, then $G^\dagger$ is a smooth affine algebraic k-subgroup which is generically smooth. In other words there exists a dense k–subscheme $U \subset G$ which is smooth. Since $G_{\overline{k}} = \bigcup_{g \in G(\overline{k})} gU$, $G^\dagger$ is smooth. We name it *the largest smooth k–subgroup of G*.

2.3.3. **Remark.** There is no reason for $G^\dagger$ to be normal in G. The simplest counterexample is $G = \mu_p \rtimes \mathbb{Z}/2\mathbb{Z}$ in characteristic $p > 0$. We have $G^\sharp = \mathbb{Z}/2\mathbb{Z}$.

2.4. **Proper morphisms.** Let $f : G \to H$ be a homomorphism of affine k–group schemes. The schematic image of f is the spectrum of the k–algebra $k[H]/\mathrm{Ker}\big(k[H] \xrightarrow{f^\sharp} k[G]\big)$. We bear in mind that its formation is compatible with field extensions. It is a Hopf subalgebra of $k[G]$ and is of finite type over k. It is then the coordinate algebra of an affine k–group scheme H' which is a closed k-subgroup scheme of H. We called it the schematic image of f; the morphism f factorizes through $f' : G \to H'$.

2.4.1. **Proposition.** *Let $f : G \to H$ be a homomorphism of affine algebraic k–groups. The following are equivalent:*

(i) f is proper;

(ii) $\ker(f)$ *is proper.*

Proof. Obviously we have $(i) \implies (ii)$. We assume (ii). Without loss of generality we can replace H by H' and also we can assume that k is algebraically closed.

First case: H is integral. Since the schematic image of $f : G \to H$ is H, $f(G)$ is dense in H [GW, prop. 10.30], that is, f is dominant. It follows that the generic point of H belongs to $f(G)$ [Stacks, Tag 01RL]. We denote by X the generic fiber of f, this is a non-empty $k(H)$–scheme. Let K be a (finite) field extension of $k(H)$ such that $X(K) \neq \emptyset$. The choice of an element $x \in X(K)$ provides an isomorphism $\ker(f)_K \xrightarrow{\sim} X_K$ so that X_K is a proper K-scheme. It follows that X is a proper $k(H)$–scheme. It follows that H admits a non-empty open subset $V \subset H$ such that $f_{|f^{-1}(V)} : f^{-1}(V) \to V$ is proper [EGA4, 8.1.3]. Since $f(G(k))$ is dense in $H(k)$, we have $H = \bigcup_{g \in G(k)} f(g)V$. We conclude that f is proper.

Second case. G and H are smooth. Similarly $f(G(k))$ is dense in $H(k)$ and $f : f^{-1}(H^0) \to H^0$ is proper by the first case. The above argument shows that f is proper.

General case. We consider the morphism $h = (f)^\dagger : G^\dagger \to H^\dagger$; its kernel is a closed subgroup of $\ker(f)$ so is proper as well. According to [GW, 12.58.5], f is proper as well[4]. $\square$

2.4.2. **Remark.** The statement holds for arbitrary algebraic groups. In the affine setting, proper is furthermore equivalent to finite.

A more advanced result is the following.

2.4.3. **Theorem.** [Pe, corollaire 4.2.5] *Let $f : G \to H$ be a homomorphism between quasi-compact k–group schemes. Then f is faithfully flat (i.e. flat and surjective) if and only if f is schematically dominant.*

In the affine case, see [DG, III.3.7.2] or [Mi2, th. 7.4].

2.5. **Monomorphisms of affine group schemes.** Let $f : \mathfrak{G} \to \mathfrak{H}$ a morphism of R-group schemes. Its kernel $\ker(f)$ is the R–group scheme $f^{-1}(1)$ which is closed in $\mathfrak{G}$.

We say that f is a *monomorphism* if $\mathfrak{G}(S) \to \mathfrak{H}(S)$ is injective for each R-algebra S/R. This is equivalent to say that $\ker(f) = 1$. If the map $f^\sharp : R[\mathfrak{H}] \to R[\mathfrak{G}]$ is surjective, then f is a monomorphism. We are interested in a suitable converse statement.

2.5.1. **Proposition.** *Let $f : G \to H$ be a monomorphism of affine k-algebraic groups. Then f is a closed immersion.*

Proof. We have $\ker(f) = 1$ so that Proposition 2.4.1 shows that f is proper. But a proper monomorphism is a closed immersion [GW, 12.92]. $\square$

2.5.2. **Remarks.** (a) In the proof, f is proper and affine so finite. We need then the following special case of the general theory: a finite monomorphism is a closed immersion. This case is slightly easier and the main input is Nakayama lemma.

(b) This statement is not true over a DVR where there are monomorphisms which are not immersions [SGA3, XVI.1.1]. However for nice enough group schemes $\mathfrak{G}$, monomorphisms are closed immersions (*ibid*, Cor. 1.5).

2.6. **Flat sheaves.** We do a long interlude for developping descent and sheafifications techniques. We use mainly the references [DG, Ro, Wa1]. Our presentation involves only rings.

2.6.1. *Covers.* A fppf (flat for short) cover of the ring R is a ring S/R which is faithfully flat and of finite presentation[5] "fppf" stands for "fidèlement plat de présentation finie".

[4] It is a consequence of the valuative criterion of properness since $h = f_{red}$.

[5] One may consider also not finitely presented covers, it is called *fpqc*, see [SGA3, IV] and [Vi].

2.6.2. **Remarks.** (1) If $1 = f_1 + \cdots + f_s$ is a partition of 1_R with $f_1, ... f_r \in R$, the ring $R_{f_1} \times \cdots \times R_{f_r}$ is a Zariski cover of R and a fortiori a flat cover.

(2) If S_1/R and S_2/R are flat covers of R, then $S_1 \otimes_R S_2$ is a flat cover of R.

(3) If S/R is a flat cover of S and S'/S is a flat cover of S, then S'/R is a flat cover of R.

(4) Finite locally free extensions S/R are flat covers, in particular finite étale surjective maps are flat covers. An example is $S = R[X]/P(X)$ where P is a monic R–polynomial.

2.6.3. *Definition.* We consider an R-functor $F : \{R - Alg\} \to Sets$. It is called *additive* if the natural map $F(S_1 \times S_2) \to F(S_1) \times F(S_2)$ is bijective for all R–rings S_1, S_2.

For each R–ring morphism $S \to S'$, we can consider the sequence

$$F(S) \longrightarrow F(S') \underset{d_{2,*}}{\overset{d_{1,*}}{\rightrightarrows}} F(S' \otimes_S S') \quad .$$

A functor of $F : \{R - Alg\} \to Sets$ is a fppf sheaf (or flat sheaf) for short if it satifies the following requirements:

(i) $F(0) = \{\bullet\}$;

(ii) F is additive;

(iii) For each R–ring S and each flat cover S'/S, and the sequence

$$F(S) \longrightarrow F(S') \underset{d_{2,*}}{\overset{d_{1,*}}{\rightrightarrows}} F(S' \otimes_S S')$$

is exact.

Requirement (iii) means that the restriction map $F(S) \to F(S')$ is injective and its image consists in the sections $\alpha \in F(S')$ satisfying $d_{1,*}(\alpha) = d_{2,*}(\alpha) \in F(S' \otimes_S S')$.

2.6.4. **Remark.** The two first conditions can be shortened in one single condition, see [DG, III.1.1] and [Stacks, Tag 006U].

Given an R–module M and S'/S as above, the theorem of faithfully flat descent states that we have an exact sequence of S–modules

$$0 \to M \otimes_R S \to (M \otimes_R S) \otimes_S S' \overset{d_{1,*} - d_{2,*}}{\longrightarrow} (M \otimes_R S) \otimes_S S' \otimes_S S' \quad .$$

This rephases by saying that the vector group functor $V(M)/R$ (which is additive) is a flat sheaf over $\mathrm{Spec}(R)$. A special case is the exactness of the sequence

$$0 \to S \to S' \overset{d_{1,*} - d_{2,*}}{\longrightarrow} S' \otimes_S S'.$$

If N is an R-module, it follows that the sequence of R–modules

$$0 \to \mathrm{Hom}_R(N, S) \to \mathrm{Hom}_R(N, S') \overset{d_{1,*}-d_{2,*}}{\longrightarrow} \mathrm{Hom}_R(N, S' \otimes_S S')$$

is exact. This shows that the vector R–group scheme $\mathfrak{W}(N)$ is a flat sheaf. More generally we have

2.6.5. **Proposition.** *Let $\mathfrak{X}/R$ be an affine scheme. Then the R–functor of points $h_{\mathfrak{X}}$ is a flat sheaf.*

Proof. The functor $h_{\mathfrak{X}}$ is additive. We are given a R–ring S and a flat cover S'/S. We write the sequence above with the R-module $R[\mathfrak{X}]$. It reads

$$0 \to \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S) \to \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S') \xrightarrow{d_{1,*}-d_{2,*}} \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S' \otimes_S S').$$

It follows that $\mathfrak{X}(S)$ injects in $\mathfrak{X}(S')$ and identifies with $\mathrm{Hom}_{R-rings}(R[\mathfrak{X}], S') \cap \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S)$. Hence the exact sequence

$$\mathfrak{X}(S) \longrightarrow \mathfrak{X}(S') \underset{d_{2,*}}{\overset{d_{1,*}}{\rightrightarrows}} \mathfrak{X}(S' \otimes_S S') \quad .$$

□

2.6.6. **Remark.** More generally, the proposition holds with a scheme $\mathfrak{X}/R$, see [Ro, 2.4.7] or [Vi, 2.5.4].

2.6.7. **Examples.** If E, F are flat sheaves over R, the R–functor $\mathrm{Hom}(E, F)$ of morphisms from E to F is a flat sheaf. Also the R–functor $\mathrm{Isom}(E, F)$ is a flat sheaf and as special case, the R–functor $\mathrm{Aut}(F)$ is a flat sheaf.

2.7. **Monomorphisms and epimorphisms.** A morphism $u : F \to E$ of flat sheaves over R is a monomorphism if $F(S) \to E(S)$ is injective for each S/R. It is an epimorphism if for each S/R and each element $e \in E(S)$, there exists a flat cover S'/S and an element $f' \in F(S')$ such that $e_{|S'} = u(f')$.

A morphism of flat sheaves which is a monomorphism and and an epimorphism is an isomorphism (exercise, solution [SGA3, IV.4.4]).

We say that a sequence of flat sheaves in groups over R $1 \to F_1 \to F_2 \to F_3 \to 1$ is exact if the map of sheaves $F_2 \to F_3$ is an epimorphism and if for each S/R the sequence of abstract groups $1 \to F_1(S) \to F_2(S) \to F_3(S)$ is exact.

2.7.1. **Examples.** (1) For each $n \geq 1$, the Kummer sequence $1 \to \mu_{n,R} \to \mathbb{G}_{m,R} \overset{f_n}{\to} \mathbb{G}_{m,R} \to 1$ is an exact sequence of flat sheaves where f_n is the n–power map. The only thing to check is the epimorphism property. Let S/R be a ring and $a \in \mathbb{G}_m(S) = S^\times$. We put $S' = S[X]/(X^n - a)$, it is finite free over S, hence is faithfully flat of finite presentation. Then $f_n(X) = a_{|S'}$ and we conclude that f_n is an epimorphism of flat sheaves.
(2) More generally, let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of f.g. abelian groups. Then the sequence of R–group schemes

$$1 \to \mathfrak{D}(A_3) \to \mathfrak{D}(A_2) \to \mathfrak{D}(A_1) \to 0$$

is exact.

2.8. **Sheafification.** Given an R-functor F, there is natural way to sheafify it in a flat sheaf $\widetilde{F}$. The first thing is to make the functor additive. For each decomposition $S = \prod_{j\in J} S_j$ (J finite, possibly empty), we have a map

$$F(S) \to \prod_{j\in J} F(S_j)$$

with the convention $\prod\limits_{j\in\emptyset} = \{\bullet\}$. We define

$$F_{add}(S) = \varinjlim \prod_{j\in J} F(S_j)$$

where the limit is taken on finite decompositions of S. By construction, F_{add} is an additive functor which satisfies $F(0) = \{\bullet\}$ and there is a natural map $F \to F_{add}$.

Now, for each S/R, we consider the "set" $\mathrm{Cov}(S)$ of flat covers[6]. Also if $f : S_1 \to S_2$ is an arbitrary R-ring map, the tensor product defines a natural map $f_* : \mathrm{Cov}(S_1) \to \mathrm{Cov}(S_2)$. We define then

$$\widetilde{F}(S) = \varinjlim_{I\subset \mathrm{Cov}(S)} \ker\Big(\quad \prod_{i\in I} F_{add}(S_i) \quad \overset{d_{1,*}}{\underset{d_{2,*}}{\rightrightarrows}} \quad F_{add}(S_i \otimes_S S_j) \Big)$$

where the limit is taken on finite subsets I of $\mathrm{Cov}(S)$. It is an R-functor since each map $f : S_1 \to S_2$ defines $f_* : \widetilde{F}(S_1) \to \widetilde{F}(S_2)$. We have also a natural mapping $u_F : F \to F_{add} \to \widetilde{F}$.

2.8.1. **Proposition.** *(1) For each R–functor F, the R–functor $\widetilde{F}$ is a flat sheaf.*

(2) The functor $F \to \widetilde{F}$ is left adjoint to the forgetful functor applying a flat sheaf to its underlying R–functor. For each R–functor F and each flat sheaf E, the natural map

$$\mathrm{Hom}_{flat\, sheaves}(\widetilde{F}, E) \xrightarrow{\sim} \mathrm{Hom}_{R-functor}(F, E)$$

(applying a morphism $u : \widetilde{F} \to E$ to the composite $F \to \widetilde{F} \to E$) is bijective.

(1) follows essentially by construction [DG, III.1.8]. Note that in this reference, the two steps are gathered in one. For (2) one needs to define the inverse mapping. Observe that the sheafification of E is itself, so that the sheafification of $F \to E$ yields a natural morphism $\widetilde{F} \to E$.

Given a morphism of flat R-sheaves $f : E \to F$, we can sheafify the functors

$$S \mapsto E(S)/R_f(S), \;\; S \mapsto \mathrm{Im}\big(E(S) \to F(S)\big),$$

[6] We do not enter in set-theoric considerations but the reader can check there is no problem there.

where $R_f(S)$ is the equivalence relation defined by $f(S)$. We denote by $\mathrm{Coim}(f)$ and $\mathrm{Im}(f)$ their respective sheafifications. We have an induced mapping

$$f_* : \mathrm{Coim}(f) \to \mathrm{Im}(f)$$

between the coimage sheaf and the image sheaf. We say that f is strict when f_* is an isomorphism of flat sheaves.

2.8.2. **Lemma.** *If f is a monomorphism (resp. an epimorphism), then f is strict.*

In the first case, we have $E \xrightarrow{\sim} \mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$; in the second case, we have $\mathrm{coker}(f) \xrightarrow{\sim} \mathrm{Im}(f) \xrightarrow{\sim} F$, see [DG, III.1.2].

2.9. **Group actions, quotients sheaves.** Let $\mathcal{G}$ be a flat sheaf in groups and let $\mathcal{F}$ be a flat sheaf equipped with a right action of $\mathcal{G}$. We consider the quotient functor $S \mapsto \mathcal{F}(S)/\mathcal{G}(S)$ and its fppf sheafification which is denoted by $\mathcal{F}/\mathcal{G}$. It is called the quotient sheaf[7].

When $\mathcal{G}$ and $\mathcal{F}$ are representable, the natural question is to investigate whether the quotient sheaf $\mathcal{G}/\mathcal{F}$ is representable. It is quite rarely the case. A first evidence to that is the following fact.

2.9.1. **Proposition.** *We are given an affine R–group scheme $\mathfrak{G}$ and a monomorphism $\mathfrak{G} \to \mathfrak{H}$ into an affine group scheme. Assume that the quotient sheaf $\mathfrak{H}/\mathfrak{G}$ is representable by an R–scheme $\mathfrak{X}$. We denote by $p : \mathfrak{H} \to \mathfrak{X}$ the quotient map and by $\epsilon_X = p(1_{\mathfrak{G}}) \in \mathfrak{X}(R)$.*

(1) The R–map $\mathfrak{H} \times_R \mathfrak{G} \to \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H}$, $(h, g) \to (h, hg)$, is an isomorphism.

(2) The diagram

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{i} & \mathfrak{H} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{\epsilon_{\mathfrak{X}}} & \mathfrak{X} \end{array}$$

is carthesian.

(3) The map i is an immersion. It is a closed immersion iff $\mathfrak{X}/R$ is separated.

(4) $\mathfrak{G}/R$ is flat iff p is flat.

(5) $\mathfrak{G}/R$ is smooth iff p is smooth.

The general statement is [SGA3, VI_B.9.2].

Proof. (1) The map $\mathfrak{H} \times_R \mathfrak{G} \to \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H}$ is a monomorphism. Let us show that it is an epimorphism of flat sheaves. We are given S/R and $(h_1, h_2) \in \mathbb{H}(S)^2$ such that $p(h_1) = p(h_2)$. There exists a flat cover S'/S and $g \in \mathfrak{G}(S')$ such that $h_{1|S'} = h_{2|S'}\, g$. Hence $g \in \mathfrak{G}(S') \cap \mathfrak{H}(S)$. Since i is a monomorphism, we conclude by descent that $g \in \mathfrak{G}(S)$ whence (h_1, h_2) comes from (h_1, g).

[7] One can work in a larger setting, that of equivalence relations and groupoids, see [DG, §III.2].

(2) It follows that the diagram

$$\begin{array}{ccccc}
\mathfrak{G} & \xrightarrow{1_{\mathfrak{H}} \times id} & \mathfrak{H} \times_R \mathfrak{G} \cong \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H} & \longrightarrow & \mathfrak{H} \\
\downarrow & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\
\mathrm{Spec}(R) & \xrightarrow{1_{\mathfrak{H}}} & \mathfrak{H} & \xrightarrow{p} & \mathfrak{X}
\end{array}$$

is carthesian as desired.

(3) If $\mathfrak{X}$ is separated, $\epsilon_{\mathfrak{X}}$ is a closed immersion and so is i.

(4) and (5) If p is flat (resp. smooth), so is i by base change. The converse is a consequence of "permanences properties" of failthfully flat descent. $\square$

2.10. **Quotients.**

2.10.1. **Proposition.** *Let G be an affine algebraic k–group acting on a quasi-projective k–variety X. Let $x \in X(k)$ and consider the orbit map $f_x : G \to X$, $g \mapsto g.x$ and the stabilizer G_x. Then the fppf sheaf G/G_x is representable by a k-variety. Furthermore the induced map $G/G_x \to X$ is an immersion.*

The k-subscheme G/G_x is called the G-orbit of x.

2.10.2. **Remark.** There is a suitable extension of this statement over rings, see [SGA3, XVI.2.2].

2.10.3. **Theorem.** *(Chevalley) Let G be an affine algebraic k–group and let $H \subseteq G$ be a closed subgroup.*

(1) There exists a linear representation $\rho : G \to \mathrm{GL}(V)$ and a line $\ell \subset V$ such that H is the stabilizer of l.

(2) The fppf quotient G/H is representable by a quasi-projective k–variety.

(3) If H is normal in G, then G/H is affine and carries a unique k-group structure such that $G \to G/H$ is a group homomorphism.

2.10.4. **Remark.** If R is a DVR and $H \subset G$ a closed R–group scheme which is flat, then G/H is representable by an R-scheme [A, IV, th. 4.C]. In higher dimensions, there are counterexamples to representability of quotient of affine flat group schemes [R2, X.14].

3. Lecture 3: Homogeneous spaces, Weil restriction, punctual Hilbert schemes

3.1. Formal definition of orbits and of homogeneous spaces over fields.

3.1.1. **Definition.** [B-L-R, 10.2] Let G be an affine algebraic k-group which acts on a k-variety X. A subscheme Z of X is called a k-orbit under the action of G if there exist a finite field extension k' of k and a point $x' \in Z(k')$ such that $Z \times_k k'$ is the orbit of x' under $G \times_k k'$.

This definition is taylor made to be insensible to finite field extensions.

3.1.2. **Remark.** A k–orbit is quasi-projective, since it descends by locally free morphisms [SGA1, VIII, 7.6].

3.1.3. **Definition.** Let G be an affine algebraic k-group which acts on a K-variety X. Then X is homogeneous under G if X is a k-orbit.

Equivalently, it means that there exists a finite field extension k'/k and a closed k'-subgroup H' of $G_{k'}$ such that $X_{k'}$ is $G_{k'}$-isomorphic to $G_{k'}/H'$.

Note that we do not require X to have a rational k–point. An example is $X = \mathrm{Spec}(\mathbb{C})$ seen as a $\mathbb{R}$-variety with the conjugacy action of $\mathbb{Z}/2\mathbb{Z}$.

3.1.4. *Examples of orbits and of homogeneous spaces.* (a) The Chevalley quotient $X = H/G$ is a homogeneous space under the action of H. For each $x \in X(k)$, the preimage $q^{-1}(x)$ is a k–orbit for the G action on H. We claim that all k–orbits under G are of this shape. Let $Z \subset H$ a k-orbit under the action of G. Theres exists a finite field extension k' of k and a point $h' \in H(k')$ such that $Z \times_k k'$ is the orbit of x' under $G \times_k k'$. We put $x' = q(h') \in X(k')$, it does not depend of the choice of h'. It follow that $d_1(x') = d_2(x') \in X(k' \otimes_k k')$ so that x' descends to point $x \in X(k)$. Since $Z_{k'} = q^{-1}(x') = (q^{-1}(x))_{k'}$, we conclude that $Z = q^{-1}(x)$.
(b) The pointed affine space $\mathbf{A}^n_k \setminus \{0\}$ is homogeneous for the action of GL_n ($n \geq 1$). This is the case also of $\mathbf{P}^{n-1}_k$, of grassmannians,...

3.2. **Contracted products.** We are given two flat R–sheaves in sets $\mathcal{F}_1$, $\mathcal{F}_2$ and and a flat sheaf $\mathcal{G}$ in groups. If $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) is equipped with a right (resp. left) action of $\mathcal{G}$, we have a natural right action of G on the product $\mathcal{F}_1 \times \mathcal{F}_2$ by $(f_1, f_2).g = (f_1 g, g^{-1} f_2)$. The sheaf quotient of $\mathcal{F}_1 \times \mathcal{F}_2$ under this action by $\mathcal{G}$ is denoted by $\mathcal{F}_1 \wedge^{\mathcal{G}} \mathcal{F}_2$ and is called the *contracted product* of $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to $\mathcal{G}$.

3.2.1. **Remark.** This construction occurs for group extensions. Let $1 \to \mathcal{A} \to \mathcal{E} \to \mathcal{G} \to 1$ be an exact sequence of flat sheaves in groups with A abelian. Given a map $\mathcal{A} \to \mathcal{B}$ of abelian flat sheaves equipped with compatible $\mathcal{G}$–actions, the contracted product $\mathcal{B} \wedge^{\mathcal{A}} \mathcal{E}$ is a sheaf in groups and is an extension of $\mathcal{G}$ by $\mathcal{B}$.

3.3. **Sheaf torsors.** Let $\mathcal{G}$ be a flat sheaf in groups.

3.3.1. **Definition.** A sheaf $\mathcal{G}$–torsor over R is a flat sheaf $\mathcal{E}/R$ equipped with a right action of $\mathcal{G}$ submitted to the following requirements:

(T1) The R-map $\mathcal{E} \times_R \mathcal{G} \to \mathcal{E} \times_R \mathcal{E}$, $(e,g) \mapsto (e, e.g)$ is an isomorphism of flat sheaves over R.

(T2) There exists a flat cover S/R such that $\mathcal{E}_(S) \neq \emptyset$.

The basic example of such an object is the trivial $\mathcal{G}$-torsor sheaf $\mathcal{G}$ equipped with the right action. For avoiding confusions, we denote it sometimes $\mathcal{E}_{tr}$.

3.3.2. **Remark.** The condition (T1) is called *pseudo-torsor*. We can replace condition $(T2)$ by the condition (T3) $E \to \{\bullet\}$ is an epimorphism of flat sheaves. We get then the definition which is in Giraud's book [Gir, III.1.3.7]. There is also one way to rephrase everything which is to require the existence of a flat cover S/R such that $\mathcal{E}_S \cong \mathcal{E}_{tr,S}$.

Now if $\mathcal{F}$ is a flat sheaf over R equipped with a right $\mathcal{G}$-action and $\mathcal{E}/R$ is a $\mathcal{G}$-torsor, we call the contracted product $\mathcal{E} \wedge^{\mathcal{G}} \mathcal{F}$ the twist of $\mathcal{F}$ by $\mathcal{E}$. It is denoted sometimes ${}^{\mathcal{E}}\mathcal{F}$ or ${}_{\mathcal{E}}F$. We record the two special cases:

(1) The action of $\mathcal{G}$ on $\mathcal{E}_{tr}$ by left translations, we get then $\mathcal{E} = {}^{\mathcal{E}}\mathcal{E}_{tr}$.

(2) The action of $\mathcal{G}$ on itself by inner automorphisms, the twist ${}^{\mathcal{E}}\mathcal{G}$ is called the inner twisted form of $\mathcal{G}$ associated to $\mathcal{E}$.

(3) We can twist the left action (by translation) $\mathcal{G} \times \mathcal{E}_{tr} \to \mathcal{E}_{tr}$, where $\mathcal{G}$ acts on itself by inner automorphisms. It provides a left action ${}^{\mathcal{E}}\mathcal{G} \times_R \mathcal{E} \to \mathcal{E}$.

If $\mathfrak{G}$ is an affine R-group scheme, descent theory shows that sheaf $\mathfrak{G}$-torsors are representable as well; we say that the relevant schemes are $\mathfrak{G}$–torsors. Furthermore if $\mathfrak{G}/R$ is flat (resp. smooth), so are the $\mathfrak{G}$–torsors. We give some examples of torsors.

3.3.3. **Examples.** (1) Galois covers $\mathfrak{Y} \to \mathfrak{X}$ under a finite group Γ, see below 3.4.1.

(2) The Kummer cover $\times n : \mathbb{G}_m \to \mathbb{G}_m$.

(3) The Chevalley quotient gives rise to the H–torsor $G \to G/H$.

(4) (Swan-Serre's correspondence) If P is a R-module locally free of rank $n \geq 1$, we define the fppf sheaf GL_n-torsor $\mathcal{E}_P$ by $\mathcal{E}_P(S) = \mathrm{Isom}_{S-mod}(S^n, P \otimes_R S)$. All GL_n–torsors arise in this way.

(5) (Étale algebras) If $\mathcal{A}$ is an étale R-algebra of rank $n \geq 1$, we define the fppf sheaf S_n-torsor $\mathcal{E}_{\mathcal{A}}$ by $\mathcal{E}_{\mathcal{A}}(S) = \mathrm{Isom}_{S-alg}(S^n, \mathcal{A} \otimes_R S)$. All S_n–torsors arise in this way.

3.4. **Quotient by a finite constant group.** An important case of torsor and quotients is the following

3.4.1. **Theorem.** [DG, §III.6] *Let Γ be a finite abtract group. We assume that Γ_R acts freely on the right on an affine R-scheme $\mathfrak{X}$. It means that the graph map $\mathfrak{X} \times_R \Gamma_R \to \mathfrak{X} \times_R \mathfrak{X}$ is a monomorphism. We put $\mathfrak{Y} =$* $\mathrm{Spec}(R[\mathfrak{X}]^\Gamma)$.

(1) *The map $\mathfrak{X} \to \mathfrak{Y}$ is a Γ_R–torsor, i.e. a Galois cover of group Γ;*

(2) *The scheme $\mathfrak{Y}/R$ represents the fppf quotient sheaf $\mathfrak{X}/\mathfrak{G}$.*

See also [R1, X, p. 108] for another proof.

3.5. **Weil restriction.** We are given the following equation $z^2 = 1 + 2i$ in $\mathbb{C}$. A standard way to solve it is to write $z = x + iy$ with $x, y \in \mathbb{R}$. It provides then two real equations $x^2 - y^2 = 1$ and $xy = 1$. We can abstract this method for affine schemes and for functors.

We are given a ring extension S/R or $j : R \to S$. Since a S-algebra is a R–algebra, a R-functor F defines a S-functor denoted by F_S and called the scalar extension of F to S. For each S–algebra S', we have $F_S(S') = F(S')$. If X is a R-scheme, we have $(h_X)_S = h_{X \times_R S}$.

Now we consider a S–functor E and define its Weil restriction to S/R denoted by $\prod_{S/R} E$ by

$$\Big(\prod_{S/R} E\Big)(R') = E(R' \otimes_R S)$$

for each R–algebra R'. We note the two following functorial facts.

(I) For a R-map or rings $u : S \to T$, we have a natural map

$$u_* : \prod_{S/R} E \to \prod_{T/R} E_T.$$

(II) For each R'/R, there is natural isomorphism of R'–functors

$$\Big(\prod_{S/R} E\Big)_{R'} \xrightarrow{\sim} \prod_{S \otimes_R R'/R'} E_{S \otimes_R R'}.$$

For other functorial properties, see appendix A.5 of [CGP].

At this stage, it is of interest to discuss the example of vector group functors. Let N be a S–module. We denote by $j_* N$ the scalar restriction of N from S to R [B:A1, §II.1.13]. The module $j_* N$ is N equipped with the R–module structure induced by the map $j : R \to S$. It satisfies the adjunction property $\mathrm{Hom}_R(M, j_* N) \xrightarrow{\sim} \mathrm{Hom}_S(M \otimes_R S, N)$ (*ibid*, §III.5.2) for each R-module M.

3.5.1. **Lemma.** *(1) $\prod_{S/R} W(N) \xrightarrow{\sim} W(j_* N)$.*
(2) If N is f.g. projective and S/R is finite and locally free, then $\prod_{S/R} W(N)$ is representable by the vector group scheme $\mathfrak{W}(j_* N)$.

For a more general statement, see [SGA3, I.6.6].

Proof. (1) For each R-algebra R', we have

$$\Big(\prod_{S/R} W(N)\Big)(R') = W(N)(R' \otimes_R S) = N \otimes_S (R' \otimes_R S) = (j_*N) \otimes_R R' = W(j_*N)(R').$$

(2) The assumptions implies that j_*N is f.g. over R, hence $W(j_*N)$ is representable by the vector R–group scheme $\mathfrak{W}(j_*N)$. □

If F is a R-functor, we have for each R'/R a natural map

$$\eta_F(R') : F(R') \to F(R' \otimes_R S) = F_S(R' \otimes_R S) = \Big(\prod_{S/R} F_S\Big)(R');$$

it defines a natural mapping of R–functor $\eta_F : F \to \prod\limits_{S/R} F_S$. For each S–functor E, it permits to defines a map

$$\phi : \mathrm{Hom}_{S-functor}(F_S, E) \to \mathrm{Hom}_{R-functor}\big(F, \prod_{S/R} E\big)$$

by applying a S–functor map $g : F_S \to E$ to the composition

$$F \overset{\eta_F}{\to} \prod_{S/R} F_S \xrightarrow{\prod_{S/R} g} \prod_{S/R} E.$$

3.5.2. **Lemma.** *The map ϕ is bijective.*

Proof. We apply the compatibility with $R' = S_2 = S$. The map $S \to S \otimes_R S_2$ is split by the codiagonal map $\nabla : S \otimes_R S_2 \to S, s_1 \otimes s_2 \to s_1 s_2$. Then we can consider the map

$$\theta_E : \Big(\prod_{S/R} E\Big)_{S_2} \xrightarrow{\sim} \prod_{S \otimes_R S_2/S_2} E_{S \otimes_R S_2} \overset{\nabla_*}{\to} \prod_{S/S} E = E.$$

This map permits to construct the inverse map ψ of ϕ as follows

$$\psi(h) : F_S \overset{l_S}{\to} \Big(\prod_{S/R} E\Big)_{S_2} \overset{\theta_E}{\to} E$$

for each $l \in \mathrm{Hom}_{R-functor}\big(F, \prod_{S/R} E\big)$. By construction, the maps ϕ and ψ are inverse of each other. □

In conclusion, the Weil restriction from S to R is then right adjoint to the functor of scalar extension from R to S.

3.5.3. **Proposition.** *Let $\mathfrak{Y}/S$ be an affine scheme of finite type (resp. of finite presentation). Then the R-functor $\prod_{S/R} h_{\mathfrak{Y}}$ is representable by an affine scheme of finite type (resp. finite representation).*

Again, it is a special case of a much more general statement, see [B-L-R, §7.6].

Proof. Up to localize for the Zariski topology, we can assume that S is free over R, namely $S = \oplus_{i=1,\dots,d} R\,\omega_i$. We see $\mathfrak{Y}$ as a closed subscheme of some affine space $\mathbb{A}^n_S$, that is given by a system of equations $(P_\alpha)_{\alpha \in I}$ with $P_\alpha \in S[t_1, \dots, t_n]$. Then $\prod_{S/R} h_{\mathfrak{Y}}$ is a subfunctor of $\prod\limits_{S/R} W(S^n) \xrightarrow{\sim} W(j_*(S^n)) \xrightarrow{\sim} W(R^{nd})$ by Lemma 3.5.1. For each I, we write

$$P_\alpha\Big(\sum_{i=1,..,d} y_{1,i}\omega_i, \sum_{i=1,..,d} y_{2,i}\omega_i, \dots, \sum_{i=1,..,d} y_{n,i} \Big) = Q_{\alpha,1}\,\omega_1 + \dots + Q_{\alpha,r}\,\omega_r$$

with $Q_{\alpha,i} \in R\big[y_{k,i}; i = 1, .., d; k = 1, ..., n\big]$. Then for each R'/R, $\Big(\prod\limits_{S/R} h_{\mathfrak{Y}}\Big)(R')$ inside ${R'}^{nd}$ is the locus of the zeros of the polynomials $Q_{\alpha,j}$. Hence this functor is representable by an affine R-scheme $\mathfrak{X}$ of finite type. Furthermore, if $\mathfrak{Y}$ is of finite presentation, we can take I finite so that $\mathfrak{X}$ is of finite presentation too. $\square$

In conclusion, if $\mathfrak{H}/S$ is an affine group scheme of finite type, then the R–group functor $\prod\limits_{S/R} h_{\mathfrak{H}}$ is representable by an R-affine group scheme of finite type. There are two basic examples of Weil restrictions.

(a) The case of a finite separable field extension k'/k (or more generally an étale k-algebra). Given an affine algebraic k'-group G'/k', we associate the affine algebraic k–group $G = \prod_{k'/k} G'$ which is often denoted by $R_{k'/k}(G)$, see [Vo, §3. 12]. In that case, $\prod_{k'/k}(G) \times_k k_s \xrightarrow{\sim} (G'_{k_s})^d$. In particular, the dimension of G is $[k' : k] \dim_{k'}(G')$; the Weil restriction of a finite algebraic group is a finite group.

(b) The case where $S = k[\epsilon]$ is the ring of dual numbers which is of very different nature. For example the quotient k-group $\Big(\prod_{k[\epsilon]/k}(\mathbb{G}_m)\Big)/\mathbb{G}_m$ is the additive k–group. Also if $p = \mathrm{char}(k) > 0$, $\prod_{k[\epsilon]/k} \mu_{p,k[\epsilon]}$ is of dimension 1.

Let us give an application of Weil restriction.

3.5.4. **Proposition.** *Let $\mathfrak{G}/R$ be an affine group scheme. Assume that there exists a finite locally free extension S/R such that $\mathfrak{G} \times_R S$ admits a faithful representation N f.g. locally free as S–module. Then $\mathfrak{G}$ admits a faithful representation M which is f.g. locally free as R–module.*

Proof. Let $\rho : \mathfrak{G} \times_R S \to \mathrm{GL}(N)$ be a faithful S–representation and denote by M/R the restriction of N from S to R. We consider then the R–map

$$\mathfrak{G} \to \prod_{S/R} \mathfrak{G} \times_R S \xrightarrow{\prod_{S/R} \rho} \prod_{S/R} \mathrm{GL}(N) \to \mathrm{GL}(M)$$

It is a composite of monomorphisms, hence a monomorphism. $\square$

3.5.5. **Remark.** It is natural to ask whether the functor of scalar extension from R to S admits a left adjoint. It is the case and we denote by $\prod_{S/R} E$ this left adjoint functor, see [DG, §I.1.6]. It is called the Grothendieck restriction.

If $\rho : S \to R$ is a R–ring section of j, it defines a structure R^ρ of S–ring. We have $\bigsqcup\limits_{S/R} E = \bigsqcup\limits_{\rho:S\to R} E(R^\rho)$. If $E = h_{\mathfrak{Y}}$ for a S–scheme $\mathfrak{Y}$, $\bigsqcup\limits_{S/R} \mathfrak{Y}$ is representable by the R–scheme $\mathfrak{Y}$. This is simply the following R-scheme $\mathfrak{Y} \to \mathrm{Spec}(S) \xrightarrow{j^*} \mathrm{Spec}(R)$.

4. Lecture 4: Gabber's compactifications

4.1. **Hilbert schemes.** Let k' be a finite k-algebra, of dimension $d \geq 1$. Let Q be a quasi-projective k'-variety. We denote by $\sigma : Q' \to \mathrm{Spec}(k')$ the structural morphism and consider:

- V the induced k-variety $Q' \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ (its Grothendieck restriction);
- $W = \prod\limits_{k'/k} Q'$ the Weil restriction[8] of Q' from k' to k.

For each k-scheme S, we have

$$W(S) = Q'(k' \times_k S).$$

We consider the Hilbert k–functor $V^{[d]} = \mathrm{Hilb}^d_{V/k}$ of closed subschemes of length d of V. Explicitely, for each k-scheme S, we have

$$V^{[d]}(S) \quad = \quad \Big\{ \text{subschemes} \quad Z \subset V \times_k S, \\ \qquad\qquad \text{finite locally free of rank } d \text{ over } S \Big\}.$$

Such a Z gives rise to the S-morphism

$$\varphi_Z : Z \hookrightarrow V \times_k S \xrightarrow{\sigma \times_k \mathrm{id}_S} k' \times_k S \tag{4.1.1}$$

between finite S-schemes locally free of same rank d.

4.1.1. **Remark.** We deal here with W and $V^{[d]}$ in the frameword of sheaves over $\mathrm{Spec}(k)$; we shall note use deep representability results, i.e. the representability of W by a quasi-projective k–variety ([Gro1, § 4], [B, § 2], or [Ni, § 5.5]). The only relevant case for this lecture is the case where Q' is finite over k'. In that case, the representability and the projectivity of $V^{[d]}$ are easy according to the further Lemma 4.1.4.

We remind to the reader that we have a morphism of k–functors

$$u = u_{Q'/k'/k} : W \; \to \; V^{[d]}$$

defined as follows: if S is a k-scheme, we associate to a point $w \in W(S)$ the k'-morphism $w' : S \times_k k' \to Q'$. Its graph

$$\Gamma_{w'} \subset Q' \times_{k'} (S \times_k k') = V \times_k S$$

is a closed S-subscheme of $V \times_k S$, isomorphic to $S \times_k k'$ hence finite locally free of rank d over S: this is the wished point $u_{Q'/k'/k}(S)(w) \in V^{[d]}(S)$.

4.1.2. **Lemma.** *(1) For each k-scheme S, the map $u_{Q'/k'/k}(S)$ above induces a bijection of $W(S)$ on the set of $Z \subset Q' \times_k S$ such that the S-morphism φ_Z defined in (4.1.1) is an isomorphism.*

(2) The morphism of k–functors $u : W \to V^{[d]}$ is representable by an open immersion.

[8] It is representable [B-L-R, §7.6, th. 4]; however we will apply it only in the affine case.

(3) If k' is a field, u induces a bijection of $W(k)$ on $V^{[d]}(k)$.

Proof. (1) Let $Z \subset Q' \times_k S$ be a subscheme such that φ_Z is an isomorphism. It defines then a section of $Q' \times_k S = V \times_k S \to k' \times_k S$, i.e. a point of $W(S)$. The statement follows.

(2) Let us show that the monomorphism u is an open immersion. Since a smooth monomorphism is an open immersion [EGA4, 17.9.1], it is enough to check the smoothness by the lifting criterion. Let S be a k–algebra equipped with an ideal I satisfying $I^2 = 0$. We consider the commutative diagram

$$\begin{tikzcd} W(S) \arrow[r, "u"] \arrow[d] & V^{[d]}(S) \arrow[d] \\ W(S/I) \arrow[r, "u"] & V^{[d]}(S/I). \end{tikzcd}$$

and we are given $w \in W(S/I)$ and $[Z] \in V^{[d]}(S)$ such that $[Z]_{S/I} = u(w) \in V^{[d]}(S/I)$. The S-morphism $\varphi_Z : Z \to k' \times_k S$ is such that $\varphi_Z \times_S S/I$ is an isomorphism. Nakayama lemma shows that φ_Z is an isomorphism so defines a point $\widetilde{w} \in W(S)$ such that $u(\widetilde{w}) = [Z]$. Since u is a monomorphism, $\widetilde{w}$ maps to w. Thus u is smooth.

(3) We assume that k' is a field. Let $[Z] \in V^{[d]}(k)$. The morphism $\varphi_Z : Z \to \operatorname{Spec}(k')$ is a k-morphism so is not trivial. Then $k[Z]$ is a k'-algebra of dimension 1, whece $k' = k[Z]$. The morphism φ_Z is an isomorphism so that $[Z] = u(w)$ for $w \in W(k)$. $\square$

We assume furthermore that an affine algebraic k-group acts (leftly) on V *by k'-automorphisms*, it means that $\sigma : V \to \operatorname{Spec}(k')$ is G-invariant. [This is the same data that an action of the k'-group $G_{k'}$ over Q'].

One deduces formally actions of G on the whole picture. The following statement permits to control orbits at infinity.

4.1.3. **Theorem.** *Under the preceding assumptions, let K be an extension of k and let $J \subset V_K^{[d]}$ a K-orbit for the action of G_K. If $K \otimes_k k'$ is a field, we have $J \subset W_K$.*

Proof. Up to replace k' by $K \otimes_k k'$, Q' by Q'_K, etc., we can assume that $K = k$ and that k' is a field. The map $y : J \to V^{[d]}$ defines a closed k–subscheme Z of $Q' \times_k J$, finite, locally free of rank d over J, and G-stable. According to Lemma 4.1.2.(1), we have to establish that the J-morphism

$$\varphi_Z : Z \longrightarrow k' \times_k J$$

defined in (4.1.1) is an isomorphism. It arises with the G-equivariant morphism

$$\psi_Z : k' \otimes_k \mathcal{O}_J \longrightarrow \mathcal{A} := \mathrm{pr}_{2*}(\mathcal{O}_Z)$$

of finite $\mathcal{O}_J$-algebras which are locallly free of rank d, and which are linearized for the G-action of G on J. In particular, the cokernel $\mathcal{C}$ of ψ_Z is a coherent $\mathcal{O}_J$-module which is G-linearized. For each $s \in \mathbb{N}$, we appeal to the

Fitting stratum $F_s(\mathcal{C}) \subset J$: it is a locally closed k–subscheme of J, such that the restriction of $\mathcal{C}$ to $F_s(\mathcal{C})$ is locally free of rank s [GW, §11.8]. Since $\mathcal{C}$ is G-linearized, each $F_s(\mathcal{C})$ is stable under G. But G acts transitively on J, it follows that there exists r such that $F_s(\mathcal{C}) = \emptyset$ for all $s \neq r$ and $F_r(\mathcal{C}) = J$. We conclude that $\mathcal{C}$ is *locally free of rank* r.

It remains to see that $r = 0$. Since $\mathcal{C}$ is locally free of rank r, the image of ψ_Z is a quotient algebra of $k' \otimes_k \mathcal{O}_J$, locally free of rank $d - r$, so defines a k–subscheme $T \subset k' \times_k J$, which is finite locally free on J of rank $d - r$ and G-invariant. Since G acts transitively on J and trivially on $\mathrm{Spec}(k')$, T comes of a k-subscheme T_0 of rank $d - r$ of $\mathrm{Spec}(k')$. But k' is a field, so that $T_0 = \emptyset$ and $r = d$, either $T_0 = \mathrm{Spec}(k')$ and $r = 0$. The first case is excluded since ψ_Z is a morphism of non-trivial algebras. Thus $r = 0$. □

4.1.4. **Lemma.** [G-G-MB, 2.5.5] *Let Y be a finite k-scheme and $d \in \mathbb{N}$. Then the functor $Y^{[d]} := \mathrm{Hilb}^d_{Y/k}$ is representable by a projective k-scheme equipped with an ample $\underline{\mathrm{Aut}}_k(Y)$-linearized line bundle.*

4.2. A useful fact.

4.2.1. **Lemma.** [G-G-MB, 5.1.1] *Let G be an affine algebraic k–group acting on an k-variety X. Let $Y \subseteq X$ be a k–subscheme of X stable under the action of G. Then the schematic adherence $\overline{Y}$ of Y in X is stable under G.*

Proof. We consider the following commutative diagram

$$\begin{array}{ccc} G \times_k Y & \subset & G \times_k \overline{Y} \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \alpha} \\ Y & \subset & X \end{array}$$

where the maps α and β arise from the action of G on X. We want to show that α factorizes by $\overline{Y}$ The preimage $\alpha^{-1}(\overline{Y})$ is a closed subscheme of $G \times \overline{Y}$ which contains $G \times_k Y$. On the other hand, $G \times_k Y$ is schematically dense in $G \times_k \overline{Y}$. Thus $\alpha^{-1}(\overline{Y}) = G \times_k \overline{Y}$. □

4.3. Good compactifications of homogeneous spaces.

4.3.1. **Definition.** Let X be a k-variety equipped with a left action of an affine algebraic k–group G. A G-equivariant *compactification* of X is an open immersion $j : X \to X^c$ which is G-equivariant where X^c is a proper k–variety equipped with a left action of G.

The points and subschemes of $X^c \setminus X$ are called "at infinity"; note that we do not require X to be dense in X^c.

4.3.2. **Definition.** Let G be an affine algebraic k–group and let H be a closed k–subgroup of G. Let J be an affine algebraic k–group acting on G which normalizes H. An equivariant $G \rtimes_k J$-compactification $j : X \to X^c$ of $X = G/H$ is *good* if it satisfies the following requirements:

(i) X^c admits an ample $G \rtimes_k J$-linearized line bundle;

(ii) $G(k_s)$ acts transitively on $X^c(k_s)$.

It is *very good* if there is no k_s-orbit at infinity.

4.3.3. **Theorem.** *(Gabber, see [G-G-MB, th. 1.2]) Let G be an affine algebraic k-group equipped with an action of a smooth affine algebraic k–group J. Let $X = G/G^\dagger$ be the quotient of G by its largest smooth k–subgroup.*

(1) X admits a good $G \rtimes J$-equivariant compactification.

(2) We assume furthermore the following condition:

(**) for each $G^\dagger \subset H \subset G$, any H-immersion $H/G^\dagger \hookrightarrow Y$ in an affine H–variety Y is a closed embedding.

Then X admits a very good compactification.

We shall discuss later that closedness condition (**); it is related to condition (*) of Gabber.

4.3.4. **Example.** Assume that k is imperfect of characteristic $p > 0$ and pick $a \in k \backslash k^p$. We consider the commutative unipotent k-group $G \subset \mathbb{G}_{a,k}^2$ defined by the equation $x^p - ay^p = 0$. Since $G^\dagger(k_s) = 0$, we have $G^\dagger = 0$. The result provides then a good compactification of G itself. There is another nice way (involving weighted projective spaces) to construct a good compactification of that unipotent k–group, see [B-L-R, §10, prop. 11].

The proof of the Theorem is based on an induction process and we shall start by the terminal step.

We put $H = G^\dagger$. Let k' be the field of definition of the smooth $\overline{k}$–group $(G_{\overline{k}})_{red}$ and put $d = [k' : k]$. We consider the closed immersion

$$j : G \hookrightarrow \prod_{k'/k} G_{k'}$$

and define $\widetilde{H} = G \cap \prod_{k'/k} (G_{k'})_{red}$. We have $H \subseteq \widetilde{H} \subset G$ so that $H(k_s) = \widetilde{H}(k_s)$. We observe that J acts on $\widetilde{H}$.

4.3.5. **Lemma.** *We have $H = \widetilde{H}$ if and only if G is smooth.*

Proof. If G is smooth, then $H = G$, so that $\widetilde{H} = G$. Conversely, we assume that $\widetilde{H} = G$, so that we have a commutative diagram

$$\begin{tikzcd} & \prod_{k'/k} G_{k',\mathrm{red}} \arrow[d, hook] \\ G \arrow[ur, dashed] \arrow[r, hook] & \prod_{k'/k} G_{k'}. \end{tikzcd}$$

We base change by k'/k and get the commutative diagram

$$\begin{tikzcd}
& \left(\prod_{k'/k} G_{k',\mathrm{red}}\right)_{k'} \arrow[r, "q_1"] \arrow[d, hook] & G_{k',\mathrm{red}} \arrow[d, hook] \\
G_{k'} \arrow[ur, dashed] \arrow[r, hook] & \left(\prod_{k'/k} G_{k'}\right)_{k'} \arrow[r, "q"] & G_{k'}
\end{tikzcd}$$

where q and q_1 are the adjunction morphisms [CGP, A.5.7]. We observe that the composition of the bottom maps are the identity of $G_{k'}$ it follows that $\mathrm{id}_{G_{k'}}$ factorizes through the inclusion of $G_{k',\mathrm{red}}$ in $G_{k'}$. Thus $G_{k',\mathrm{red}} = G_{k'}$ and G is smooth. $\square$

We shall compactify $G/\widetilde{H}$ which is a closed k-subscheme of the Weil restriction $W := \prod_{k'/k}\big(G_{k'}/(G_{k'})_{red}\big)$. This is an affine k-scheme since the k'-scheme $Q' = G_{k'}/(G_{k'})_{red}$ is affine. Actually Q' is the spectrum of a local k'–algebra A' of residual field k' [DG, III.3.6.4]. We denote by V the underlying k-scheme, that is Q' seen as a k-scheme.

We denote by $V^{[d]} = \mathrm{Hilb}^d_{V/k}$ the Hilbert scheme of finite subschemes of V of length $d = [k' : k]$. Since V is finite over k, $V^{[d]}$ is projective and admits an ample $G \rtimes J$–linearized line bundle. We have an open immersion

$$u : W \to V^{[d]}$$

which is $G \rtimes_k J$-equivariant. It maps the point $s \in W(k) = Q(k')$ to the reduced closed k-subscheme of V which is of degree $d = [k' : k]$.

The k-group G acts on u and the stabilizer of the point $s \in W(k)$ is $\widetilde{H}$. According to Proposition 2.10.1, we have an immersion

$$i : X = G/\widetilde{H} \hookrightarrow V^{[d]}$$

Now let X^c be the schematic closure of X, it is an equivariant $G \rtimes J$-compactification of X. The pull-back of the ample linearized bundle of $V^{[d]}$ defines an ample $G \rtimes_k J$-linearized bundle. Lemma 4.1.4.(3) implies that $X(k_s) = X^c(k_s) = \{s\}$. Thus X^c is a good $G \rtimes_k J$–compactification of X.

If $\widetilde{H} = G$ or $\widetilde{H} = H$, we have then proven Theorem 4.3.3. Else we can assume by induction that we can compactify $\widetilde{H}/H$. There is a natural way to construct a good (resp. very good) compactification of G/H from good (resp. very good) compactifications of G/H and $\widetilde{H}/H$, see [B-L-R, §10.2] (or [G-G-MB, 5.3.3]).

The proof of (2) is based on Theorem 4.1.3. Our assumption implies that $X = G/G^\dagger$ is closed in the affine G-variety W so that $X = X^c \cap W$. We are given a k-orbit $I \subset X^c$ and the quoted statement implies that $Z \subset X^c \cap W = X$.

This statement is a special case of a much general result on compactifications announced by Gabber [Ga].

5. Lecture 5: Topological henselian fields

5.1. **Topological rings.** Let R be a topological Hausdorff ring (unital commutative). Basic examples are local fields, rings of p–adic integers, rings of continuous functions of manifolds.

5.1.1. **Proposition.** [Co, prop. 2.1] *(1) There is a unique way to topologize $X(R)$ for affine finite type R-schemes X in a manner that is functorial in X, compatible with the formation of fiber products, carries closed immersions to closed embeddings, and for $X = \mathrm{Spec}(R[t])$ gives $X(R) = R$ its usual topology. Explicitly, if A is the coordinate ring of X then $X(R)$ has the weakest topology relative to which all maps $X(R) \to R$ induced by elements of A are continuous.*

(2) If R is locally compact, then $X(R)_{\mathrm{Top}}$ is locally compact.

Proof. (1) To see uniqueness, we pick a closed immersion $i : X \to \mathrm{Spec}(R[t_1, \dots, t_n])$. By forming the induced map on R-points and using compatibility with products (view affine n-space as product of n copies of the affine line), as well as the assumption on closed immersions, the induced set map $X(R)_{\mathrm{Top}} \to R^n$ is a topological embedding into R^n endowed with its usual topology. This proves the uniqueness. For the existence, we pick an R-algebra isomorphism $A = \Gamma(X, O_X) = R[t_1, \dots, t_n]/I$ for an ideal I, and identify $X(R)$ with the subset of R^n on which the elements of I (viewed as functions $R^n \to R$) all vanish. We wish to endow $X(R)$ with the subspace topology, and the main issue is to check that this construction is independent of the choice of the presentation of I and enjoys all of the desired properties. We claim that the topology defined is the same as the subspace topology defined by the canonical injection $X(R) \hookrightarrow R^A$. Let $a_1, \dots, a_n \in A$ the respective images of $t_1, \dots, t_n$. The injection $X(R) \hookrightarrow R^n$ is the composition of the natural injection $X(R) \hookrightarrow R^A$ and the map $R^A \to R^n$ defined by $(a_1, \dots, a_n) \in A^n$. Hence, every open set in $X(R)$ is induced by an open set in R^A because $R^A \to R^n$ is continuous. Since every element of A is an R-polynomial in $a_1, \dots, a_n$ and R is a topological ring (so polynomial functions $R^n \to R$ over R are continuous), it follows that the map $X(R) \to R^A$ is also continuous. Thus, indeed $X(R)$ has been given the subspace topology from R^A, so the topology on $X(R)$ is clearly well-defined and functorial in X.

Consider a closed immersion $Y \subset X$. Since $Y(R)_{\mathrm{Top}}$ is closed in R^n, it is closed in $X(R)_{\mathrm{Top}}$.

Finally, we are given maps $X \to Y$ and $Z \to Y$ between affine R-schemes of finite type. It gives rise to a continuous bijection

$$X(R)_{\mathrm{Top}} \times_{Y(R)_{\mathrm{Top}}} Z(R)_{\mathrm{Top}} \xrightarrow{\sim} \big(X \times_Y Z\big)(R)_{\mathrm{Top}}.$$

The left handside embeds topologically in $X(R)_{\mathrm{Top}} \times Z(R)_{\mathrm{Top}}$ when the right handside embeds topologically $\big(X \times_R Z\big)(R)_{\mathrm{Top}}$. But the compatibility holds for absolute products so that the continuous bijection above is a topological embedding whence is a homeomorphism.

(2) is clear. $\square$

5.1.2. **Remark.** (a) The Hausdorff property is necessary to require if we want closed immersions to go over to closed embeddings. Indeed, by considering the origin in the affine line we see that such a topological property forces the identity point in R to be closed.

(b) For establishing uniqueness, we can also add two closed embeddings of X in an affine space. The advantage of Conrad's proof is to provide in the same time the functoriality.

5.1.3. **Example.** Consider the open R-subscheme $\mathbb{G}_m \subset \mathrm{A}^1_R$. It is the closed R–subscheme of $\mathbf{A}^2_R$ defined by $xy = 1$. In other words, $R^\times$ acquires the topology of the hyperbola $xy = 1$ inside R^2. In particular the map $(R^\times)_{\mathrm{Top}} \to (R^\times)_{\mathrm{Top}}$, $x \mapsto x^{-1}$ is continuous.

On the other hand, we can consider the topology Top' induced by the embedding $R^\times \subset R$. Then the map $(R^\times)_{\mathrm{Top}} \to (R^\times)_{Top'}$ is continuous and is an homeomorphism if and only if the inversion is continuous.

This is not the case in general. Counter-examples are rings of adeles and certain rings from analysis.

5.1.4. **Proposition.** [Co, prop. 3.1] *We assume that R is local, that $R^\times$ is open in R and has continuous inversion.*

(1) Let $i : U \to X$ be an open immersion between affine R–schemes of finite type. Then $U(R)_{\mathrm{Top}} \to X(R)_{\mathrm{Top}}$ is an open embedding.

(2) There is a unique way to topologize $X(R)$ for separated R-schemes X of finite type in a manner that is functorial in X, compatible with the formation of fiber products, carries closed immersions to closed embeddings, and for $X = \mathrm{Spec}(R[t])$ gives $X(R) = R$ its usual topology. Furthermore if X is separated, then $X(R)_{\mathrm{Top}}$ is Hausdorff.

(3) If $i : U \to X$ be an open immersion between affine R–schemes of finite type, then $U(R)_{\mathrm{Top}} \to X(R)_{\mathrm{Top}}$ is an open embedding.

Proof. (1) We start with the case of a basic open subset $U = X_f$ for $f \in A = \Gamma(X, O_X)$. We consider the cartesian diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & \mathbb{G}_{m,R} \arrow[d] \\ X \arrow[r, "u"] & \mathbb{A}^1_R . \end{tikzcd}$$

The compatibility with fiber products reduces the problem to the special case $\mathbb{G}_{m,R} \subset \mathbb{A}^1_R$. In this case the topology on $R^\times$ acquires the topology of the hyperbola $xy = 1$ inside R^2 and this is homeomorphic to the induced topology on $R^\times$ under our assumptions.

In the general case, $U = U_1 \cup \dots U_t$ where U_i is a basic affine subset of X. Since R is local we have $U(R) = U_1(R) \cup \dots \cup U_t(R)$ so that $U(R)_{\mathrm{Top}}$ is open in $X(R)_{\mathrm{Top}}$.

(2) Let X be R–scheme of finite type and let $X = U_1 \cup \dots \cup U_r$ be a Zariski cover by affine R-schemes. Since $X(R) = U_1(R) \cup \dots \cup U_r(R)$, the topology on each $U_i(R)$ defines a topology on $X(R)$. By taking two covers and their refinement, (1) yields that the topology on $X(R)$ does not depend of the cover. The functorial properties are left to check to the reader.

We assume that X is separated, that is $\Delta : X \to X \times_R X$ is a closed immersion. It follows that $\Delta_{\mathrm{Top}} : X(R)_{\mathrm{Top}} \to X(R)_{\mathrm{Top}}$ is a closed topological embedding. Thus $X(R)_{\mathrm{Top}}$ is a separated topological space.

(3) The $U_i \cap U$ form an open cover of U. Since U_i is affine, (1) shows that $(U_i \cap U)(R)_{\mathrm{Top}}$ is open in $U_i(R)_{\mathrm{Top}}$ so is open in $X(F)$. Since $U(R) = \bigcup (U_i \cap U)(R)$, we conclude that $U(R)_{\mathrm{Top}}$ is open in $X(R)_{\mathrm{Top}}$. $\square$

5.2. **Ultra-paracompacity.** We recall that a topological space X is *ultra-paracompact* if each open covering $(U_i)_{i\in I}$ of X admits a refinement $(V_j)_{j\in J}$ such that $X = \bigsqcup_{j\in J} V_j$.

5.2.1. **Proposition.** *Let K be a valued field (for a non-trivial valuation v). Let X be a K-variety. Let Ω be an open subset of $X(K)_{\mathrm{Top}}$. Let (Ω_α) be a open covering of Ω.*

(1) Ω admits a basis of open neighboroods $\mathcal{B} = (\mathcal{B}_\lambda)_{\lambda\in\Lambda}$ finer than (Ω_α) such that for all $\lambda, \lambda' \in \Lambda$, satisfying $\mathcal{B}_\lambda \cap \mathcal{B}_{\lambda'} \neq \emptyset$, we have $\mathcal{B}_\lambda \subseteq \mathcal{B}_{\lambda'}$ or $\mathcal{B}_{\lambda'} \subseteq \mathcal{B}_\lambda$.

(2) Each subset of $X(K)_{\mathrm{Top}}$ is ultra-paracompact.

The terminology for (1) is that Ω admits a non-archimedean basis of open neighboroods [Mo, déf. 5]. This occurs for closed balls of K^n. For $\gamma \in \Gamma$, we put $\mathcal{B}(x,\gamma) = \{ w \in K^n \mid v(w_l - x_l) \geq \gamma \,\text{for}\, l = 1,...,n_i \}$. Assume that $\mathcal{B}(x,\gamma) \cap \mathcal{B}(x',\gamma') \neq \emptyset$, that $\gamma' \leq \gamma$ and pick $z \in \mathcal{B}(x,\gamma) \cap \mathcal{B}(x',\gamma')$. We claim that $\mathcal{B}(x,\gamma) \subset \mathcal{B}(x',\gamma')$. If $y \in \mathcal{B}(x,\gamma)$, we write $y - x' = (y-x) + (x-z) + (z-x')$. It follows that $v(y_i - x'_i) \geq \mathrm{Inf}\big(v(y_i - x_i), v(x_i - z_i), v(z_i - x'_i)\big) \geq \mathrm{Inf}(\gamma,\gamma,\gamma') = \gamma'$ for $i = 1,...,n$ so that $y \in \mathcal{B}(x',\gamma')$.

The general case is based on the use of integral models, see [G-MB, Appendice].

5.3. **Topologically henselian fields.** Let F be an Hausdorff topological field.

5.3.1. **Definition.** The field F is *topologically henselian* if for each étale map $X \to Y$ between F–varieties, the induced map $f_{\mathrm{Top}} : X(F)_{\mathrm{Top}} \to Y(F)_{\mathrm{Top}}$ is a local homeomorphism.

Basic examples are of course $\mathbb{R}$ and $\mathbb{C}$ but also their subfields $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}} \cap \mathbb{R}$.

5.3.2. **Lemma.** *Assume that F is topologically henselian. Let $f : X \to Y$ be a smooth map between F–varieties.*

(1) Let $\Omega \subset Y(F)_{\mathrm{Top}}$ be the image of f_{Top}. Then the induced map $X(F)_{\mathrm{Top}} \to \Omega$ admits locally continuous sections.

(2) The map $f_{\mathrm{Top}} : X(F)_{\mathrm{Top}} \to Y(F)_{\mathrm{Top}}$ is open.

Proof. (1) The ultra-paracompactness property reduces to show the existence of sections locally on I.

Let $x \in X(F)$ with image $y = f(x) \in Y(F)$. According to [B-L-R, §2.2, prop. 14], there exists a morphism $h : Z \to X$, a point $z \in Z(F)$ such that $h(z) = x$ such that $f \circ h : Z \to X$ is étale. The diagram

$$\begin{tikzcd} & Z \arrow[dl, "h"'] \arrow[d, "f\circ h"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

gives rise to

$$\begin{tikzcd} & Z(F)_{\mathrm{Top}} \arrow[dl, "h_{\mathrm{Top}}"'] \arrow[d, "F"] \\ X \arrow[r, "f_{\mathrm{Top}}"] & Y(F)_{\mathrm{Top}}. \end{tikzcd}$$

Our assumption implies that $(f \circ h)_{\mathrm{Top}}$ is a local homeomorphism. There exist an open neighborood V_y of y and a continuous $s_y : V_y \to Z(F)_{\mathrm{Top}}$. It follows that $h_{\mathrm{Top}} \circ s_y : V_y \to X(F)_{\mathrm{Top}}$ is a continuous section of f_{Top}.

(2) It is a consequence of (1). □

5.3.3. **Remark.** In the statement, if F is furthermore a valued field, the ultra-paracompactness property (Remark 5.3.3) implies that $X(K)_{\mathrm{Top}} \to \Omega$ admits a continuous sections.

We remind to the reader the following definition.

5.3.4. **Definition.** Let $(R, \mathfrak{m}, \kappa)$ be a local ring. We say R is *henselian* if for every monic $f \in R[T]$ and every root a_0 of f such that $f'(a_0) \in \kappa^\times$ there exists an $a \in R$ such that $f(a) = 0$ and $\overline{a} = a_0$.

An important point is that such a root a of f is unique. This is clear if R is integral and the general case goes as follows. We write $f(x+y) - f(x) = f'(x)y + g(x,y)y^2$ in $R[x,y]$. Let b another root of f such that $\overline{b} = a_0$. We have $0 = f(b) - f(a) = f(a + (b-a)) - f(a) = f'(a)(b-a) + c(b-a)^2$ for some $c \in R$. By assumption $f'(a)$ is a unit in R. Hence $(b-a)(1 + f'(a)^{-1}c(b-a)) = 0$. By assumption $b - a \in \mathfrak{m}$, hence $1 + f'(a)^{-1}c(b-a)$ is a unit in R so that $ba = 0$ in R.

There are several equivalent definitions, see [Stacks, Tag 04GE]. Here we shall use only that we can remove the “monic” assumption in the definition (and the unicity argument holds as well).

Complete discretly valued fields are henselian (Newton's method [Wa1, th. 32.11]) and also henselizations of local rings. Another examples are iterated Laurent fields $k((t_1))((t_2))\dots((t_n))$ [Wd, §3]. The next statement justifies the terminology.

5.3.5. **Proposition.** *Let F be an henselian valued field, i.e. its valuation ring O_v is henselian. Then F is topologically henselian.*

This applies in particular to p-adic fields.

Proof. We are given an étale morphism $f : X \to Y$ between F-varieties. Let $x \in X(F)$ and put $y = f(x)$. Without loss of generality we can assume that $Y = \mathrm{Spec}(A)$ is affine and that X is an open subset of $\mathrm{Spec}(A[t]/P(t))$ where P is a monic separable polynomial such that $P'(x) \in F^\times$ [B-L-R, §2.3, prop. 3]. We can even assume that $X = \mathrm{Spec}(A[t]/P(t))$.

We embed Y is an affine K–space $\widetilde{Y} = \mathbf{A}^n_F = \mathrm{Spec}(\widetilde{A})$ and lift P in an unitary polynomial $\widetilde{P}(t) \in \widetilde{A}[t]$. We claim that we can replacing Y by $\widetilde{Y}$ and P by $\widetilde{P}$. We have indeed a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & \mathrm{Spec}(\widetilde{A}[t]/\widetilde{P}(t)) = \widetilde{X} \arrow[d, "\widetilde{f}"] \\ Y \arrow[r] & \widetilde{Y} = \mathrm{Spec}(\widetilde{A}) \end{tikzcd}$$

where the horizontal map are closed immersions. It gives rise to the following cartesian diagram of topological spaces

$$\begin{tikzcd} X(F)_{\mathrm{Top}} \arrow[r] \arrow[d, "f"] & \widetilde{X}(F)_{\mathrm{Top}} \arrow[d, "\widetilde{f}"] \\ Y(F)_{\mathrm{Top}} \arrow[r] & \widetilde{Y}(F)_{\mathrm{Top}}. \end{tikzcd}$$

If $\widetilde{f}_{\mathrm{Top}}$ is a local isomorphism at x, so is f_{Top} by base change. We can then assume that $Y = \mathbf{A}^n_F = \mathrm{Spec}\, F[\underline{Z}] = \mathrm{Spec}\, F[Z_1, \dots, Z_n]$, $X = \mathrm{Spec}\,(F[\underline{Z}, T]/(P))$ where P is monic in T, and where y (resp. x) is the origin of $\mathbf{A}^n_F$ (resp. of $\mathbf{A}^{n+1}_F$). Furthermore the projection $X \to Y$ is étale at x. We can choose then coordinates such that the tangent hyperplane at x in X is given by the equation $T = 0$, so that P is (up scaling) of the shape

$$P(\underline{Z}, T) = T + \sum_{|I|+j \geq 2} a_{I,j}\, \underline{Z}^I T^j \qquad (a_{I,j} \in F).$$

Let R be the valuation ring of v and denote by $\mathfrak{m}$ its maximal ideal. For each $\alpha \in F^\times$, we may replace P by $P_\alpha(\underline{Z}, T) := \frac{1}{\alpha} P(\alpha \underline{Z}, \alpha T)$. The coefficient of $\underline{Z}^I T^j$ is P_α is $\alpha^{|I|+j-1} a_{I,j}$: in that formula the exposant of α is > 0. Taking α close enough of 0, we can assume that the coefficients $a_{I,j}$ belong to $\mathfrak{m}$. The Hensel property shows that for each $\underline{z} \in R^n$, the polynomial $P(\underline{z}, T) \in R[T]$ admits a unique root $t(\underline{z})$ in $\mathfrak{m}$. In other words, f_{Top} induces a bijection between ${f_{\mathrm{Top}}}^{-1}(R^n) \cap (R^n \times \mathfrak{m})$ (neighborood of x in $X_{{}_{\mathrm{Top}}}$) and R^n (neighborood of y in F^n).

It remains to see that the map $\underline{z} \mapsto t(\underline{z})$ is continous at the origin. This is enough to establish the inequality

$$v(t(\underline{z})) \geq \min_{i=1,\dots,n} v(z_i).$$

In other words, we have to show for $\underline{z} = (z_1, \dots, z_n) \in R^n$ and $t \in \mathfrak{m}$ satisfying

$$v(t) < \min_{i=1,\dots,n} v(z_i),$$

(note that it implies that $t \neq 0$), then $P(\underline{z}, t) \neq 0$. This is enough to observe that each term $\underline{z}^I \, t^j$, for $|I| + j \geq 2$ has valuation $> v(t)$ (recall that $v(a_{I,j}) > 0$). This holds if $|I| \geq 1$ from the assumption on the $v(z_i)$, and it holds also for $|I| = 0$: in this case we have $j \geq 2$ and $t \in \mathfrak{m}$. $\square$

Together with Remark 5.3.3, Proposition 5.3.5 has the following nice consequence.

5.3.6. **Corollary.** *Assume that F is an henselian valued field. Let $f : X \to Y$ be a smooth morphism and denote by $\Omega = f(X(F))$. Then the induced map $X(F)_{\mathrm{Top}} \to \Omega$ admits a continuous section.*

5.4. Application to torsors.

5.4.1. **Proposition.** *We assume that F is a topologically henselian field. Let $f : X \to Y$ be a morphism between of F–varieties which is a G-torsor for a smooth affine algebraic F-group G.*

(1) The map $f_{\mathrm{Top}} : X(F)_{\mathrm{Top}} \to Y(F)_{\mathrm{Top}}$ is open.

(2) The characteristic map $\varphi : Y(F) \to H^1(F, G)$ is locally constant.

(3) Let I the image of f_{Top}. Then I is clopen and the induced map $f_{\mathrm{Top}} : X(F)_{\mathrm{Top}} \to I$ is a principal $G(K)_{\mathrm{Top}}$-fibration.

Proof. (1) The map f is smooth so f_{Top} is open.

(2) We need to show that the non-trivial fibers of φ are open in $Y(F)_{\mathrm{Top}}$. This is the case for $I = \varphi^{-1}(1)$. We are given $c = \varphi(y_0)$ for $y_0 \in Y(F)$. We consider the G–torsor $E = E_{y_0} =$. If $y' \in Y(F)$ we have $\varphi(y') = c$ if and only if the G–torsors E and E_y are isomorphic. This condition is equivalent to require that the ${}^E G$-torsor $\mathrm{Isom}_G(E, E_{y'})$ is trivial where ${}^E G$ stands for the twisted F–group. This rephrases to say that the ${}^E G$-torsor $P = \mathrm{Isom}_G(E_Y, E)$ over Y satisfies $P_y(k) \neq 0$.

Since the twisted F-group ${}^E G$ is smooth as well, the preceding case enables us to conclude that $\varphi^{-1}(c)$ is open in $Y(F)_{\mathrm{Top}}$.

(3) We have $I = \varphi^{-1}(1)$, so I is clopen in $Y(F)_{\mathrm{Top}}$. The induced map admits local sections, so is a principal $G(K)_{\mathrm{Top}}$-fibration.

$\square$

5.4.2. **Remark.** If (K, v) is an henselian valued field, we can say more. Since $X(F)_{\mathrm{Top}} \to I$ admits locally sections, the paracompactness property yields that $X(F)_{\mathrm{Top}} \to I$ admits a continuous section. The principal fibration is then trivial.

5.4.3. **Corollary.** *We assume that F is a topologically henselian field. Let G be an affine algebraic F–group acting on a F–variety. Let $x \in X(F)$ such that the stabilizer G_x is smooth.*

(1) The orbit $G(F).x$ *is open in* $X(F)_{\mathrm{Top}}$.

(2) If $X \cong G/G_x$, *then* $G(F).x$ *is clopen in* $X(F)_{\mathrm{Top}}$ *and the map* $G(F) \to G(F).x$ *is a principal* $G(K)_{\mathrm{Top}}$*-fibration.*

Note that it applies to group morphisms $G \to G/N$ whenever N is a smooth normal F–subgroup of G.

6. Lecture 6: Using integral models

6.1. **Integral models.** We are given a valued field (K, v) and denote by A its valuation ring (not necessarily noetherian). We recall the following fact. If X is a K–variety, then there exists a separated A–scheme $\mathfrak{X}$ of finite presentation such that $\mathfrak{X} \times_A K \cong X$. Such a A-scheme $\mathfrak{X}$ is called an *integral model* of X.

If X is affine and embedded in $\mathbf{A}^n_K$, it is enough take the schematic adherence of We can do that also in the projective case but the general case is an application of Nagata's compactification theorem [GW, th. 12.70]. We compactify X in a proper K-variety X^c and write $K = A_f$ with $f \in \mathfrak{m}$. We apply Nagata's theorem to the map $X^c \to \operatorname{Spec}(A_f) \to \operatorname{Spec}(A)$. There exists a proper A–scheme $\mathfrak{X}^c$ such that X^c is isomorphic to an open schematically dense subscheme of $\mathfrak{X}^c$. Then X^c is open dense in $\mathfrak{X}^c_K$ so we have $X^c = \mathfrak{X}^c_K$. Finally $\mathfrak{X} = \mathfrak{X}^c \setminus \mathfrak{Z}$ where $\mathfrak{Z}$ stands for the schematic adherence of $Z = X^c \setminus X$.

6.1.1. **Lemma.** *Let $\mathfrak{X}$ be an A-model of the K–variety X.*

(1) The map $\mathfrak{X}(A)_{\mathrm{Top}} \to X(K)_{\mathrm{Top}}$ is topological open embedding.

(2) If $\mathfrak{X}$ is proper, then $\mathfrak{X}(A)_{\mathrm{Top}} \to X(K)_{\mathrm{Top}}$ is a homeomorphism.

Proof. (1) Since $\mathfrak{X}$ is separated, the map $\mathfrak{X}(A) \to X(K)$ is injective.

Reduction to the affine case. We assume that the statement is known in the affine case. Let $\mathfrak{U}_1, \dots, \mathfrak{U}_n$ be an affine covering of $\mathfrak{X}$. Our assumption implies that $\mathfrak{U}_i(A)_{\mathrm{Top}} \to U_i(K)_{\mathrm{Top}}$ is a topological embedding and so is $\mathfrak{U}_i(A)_{\mathrm{Top}} \to X(K)_{\mathrm{Top}}$. Since the $U_i(K)$(s form an open cover of $X(K)_{\mathrm{Top}}$ and $\mathfrak{U}_i(A) = \mathfrak{X}(A) \cap \mathfrak{U}_i(K)$, we conclude that $\mathfrak{X}(A)_{\mathrm{Top}} \to X(K)_{\mathrm{Top}}$ is a topological embedding.

We assume that $\mathfrak{X}$ is affine and consider a closed embedding Let $j : \mathfrak{X} \subset \mathbf{A}^N_A$. We have the cartesian diagram

$$\begin{tikzcd} \mathfrak{X}(A) \arrow[r, phantom, "\subset"] \arrow[d, hook] & A^N \arrow[d, hook] \\ \mathfrak{X}(K) \arrow[r, phantom, "\subset"] & K^N. \end{tikzcd}$$

We observe that the horizontal maps are topological embeddings and that the map $A^N \to K^n$ is continuous and an open embedding. It follows that $\mathfrak{X}(A)_{\mathrm{Top}} \to \mathfrak{X}(K)_{\mathrm{Top}}$ is continuous and is an open embedding.

(2) In this case we have $\mathfrak{X}(A) = \mathfrak{X}(K)$. $\square$

6.2. **The locally compact case.**

6.2.1. **Proposition.** *Assume that K is locally compact. Let X be a K-variety together with an A-model $\mathfrak{X}$.*

(1) $\mathfrak{X}(A)_{\mathrm{Top}}$ is a compact subset of $X(K)_{\mathrm{Top}}$ and the topological space $X(K)_{\mathrm{Top}}$ is locally compact.

(2) Let $f : X \to Y$ be a proper morphism between K–varieties. Then $f_{\mathrm{Top}} : X(K)_{\mathrm{Top}} \to Y(K)_{\mathrm{Top}}$ is a proper topological map.

In particular, if X is proper over K, then $X(K)_{\mathrm{Top}}$ is compact.

Proof. (1) Once again, we can assume that $\mathfrak{X}$ is affine and choose a closed A-immersion $j : \mathfrak{X} \to \mathbf{A}^n_A$. Then $\mathfrak{X}(A)$ is a closed subset of A^N so is compact. Since $X(K)$ is a closed subset of the locally compact space K^n, $X(K)_{\mathrm{Top}}$ is locally compact.

(2) We assume firstly that f is projective, that is there exists a closed immersion $i : X \to \mathbf{P}^n_Y$ such that $f = p_2 \circ i$. Since $\mathbf{P}^n(K)_{\mathrm{Top}}$ is compact, the map $(p_2)_{\mathrm{Top}} : \mathbf{P}^n(K)_{\mathrm{Top}} \times Y(K)_{\mathrm{Top}} \to Y(K)_{\mathrm{Top}}$ is proper [B:TG1, I.10.2, cor. 5]. On the other hand, i_{Top} is proper and so is f_{Top} by composition.

Since the spaces are locally compact, we can use the criterion of properness with preimages of compact [B:TG1, I.10.3, prop. 7]. In general, we shall argue by induction on $\dim(X)$ (allowing any Y), the case of dimension 0 being clear (for all Y). We may assume that X is reduced and irreducible, so by Chow's Lemma [GW, 13.100], there is a surjective projective birational K-map $h : X' \to X$ with X' a reduced and irreducible K-scheme such that X' is also projective over Y. Choose a proper closed subset $Z \subsetneq X$ such that h is an isomorphism over $X \setminus Z$. Clearly $X(K) = Z(K) \cup h(X'(K))$, and $Z(K)$ is $Y(K)$-proper since $\dim(Z) < \dim(X)$. Also, $X'(K)_{\mathrm{Top}}$ is $Y(K)$-proper and $X'(K)$-proper since X' is projective over Y and X, so the maps $Z(K)_{\mathrm{Top}} \sqcup X'(K)_{\mathrm{Top}} \to Y(K)_{\mathrm{Top}}$ and $Z(K)_{\mathrm{Top}} \sqcup X'(K)_{\mathrm{Top}} \twoheadrightarrow X(K)_{\mathrm{Top}}$ are proper. Hence, the map $X(K)_{\mathrm{Top}} \to Y(K)_{\mathrm{Top}}$ between Hausdorff spaces is proper [B:TG1, I.10.1, prop. 3]. $\square$

In particular, if $f : X \to Y$ is proper, the image $f(X(K))$ is closed in $Y(K)_{\mathrm{Top}}$.

6.3. **Completions.** We come back in the general framework of a valued field (K, v) with valuation ring A. We denote by Γ the valuation group. For each $\gamma \geq 0$, the ideal

$$\mathfrak{m}_\gamma = \bigl\{ x \in A \mid v(x) \geq \gamma \bigr\}$$

is principal and is closed in A. All finitely generated ideals of A are of this shape (in particular it is always the case when A is noetherian).

The completion $\widehat{K}$ (resp. $\widehat{A}$) of K (resp. A) is defined by means of Cauchy filters [B:AC, VII.3]. This is equivalent to the following construction. We define $\widehat{A} = \varprojlim_{\gamma > 0} A/\mathfrak{m}_\gamma$.

Let $\widehat{a}$ be a non-zero element of $\widehat{A}$, i.e. there exists $\gamma > 0$ such that its projection in $A/\mathfrak{m}_\gamma$ is non zero. If $a, a' \in A$ lift $\widehat{a} \mod m_\gamma$, we have $v(a) > \gamma$ and $v(a - a') > \gamma$ whence $v(a) \geq \mathrm{Min}(v(a'), v(a' - a)) \geq v(a')$. Similarly we have $v(a') \geq v(a)$ so that $v(a') = v(a)$. We define then $\widehat{v}(\widehat{a}) = v(a)$ for an arbitrary lift a of $\widehat{a} \mod \mathfrak{m}_\gamma$ and it is straightforward to check that is does not depend of the choice of γ. We extended then v to a valuation

$\widehat{v}$ on $\widehat{A}$ and we write $v = \widehat{v}$. The ring $\widehat{A}$ is local with maximal ideal $\widehat{m} = \{a \in \widehat{A} \mid v(a) > 0\} = \mathfrak{m}\widehat{A}$; it is reduced so is a domain. Its fraction field $\widetilde{K}$ is valued field with contains (K, v).

By construction, the ring A is dense in $\widehat{A}$ and so is K in $\widehat{K}$.

6.3.1. **Remark.** Note that we have also $\widehat{A} = \varprojlim_{J \in \mathfrak{J}} A/J$ where $\mathfrak{J}$ stands for all proper non-zero ideals of A.

6.3.2. **Examples.** (a) The completion of $k(t)$ for the valuation v_t is the Laurent serie field $k((t))$.

(b) More generally the completion $\widehat{K}$ of (K, v) when v is of rank one is henselian [Wa1, th. 32.11]. Furthermore the separable closure L of K in $\widehat{K}$ is henselian [Wa1, th. 32.18].

(c) This is not true for higher valuations [Wa1, ex. 32.3].

6.3.3. **Lemma.** [Wa1] *Let (K, v) be a henselian valued field.*

(1) $\widehat{K}$ is henselian.

(2) The field K is separably closed in $\widehat{K}$.

(3) $\widehat{K} \otimes_K K_s$ is a field which is a separable closure of $\widehat{K}$.

Assertion (1) is straightforward (by approximation of coefficients of the relevant polynomial) and for (2), (3), see [Wa1, 32.19 and 32.20].

6.3.4. **Proposition.** *Let (K, v) be a henselian valued field. Let X be a smooth K-variety. Then $X(F)$ is dense in $X(\widehat{K})_{\mathrm{Top}}$.*

Proof. Let Ω be an non-empty open subset of $X(\widehat{K})_{\mathrm{Top}}$. We want to show that $\Omega \cap X(K)$ is non-empty. We pick $x \in X(\widehat{K})$. Up to shrink X, we can assume that there exists an étale morphism $f : X \to \mathbf{A}^d_K$. Since $\widehat{K}$ is henselian, the map $f_{\mathrm{Top}} : X(\widehat{K})_{\mathrm{Top}} \to (\widehat{K})^n$ is open so that $f(\Omega)$ is open in $(\widehat{K})^n$. It follows that $f(\Omega) \cap (K)^n \neq \emptyset$. We pick $a \in f(\Omega) \cap (K)^n$. Since K is separably closed in $\widehat{K}$ and f is étale, we have that $X_a(K) = X_a(\widehat{K})$. Thus $X(K) \cap \Omega \neq \emptyset$. □

6.4. **Admissible valuation fields.** We assume that the valued field K is of characteristic $p > 0$. The Frobenius morphism $\mathrm{Fr} : \mathbb{G}_{a,K} \to \mathbb{G}_{a,K}$, $x \mapsto x^p$, is a faithfully flat morphism of K–algebraic groups whose kernel is $\mathbb{F}_p$. According to Proposition 2.4.1, Fr is then a proper map. We have

$$\mathrm{Fr}(K) = K^p \subset K.$$

The natural question to address is whether K^p is closed in K.

6.4.1. **Lemma.** *Let $\widehat{K}$ be the completion of K.*

(1) $\widehat{A}^p$ is closed in $\widehat{A}$ and $\widehat{K}^p$ is closed in $\widehat{K}$.

(2) If the completion $\widehat{K}$ is separable over K, then K^p is closed in K.

(3) If (K, v) is henselian, the following are equivalent:

(i) K^p *is closed in* K*;*

(ii) K *is algebraically closed in* $\widehat{K}$*;*

(iii) $\widehat{K}$ *is separable over* K*.*

Proof. (1) Let $b \in \widehat{A}$ an element adherent to $\widehat{A}^p$. For each $\gamma > 0$, there exists $a \in \widehat{A}$ such that $v(a^p - a) \geq p\,\gamma$. For another choice a', we have

$$p\,v(a'-a) = v((a')^p - a^p) \geq \mathrm{Min}\big(v((a')^p - b), v((a)^p - b)\big) \geq p\gamma.$$

If follows that $a - a' \in \mathfrak{m}_\gamma$ so that the image of a in $A/\mathfrak{m}_\gamma$ does not depend of the choice of the lift. We have defined a coherent family $(a_\gamma)_{\gamma>0}$ that is a point $a \in \widehat{A}$ which satisfies $a^p = b$. We have shown that $(\widehat{A})^p$ is closed in $\widehat{A}$ and so is $(\widehat{K})^p$ in $\widehat{K}$.

(2) We assume that $\widehat{K}$ is separable over K. If an element a of K is adherent to K^p, (1) implies that $a = b^p$ with $b \in \widehat{K}$. The assumption implies that $a \in K^p$.

(3) The implication $(iii) \implies (i)$ is a special case of (2).

$(i) \implies (ii)$ Let L/K a finite subextension of $\widehat{K}$. Let E be the maximal separable subextension of E/K. According to Lemma 6.4.1.(2), we have $E = K$ so that L is purely inseparable. If $K \subsetneq L$, there exists $x \in L \setminus K$ such that $x^p \in K$. Since K^p is dense and closed in $K \cap (\widehat{K})^p$, we have $K^p = K \cap (\widehat{K})^p$ whence $x \in K^p$. We conclude that $x \in K$ and that $K = L$.

$(ii) \implies (iii)$: obvious.

□

6.4.2. **Definition.** A valued field (K, v) is *admissible* if it henselian and if $\widehat{K}$ is a separable extension of K.

6.4.3. **Example.** (a) The henselian field $\mathbb{F}_p((t))$ is obviously admissible since it is complete. Let K be the algebraic closure of $\mathbb{F}_p(t)$ in $\mathbb{F}_p((t))$, this is an henselian valued field. Since $\mathbb{F}_p((t))$ is separable over $\mathbb{F}(t)$ (use for example MacLane criterion), it follows that K is admissible.

(b) (F.K. Schmidt) Since $\mathbb{F}_p((t))$ has infinite transcendence degree over $\mathbb{F}_p(t)$, we can choose some element $s \in \mathbb{F}_p[[t]]$ which is transcendental over $\mathbb{F}_p(t)$. We consider the subfields $K = \mathbb{F}_p(t, s^p)$ and $K' = \mathbb{F}_p(t, s)$ of $\mathbb{F}_p((t))$. Since $\widehat{K}$ is the adherence of K in $\mathbb{F}_p((t))$, we see that $K' \subset \widehat{K}$ so that $\widehat{K}$ is not separable over K.

Those fields are not henselian. Let K^h (resp. $(K')^h$) be the separable closure of K (resp. K') in $\mathbb{F}_p((t))$. Then K^h (resp. $(K')^h$) is henselian (and separable) over K (resp. over K') [Wa1, th. 32.18]. This shows that $\widehat{(K^h)} = \widehat{K}$ is not separable over K^h.

Admissible fields have further nice properties (e.g. [G-G-MB, 3.5.3, 4.1.1]). We record the following refinement of Proposition 6.3.4.

6.4.4. **Theorem.** *(Moret-Bailly,* [MB2, cor. 1.2.1]*) Let* (K, v) *be an admissible valued field. Let* X *be a* K*-variety. Then* $X(K)$ *is dense in* $X(\widehat{K})_{\mathrm{Top}}$*.*

6.5. **The strong approximation theorem.** Let (K, v) be an *admissible* valued field and denote by A its valuation ring.

6.5.1. **Theorem.** *(Moret-Bailly,* [MB2]*) Let* $\mathfrak{X}$ *be a separated* A*–scheme of finite presentation. Let* $\gamma \in \Gamma$*,* $\gamma > 0$*. Then there exists* $\gamma' \geq \gamma$ *such that*

$$\mathrm{Im}\Big(\mathfrak{X}(A) \to \mathfrak{X}(A/\mathfrak{m}_\gamma)\Big) = \mathrm{Im}\Big(\mathfrak{X}(A/\mathfrak{m}_{\gamma'}) \to \mathfrak{X}(A/\mathfrak{m}_\gamma)\Big).$$

This extends a result of Greenberg [Gre] which is the case when v is a discrete valuation.

6.5.2. **Corollary.** *(Infinitesimal Hasse principle) Let* $\mathfrak{X}$ *be a separated* A*–scheme of finite presentation. Then the following are equivalent:*

(i) $\mathfrak{X}(A) \neq \emptyset$*;*

(ii) $\mathfrak{X}(A/\mathfrak{m}_\gamma) \neq \emptyset$ *for all* $\gamma > 0$*.*

6.5.3. **Remark.** Example 6.4.3.(2) provides an henselian valued field K such that $K^p \subsetneq K \cap (\widehat{K})^p$ so that $A^p \subsetneq A \cap (\widehat{A})^p$. In this case, the infinitesimal Hasse principle fails and a fortiori the strong approximation theorem.

We come back to properness issues.

6.5.4. **Theorem.** *Let* $f : X \to Y$ *be a proper morphism between* K*-varieties.*
(1) (Moret-Bailly [MB2]*) The image of* f_{Top} *is closed in* $Y(K)_{\mathrm{Top}}$*.*
(2) [G-G-MB, Th. 4.2.3] *Let* $y \in Y(K)$ *such that the fiber* $C = X_y(K)_{\mathrm{Top}}$ *is compact. Then each neighborhood* Ω *of* C *in* $X(K)_{\mathrm{Top}}$ *contains a neighborhood of the shape* $f^{-1}(\Upsilon)$ *where* Υ *is a neighborhood of* y *in* $Y(K)_{\mathrm{Top}}$*.*

Proof. Both statements are local on the target so we can assume that Y is affine.
(1) We are given an element $y \in Y(K)$ which is adherent to $f(X(K))$ and want to show that $X_y(K) \neq \emptyset$. There exists an affine A–model $\mathfrak{Y}$ of Y such that $y \in \mathfrak{Y}(A) \subset Y(K)$. By using Nagata's compactification theorem, there exists an A-model $\mathfrak{X}$ of X together with a proper A-map $\mathfrak{X} \to \mathfrak{Y}$. Since f is proper, we have $\mathfrak{X}(A) = X(K) \cap f^{-1}(\mathfrak{Y}(A))$. Then y is adherent to $f(\mathfrak{X}(A))$. We put $\mathfrak{X}_y = f^{-1}(y)$.

Our assumption implies that $\mathfrak{X}_y(A/\mathfrak{m}_\gamma) \neq \emptyset$ for all $\gamma > 0$. The infinitesimal Hasse principle yields $\mathfrak{X}_y(A) \neq \emptyset$. We conclude that y belongs to $f(X(K))$.
(2) We have $C = \mathfrak{X}_y(A)$ which is assumed compact. Since $\mathfrak{Y}(A)$ is a neighborhood of $y \in Y(K)$ (Lem. 6.1.1), we can assume (up to shrink Ω) that $U \subset \mathfrak{Y}(A)$.
(2) We have $C = \mathfrak{X}_y(A)$. For each $\xi \in C$, there exists $\gamma_\xi > 0$ such that

$$B_{\mathfrak{X}}(\xi, \gamma_\xi) = \Big\{\xi' \in \mathfrak{X}(A) \mid \xi' \equiv \xi \quad \text{in} \quad \mathfrak{X}(A/\mathfrak{m}_{\gamma_\xi})\Big\} \ \subset \ \Omega.$$

But C is compact so is covered by finitely many open subsets as above. Up to shrink Ω, we can assume that there exists $\gamma > 0$ such that

(6.5.1)
$$\Omega = \bigcup_{\xi \in C} B_{\mathfrak{X}}(\xi, \gamma) = \Big\{ x \in \mathfrak{X}(A) \mid \exists \xi \in \mathfrak{X}_y(A) \quad \text{satisfying} \quad x \equiv \xi \quad \text{in} \quad \mathfrak{X}(A/\mathfrak{m}_\gamma) \Big\}.$$

The strong approximation theorem provides $\gamma' \geq \gamma$ such that

$$\mathrm{Im}\Big(\mathfrak{X}_y(A) \to \mathfrak{X}_y(A/\mathfrak{m}_\gamma) \Big) = \mathrm{Im}\Big(\mathfrak{X}_y(\mathfrak{m}_{\gamma'}) \to \mathfrak{X}_y(\mathfrak{m}_\gamma) \Big).$$

We define $\Upsilon = B_{\mathfrak{Y}}(y, \gamma')$ and claim that $f^{-1}(\Upsilon) \subset \Omega$. Let $x \in f^{-1}(\Upsilon) \subset \mathfrak{X}(A)$. Then $f(x)$ agrees with y modulo $\mathfrak{m}_{\gamma'}$. Its image in $\mathfrak{X}_y(A/\mathfrak{m}_\gamma)$ admits a lifting $\xi \in \mathfrak{X}_y(A)$. With the shape of Ω in (6.5.1), we get that $x \in \Omega$. $\square$

6.5.5. **Remark.** In (2), $C = \emptyset$ is allowed so that (2) covers (1).

6.6. **Application to group homomorphisms.** The valued field (K, v) is still assumed admissible.

6.6.1. **Corollary.** *Let $f : G \to H$ be a finite homomorphism of affine algebraic K–groups. Then $f(G(K))$ is a closed subgroup of $H(K)_{\mathrm{Top}}$.*

With some more work, one can show that f_{Top} is a topologically proper [G-G-MB, Cor. 4.2.5].

7. Lecture 7: Proof of the main result

7.1. Gabber's condition $(*)$.

7.1.1. **Definition.** Let G be an affine algebraic k–group. We say that G satisfies the $(*)$-condition if the $\overline{k}$-tori of $G_{\overline{k}}$ are tori of $(G^\dagger)_{\overline{k}}$.

If G is smooth, commutative or unipotent, then G satisfies $(*)$.

7.1.2. **Lemma.** *We assume that G satifies condition $(*)$. Then G satisfies the following condition*

(**) for each $G^\dagger \subset H \subset G$, any H-immersion $H/G^\dagger \hookrightarrow Y$ in an affine H–variety Y is a closed embedding.

Proof. An easy reduction permits to assume G and H connected. It is enough to show that the orbit of the smooth $\overline{k}$–group $(H_{\overline{k}})_{red}$ on the origin $y_0 \in Y_{\overline{k}}$ is closed. Our assumption implies that the maximal tori of $(H_{\overline{k}})_{red}$ fix y_0 so that $(H_{\overline{k}})_{red}$ acts through its maximal unipotent quotient U (defined in [CGP, A.2.8]). Rosenlicht's lemma [SGA3, XVII.5.7.3] yields that $(H_{\overline{k}})_{red}.y_0 = U.y_0$ is closed in $Y_{\overline{k}}$. □

7.2. The statement.

Let (K, v) be an admissible valuation field.

7.2.1. **Theorem.** [G-G-MB, 1.2, 1.4] *Let G be an affine algebraic K-group. Let $f : X \to Y$ be a G–torsor where X, Y are algebraic K-varieties.*

(1) The image $I := f(X(K))$ is locally closed in $Y(K)_{\mathrm{Top}}$ and the induced map $X(K)_{\mathrm{Top}} \to I$ is a trivial principal topological $G(K)_{\mathrm{Top}}$-fibration. In particular f_{Top} it is strict.

(2) If G satisfies the condition $(*)$, then I is clopen in $Y(K)_{\mathrm{Top}}$.

This applies to orbits and also to group homomorphisms.

7.2.2. **Corollary.** *Let $f : G \to H$ be a morphism of affine algebraic K-groups. Then $I := f(G(K))$ is a closed subgroup of $Y(K)_{\mathrm{Top}}$ and f_{Top} is strict.*

7.2.3. **Remark.** For local non-archimedean fields, there is then no need to appeal to the Baire's theorem.

7.3. Proof of (1).

We are given a G-torsor $f : X \to Y$. We denote by $H = G^\dagger$ the largest smooth K–subgroup of G. According to Theorem 4.3.3, the homogeneous space G/H admits a G-equivariant compactification $(G/H)^c$ such that $\{\bullet\} = (G/H)(F) = (G/H)^c(F)$ for each separable field extension F/K. We consider the contracted product $Z^c := X \overset{G}{\wedge} (G/H)^c$, it is proper over Y and this is then a relative compactification of $Z := X \overset{G}{\wedge} (G/H) \cong X/H$. We put $Z^\infty := Z^c \setminus Z$ (with its reduced structure). We have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\pi"] \arrow[drr, "f"'] & Z \arrow[r, hook, "j"] \arrow[dr, "h"] & Z^c \arrow[d, "h^c"] & Z^\infty \arrow[l, hook', "i"'] \arrow[dl, "h^\infty"] \\
 & & Y &
\end{tikzcd}$$

where π is a torsor under the *smooth* K–group H, j is an open immersion, i is the complementary open immersion, h^c are h^∞ proper.

7.3.1. **Lemma.** *(a) $Z(K_s) \to Y(K_s)$ is injective;*
(b) The images of h_{Top} and of h^∞_{Top} are disjoint.

Proof. (a) Since π is smooth, the map $X(K_s) \to Z(K_s)$ is onto. We are given $z_1, z_2 \in Z(K_s)$ having same image in $Y(K_s)$. We lift z_i in $x_i \in X(K_s)$ for $i = 1, 2$. Since x_1, x_2 have same image in $Y(K_s)$, it follows that there exists $g \in G(K_s)$ such that $x_1 = x_2.g$. Since $H(K_s) = G(K_s)$, we have $g \in H(K_s)$ so that $z_1 = z_2$.
(b) We denote by $L = h_{\text{Top}}(Z(K))$. Let $y \in L$ and consider the fibers $Z_y \subset Z^c_y$ of h anf h^c at y. By (1), $Z_y(K) = \{z\}$ and z is the only separable point of Z_y. Since π is smooth surjective, $\pi^{-1}(z)(K_s) \neq \emptyset$, so that the G torseur X_y becomes trivial over K_s. It follows that $(Z^c_y)_{K_s}$ is isomorphic to $(G/H)^c_{K_s}$ and has also only a single separable point. We conclude that $Z^\infty_y(K_s) = \emptyset$. Thus $L \cap \mathrm{Im}(h^\infty_{\text{Top}}) = \emptyset$. $\square$

This implies that

$$L = \mathrm{Im}(h^c_{\text{Top}}) \setminus \mathrm{Im}(h^\infty_{\text{Top}}).$$

Since h^c and h^∞ are proper, $\mathrm{Im}(h^c_{\text{Top}})$ and $\mathrm{Im}(h^\infty_{\text{Top}})$ are closed according to Theorem 6.5.4.(1). We have proven that L is locally closed.

7.3.2. **Claim.** *The induced map $\varphi : Z(K)_{\text{Top}} \to L$ is a homeomorphism.*

Let $z \in Z(K)$ and put $y = h(z)$. We want to show that φ^{-1} is continuous at y. Let Ω be an open neighborhood of $z \in Z(K)_{\text{Top}}$, it is also an open neighborood of $z \in Z^c(K)_{\text{Top}}$. We apply Theorem 6.5.4.(2) to $C = \{z\}$ and to the proper map $h^c : Z^c \to Y$. It provides a neighborhood Υ of y in $Y(K)_{\text{Top}}$ such that $(h^c)^{-1}(\Upsilon) \subset \Omega$. It follows that $\varphi^{-1}(\Upsilon \cap L) \subset (h^c)^{-1}(\Upsilon \cap L) \subset \Omega$. Since $\Upsilon \cap L$ is a neighborhood of y in L, we conclude that φ^{-1} is continuous at y.

Summarizing f_{Top} decomposes as follows:

$$X(K)_{\text{Top}} \xrightarrow{\pi_{\text{Top}}} \Omega \hookrightarrow Z(K)_{\text{Top}} \xrightarrow{\sim} L \hookrightarrow Y(K)_{\text{Top}}$$

where $\Omega_\pi := \mathrm{Im}(\pi_{\text{Top}}) \subset Z(K)_{\text{Top}}$. Furthermore we have the following properties:

- the first map $X_{\text{Top}} \to \Omega_\pi$ is open and surjective, this is a trivial principal G_{Top}-fibration (Remark 5.3.3).
- $\Omega_\pi \hookrightarrow Z_{\text{Top}}$ is a clopen embedding;

- $Z_{\text{Top}} \xrightarrow{\sim} L$ is an homeomorphism;
- $L \hookrightarrow Y_{\text{Top}}$ is a locally closed topological embedding.

Thus f_{Top} is strict.

7.4. **Sketch of proof of (2).** We assume that G satisfies condition $(*)$. Lemma 7.1.2 shows that it satisfies condition $(**)$. Theorem 4.3.3.(2) shows that in (1) we can assume that G has no separable orbit on $(G/H)^c \setminus (G/H)$. This ensures that $Z(k) = Z^c(k)$ [G-G-MB], so that $\mathrm{Im}(h_{\mathrm{Top}}) = \mathrm{Im}(h^c_{\mathrm{Top}})$. By inspection of the proof of (1), we conclude that the image $\mathrm{Im}(f_{\mathrm{Top}})$ is closed.

References

[A] S. Anantharaman, *Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1*, Bull. Soc. Math. France, Mem. **33** (1973).

[B-Z] I. N. Bernstein et A. V. Zelevinskiĭ, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), 5–70.

[B] J. Bertin, *The punctual Hilbert scheme, an introduction*, HAL : cel-00437713.

[Bo] A. Borel, *Linear Algebraic Groups (Second enlarged edition)*, Graduate text in Mathematics **126** (1991), Springer.

[BS] A. Borel et J.-P. Serre, *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv. 39 (1964), 111–164.

[BT] A. Borel and J. Tits, *Groupes réductifs*, Publ. Math. I.H.É.S. **27**(1965), 55–150.

[B:A1] N. Bourbaki, *Algèbre*, Ch. 1 à 3, Springer.

[B:A4] ———, *Algèbre*, Ch. 4 à 7, Masson, Paris, 1981.

[B:A8] ———, *Algèbre*, Ch. 8, second edition, Springer 2012.

[B:A9] ———, *Algèbre*, Ch. 9, (French), reprint of the 1959 original, Springer-Verlag, Berlin, 2007.

[B:AC] ——— *Algèbre commutative*, Ch. 1 à 4, Springer.

[B:AC] ——— *Algèbre commutative*, Ch. 5 à 7, Springer.

[B:TG1] ——— *Topologie générale*, Ch. 1 à 4, Springer.

[B:TG2] ——— *Topologie générale*, Ch. 5 à 9, Springer.

[B-HC] A. Borel, Harish-Chandra *Arithmetic subgroups of algebraic groups*, Ann. of Math. **75** (1962), 485-535.

[B-L-R] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Springer-Verlag, Berlin, 1990.

[Bn] R. Bremigan, *Quotients for algebraic group actions over non-algebraically closed fields*, Journal für die reine und angewandte Mathematik **453** (1994), 21-47.

[Br] S. Brochard, *Topologies de Grothendieck, descente, quotients*, in *Autour des schémas en groupes, vol. I*, Panoramas et Synthèses, Soc. Math. France 2014.

[CF] B. Calmès, J. Fasel, *Groupes classiques*, in *Autour des schémas en groupes*, vol II, Panoramas et Synthèses (2015), arXiv:1401.1992v2.

[C-TS] J.-L. Colliot–Thélène, J.-J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Annalen **244** (1979), 105–134.

[Co1] B. Conrad, *Reductive group schemes*, in *Autour des schémas en groupes, vol. I*, Panoramas et Synthèses **42-43**, Soc. Math. France 2014.

[Co2] ———, *Non-split reductive groups over $\mathbb{Z}$*, to appear in *Autour des schémas en groupes, vol II*, Panoramas et Synthèses (2015).

[Co] B. Conrad, *Weil and Grothendieck Approaches to Adelic Points*, L'Ens. Math. (2) **58** (2012), 61–97.

[CGP] B. Conrad, O. Gabber, G. Prasad, *Pseudo-reductive groups*, Cambridge University Press (2010) and second edition (2015), to appear.

[DG] M. Demazure, P. Gabriel, *Groupes algébriques*, North-Holland (1970).

[Ga] O. Gabber, *On pseudo-reductive groups and compactification theorems*, Oberwolfach Reports (2012), 2371–2374.

[G-G-MB] O. Gabber, P. Gille, L. Moret-Bailly, *Fibrés principaux sur les corps valués henséliens*, Algebr. Geom. **1** (2014), 573-612.

[Gir] J. Giraud, *Cohomologie non-abélienne*, Springer (1970).

[GW] U. Görtz and T. Wedhorn, *Algebraic Geometry I*, second edition Viehweg+Teubner 2020.

[Gro1] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébriques IV: les schémas de Hilbert*, Séminaire Bourbaki **6** (1960-1961), Exposé No. 221, 28 p.

[EGA4] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique IV*, Publications mathématiques de l'I.H.É.S. no. 20, 24, 28 and 32 (1964 - 1967).

[EGA-I] ———, *Eléments de Géométrie Algébrique I*, Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen **166**, Springer-Verlag 1971.

[G-MB] P. Gille, L. Moret-Bailly, *Fibrés principaux et adèles*, preprint (2021).

[Gre] M.J. Greenberg, *Rational points in Henselian discrete valuation rings.* Pub. Math. I.H.E.S. **31** (1966), 59–64.

[Mi1] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series **33** (1980), Princeton University Press, Princeton, N.J.

[Mi2] J. S. Milne, *Basic theory of affine group schemes*, http://stacks.math.columbia.edu/

[Mo] A. F. Monna, *Remarques sur les métriques non-archimédiennes. II.*, Nederl. Akad. Wetensch., Proc. **53**, (1950) 625-637 = Indagationes Math. 179–191 (1950).

[MB2] L. Moret-Bailly, *An extension of Greenberg's theorem to general valuation rings*, Manusc. Math. **139** (2012) n^0 **1**, 153–166.

[Ni] N. Nitsure, *Construction of Hilbert and Quot schemes*, in *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*, Math. surveys and Monographs vol. **123** (2005), American Mathematical Society.

[Ni02] N. Nitsure, *Representability of* GL_E, Proc. Indian Acad. Sci. **112** (2002), 539-542.

[Ni04] N. Nitsure, *Representability of* Hom *implies flatness*, Proc. Indian Acad. Sci. **114** (2004), 7-14.

[Pe] D. Perrin, *Approximation des schémas en groupes quasi compacts sur un corps*, Bull. Soc. Math. France **104** (1976), 323-335.

[R1] M. Raynaud, *Anneaux locaux henséliens*, Lecture in Note in Math. **169** (1970), Springer.

[R2] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Lecture Notes in Mathematics **119** (1970), Springer-Verlag.

[Ro] M. Romagny, *Cours "Géométrie Algébrique II"*, http://perso.univ-rennes1.fr/matthieu.romagny/M2_1112/cours.pdf

[Se1] J.P. Serre, *Cohomologie galoisienne*, cinquième édition, Lecture Notes in Math. **5**, Springer Verlag, Berlin-Heidelberg-New York.

[Se2] J.P. Serre, *Lie algebras and Lie groups*, 1964, Lectures given at Harvard University. Corrected fifth printing of the second (1992) edition. Lecture Notes in Mathematics, 1500. Springer-Verlag, Berlin, 2006.

[SGA1] *Séminaire de Géométrie algébrique de l'I.H.E.S., Revêtements étales et groupe fondamental, dirigé par A. Grothendieck*, Lecture Notes in Math. 224. Springer (1971).

[SGA3] *Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).

[Sp] T.A. Springer, *Linear algebraic groups*, Second edition (1998), Birkhäuser.

[Stacks] Stacks project, *Commutative Algebra*, http://stacks.math.columbia.edu/

[VN] J. Van Name, *Ultraparacompactness and Ultranormality*, preprint arXiv:1306.6086.

[Vi] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, Fundamental Algebraic Geometry: Grothendieck's FGA explained, AMS, pages 1-103.

[Vo] V.I. Voskresenskiĭ and B. Kunyavskiĭ, *Algebraic tori*, second edition, AMS.

[Wd] A. R. Wadsworth *Valuation theory on finite dimensional division algebras*, Valuation theory and its applications, Vol. I (Saskatoon, SK, 1999), 385-449, Fields Inst. Commun. **32** (2002), Amer. Math. Soc.

[Wa] S. Warner, *Topological fields*, North-Holland Mathematics Studies **157** (1989), North-Holland Publishing Co., Amsterdam.

[Wa1] W. C. Waterhouse, *Introduction to Affine Group Schemes*, Graduate Texts in Mathematics, **66**, Springer-Verlag 1979.

[We] A. Weil, *Foundations of Algebraic Geometry*, American Mathematical Society (1962).

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