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A LOCAL LIMIT THEOREM FOR RANDOM WALKS IN RANDOM SCENERY AND ON RANDOMLY ORIENTED LATTICES[1]

BY FABIENNE CASTELL, NADINE GUILLOTIN-PLANTARD, FRANÇOISE PÈNE AND BRUNO SCHAPIRA

Université Aix-Marseille I, Université de Lyon, Université de Brest and Université Paris-Sud 11

Random walks in random scenery are processes defined by $Z_n := \sum_{k=1}^{n} \xi_{X_1+\cdots+X_k}$, where $(X_k, k \geq 1)$ and $(\xi_y, y \in \mathbb{Z})$ are two independent sequences of i.i.d. random variables. We assume here that their distributions belong to the normal domain of attraction of stable laws with index $\alpha \in (0, 2]$ and $\beta \in (0, 2]$, respectively. These processes were first studied by H. Kesten and F. Spitzer, who proved the convergence in distribution when $\alpha \neq 1$ and as $n \to \infty$, of $n^{-\delta} Z_n$, for some suitable $\delta > 0$ depending on α and β. Here, we are interested in the convergence, as $n \to \infty$, of $n^{\delta}\mathbb{P}(Z_n = \lfloor n^{\delta} x \rfloor)$, when $x \in \mathbb{R}$ is fixed. We also consider the case of random walks on randomly oriented lattices for which we obtain similar results.

1. Introduction.

1.1. *About the model.* Random walks in random scenery (RWRS) are simple models of processes in disordered media with long-range correlations. They have been used in a wide variety of models in physics to study anomalous dispersion in layered random flows [30], diffusion with random sources, or spin depolarization in random fields (we refer the reader to Le Doussal's review paper [27] for a discussion of these models).

On the mathematical side, motivated by the construction of new self-similar processes with stationary increments, Kesten and Spitzer [24] and Borodin [3, 4] introduced RWRS in dimension one and proved functional limit theorems. These processes are defined as follows. Let $\xi := (\xi_y, y \in \mathbb{Z})$ and $X := (X_k, k \geq 1)$ be two independent sequences of independent identically distributed random variables taking values in $\mathbb{R}$ and $\mathbb{Z}$, respectively. The sequence ξ is called the *random scenery*. The sequence X is the sequence of increments of the *random walk* $(S_n, n \geq 0)$ defined by $S_0 := 0$ and $S_n := \sum_{i=1}^{n} X_i$, for $n \geq 1$. The *random walk in*

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random scenery Z is then defined for all $n \geq 1$ by

$$Z_n := \sum_{k=0}^{n-1} \xi_{S_k}.$$

Denoting by $N_n(y)$ the local time of the random walk S:

$$N_n(y) = \#\{k = 0, \dots, n-1 : S_k = y\},$$

it is straightforward to see that Z_n can be rewritten as $Z_n = \sum_y \xi_y N_n(y)$.

As in [24], the distribution of ξ_1 is assumed to belong to the normal domain of attraction of a strictly stable distribution $\mathcal{S}_\beta$ of index $\beta \in (0, 2]$, with characteristic function given by

$$\phi(u) = e^{-|u|^\beta (A_1 + i A_2 \operatorname{sgn}(u))}, \qquad u \in \mathbb{R},$$

where $0 < A_1 < \infty$ and $|A_1^{-1} A_2| \leq |\tan(\pi\beta/2)|$. When $\beta \neq 1$, this is the most general form of a strictly stable distribution. In the case $\beta = 1$, this is the general form of a random variable Y with strictly stable distribution satisfying the following symmetry condition:

$$\sup_{M>0} \left|\mathbb{E}\big(Y \mathbf{1}_{\{|Y|<M\}}\big)\right| < +\infty. \tag{1}$$

We will denote by f_β the density function of the law $\mathcal{S}_\beta$.

Concerning the random walk, the distribution of X_1 is assumed to belong to the normal domain of attraction of a strictly stable distribution $\mathcal{S}_\alpha$ with index $\alpha \in (0, 2]$. In this paper, we will actually not consider the case $\alpha = 1$ (see Remark 2 in [24] for some discussion on this case).

Then the following weak convergences hold in the space of càdlàg real-valued functions defined on $[0, \infty)$ and on $\mathbb{R}$, respectively:

$$\big(n^{-1/\alpha} S_{\lfloor nt \rfloor}\big)_{t \geq 0} \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (U(t))_{t \geq 0}$$

and

$$\left(n^{-1/\beta} \sum_{k=0}^{\lfloor nx \rfloor} \xi_k\right)_{x \in \mathbb{R}} \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (Y(x))_{x \in \mathbb{R}},$$

where U and Y are two independent Lévy processes such that $U(0) = 0$, $Y(0) = 0$, $U(1)$ has distribution $\mathcal{S}_\alpha$, $Y(1)$ and $Y(-1)$ have distribution $\mathcal{S}_\beta$. When $\alpha \in (1, 2]$, the random walk $(S_n, n \geq 0)$ is recurrent, and the limiting process U admits a local time process. We denote by $(L_t(x), t \in \mathbb{R}^+, x \in \mathbb{R})$ the jointly continuous version of this local time.

Let

$$\delta := 1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta}.$$

Papers [3, 4, 24] proved that the following weak convergences hold in the space of continuous real-valued functions defined on $[0, \infty)$:

(2) $$\text{if } \alpha > 1, \qquad (n^{-\delta} Z_{nt})_{t\geq 0} \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (\Delta(t))_{t\geq 0},$$

(3) $$\text{if } \alpha < 1, \qquad (n^{-1/\beta} Z_{nt})_{t\geq 0} \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (Y(t)\, \mathbb{E}[\widetilde{N}_\infty^{\beta-1}(0)]^{1/\beta})_{t\geq 0},$$

where

- Z_s is defined as the linear interpolation $Z_s = Z_n + (s-n)(Z_{n+1} - Z_n)$ when $n \leq s \leq n+1$,
- Δ is the process defined by

$$\Delta(t) = \int_{-\infty}^{+\infty} L_t(x)\, dY(x),$$

- $\widetilde{N}_\infty(0)$ is the total time spent in 0 by the two-sided random walk $(S_k, k \in \mathbb{Z})$ with $S_{-k} = -\sum_{m=1}^{k} X_{-m}$ [where $(X_{-k}, k \geq 1)$ is independent of $(X_k, k \geq 1)$ and with the same distribution].

The limiting process Δ is known to be a continuous δ-self-similar process with stationary increments. It can be seen as a mixture of β-stable processes, but it is not a stable process.

Since these seminal papers, RWRS have been extensively studied. Far from being exhaustive, we can cite limit theorems in higher dimension [2], strong approximation results and laws of the iterated logarithm [12, 13, 25], limit theorems for correlated sceneries or walks [11, 19], large and moderate deviations results [1, 7, 8, 17], ergodic and mixing properties (see the survey [14]). Our contribution in this paper is a local version of the convergence results from [24], as we make more precise in the next subsection.

1.2. *The results.* Our first statement is obtained in the case when the ξ's are $\mathbb{Z}$-valued random variables. Let $\varphi_\xi(u) := \mathbb{E}[e^{iu\xi_1}]$ be the characteristic function of ξ_1. Remember that there exists an integer $d \geq 1$ such that $\{u : |\varphi_\xi(u)| = 1\} = \frac{2\pi}{d}\mathbb{Z}$ (d is the g.c.d. of the set of $b - c$ where b and c belong to the support of the distribution of ξ_1).[1]

Our first result concerns the case $\alpha > 1$.

THEOREM 1 (Lattice case, $\alpha > 1$). *Assume that* $\alpha \in (1, 2]$ *and* $\beta \in (0, 2]$. *Let* $C(x)$ *be the continuous function defined by*

$$C(x) := \mathbb{E}[|L|_\beta^{-1} f_\beta(|L|_\beta^{-1} x)] \qquad \text{for all } x \in \mathbb{R},$$

where $|L|_\beta := (\int_{\mathbb{R}} L_1^\beta(y)\, dy)^{1/\beta}$. *Then, for every* $x \in \mathbb{R}$, *we have* $0 < C(x) < \infty$ *and*

[1] Note that ξ is said to be nonarithmetic if $d = 1$.

- *if* $\mathbb{P}(n\xi_1 - \lfloor n^\delta x \rfloor \notin d\mathbb{Z}) = 1$, *then* $\mathbb{P}(Z_n = \lfloor n^\delta x \rfloor) = 0$;
- *if* $\mathbb{P}(n\xi_1 - \lfloor n^\delta x \rfloor \in d\mathbb{Z}) = 1$, *then*

$$\mathbb{P}(Z_n = \lfloor n^\delta x \rfloor) = d\frac{C(x)}{n^\delta} + o(n^{-\delta}),$$

where the $o(n^{-\delta})$ *is uniform in* x.

REMARK. There is no other alternative for the law of ξ_1. Indeed, let b be in the support of ξ_1. Then $n\xi_1$ belongs to $nb + d\mathbb{Z}$. Hence, the condition $n\xi_1 - \lfloor n^\delta x \rfloor \in d\mathbb{Z}$ is equivalent to $\lfloor n^\delta x \rfloor - nb \in d\mathbb{Z}$.

Our second result concerns the case $\alpha < 1$.

THEOREM 2 (Lattice case, $\alpha < 1$). *Assume that* $\alpha \in (0, 1)$, $\beta \in (0, 2]$ *and* $x \in \mathbb{R}$. *Let* $D(x) := rf_\beta(rx)$, *with* $r := \mathbb{E}[\widetilde{N}_\infty^{\beta-1}(0)]^{-1/\beta}$.

- *If* $\mathbb{P}(n\xi_1 - \lfloor n^{1/\beta} x \rfloor \notin d\mathbb{Z}) = 1$, *then* $\mathbb{P}(Z_n = \lfloor n^{1/\beta} x \rfloor) = 0$;
- *if* $\mathbb{P}(n\xi_1 - \lfloor n^{1/\beta} x \rfloor \in d\mathbb{Z}) = 1$, *then*

$$\mathbb{P}(Z_n = \lfloor n^{1/\beta} x \rfloor) = d\frac{D(x)}{n^{1/\beta}} + o(n^{-1/\beta}),$$

where the $o(n^{-1/\beta})$ *is uniform in* x.

REMARK. We notice that in (2) and (3) it was assumed for simplicity that $d = 1$, but analogous results hold with general d. It can then be seen by using the Fourier inverse transform that $C(x)$ and $D(x)$ are the densities, respectively, of $\Delta(1)$ and $r^{-1}Y(1)$.

Finally, we get the local limit theorem when ξ is strongly nonlattice, that is, when $\limsup_{|u|\to+\infty} |\varphi_\xi(u)| < 1$.

THEOREM 3 (Strongly nonlattice case).

- *If* $\alpha > 1$ *and* $\beta \in (0, 2]$, *then for all* $a, b \in \mathbb{R}$ *such that* $a < b$,

$$\lim_{n\to\infty} n^\delta \mathbb{P}\big[Z_n \in [n^\delta x + a; n^\delta x + b]\big] = C(x)(b - a).$$

- *If* $\alpha < 1$ *and* $\beta \in (0, 2]$, *then for all* $a, b \in \mathbb{R}$ *such that* $a < b$,

$$\lim_{n\to\infty} n^{1/\beta} \mathbb{P}\big[Z_n \in [n^{1/\beta} x + a; n^{1/\beta} x + b]\big] = D(x)(b - a).$$

On the one hand, these results give some qualitative information about the behavior of Z. For instance, the transience of the process Z is easily deduced (with Borel–Cantelli lemma) when $\beta < 1$. Note that since Z is not a Markov chain, the recurrence property when $\beta > 1$ does not directly follow from the above local limit theorems. However, this can be proved by using an argument from ergodic theory (see [31]). Indeed, it is enough to remark that when $\beta \in (1, 2]$, the random variables ξ_{S_k}, $k \in \mathbb{N}$ form an ergodic and stationary sequence of integrable and centered random variables.

On the other hand, this work was motivated by the study of random walks on randomly oriented lattices. In the simplest case, one should think to the simple random walk defined on a random sublattice of the oriented lattice $\mathbb{Z}^2$, which is constructed as follows. On each horizontal line, one removes all edges oriented to the right with probability $1/2$ or those oriented to the left with probability $1/2$, and so independently on each level. Then it is known, and not difficult to see, that the first coordinate of the resulting random walk is closely related to a random walk in random scenery $Z = \sum_k \xi_{S_k}$, with S the simple random walk on $\mathbb{Z}$ and the ξ_y i.i.d. random variables with geometric distribution (see Section 5 or [18] for more explanations). In [18], it was conjectured that the probability of return to the origin of this random walk is equivalent to a constant times $n^{-5/4}$. Here, we prove a local limit theorem for even more general random walks, giving in particular a proof of this conjecture. We refer the reader to Section 5 for more precise statements of our results.

1.3. *Outline of the proof.* Let us give a very rough description of the proofs for RWRS. To fix ideas, we do it for $x = 0$ and $\alpha > 1$. By the Fourier inverse transform, we have to study the asymptotic behavior of

$$\int \mathbb{E}[e^{itZ_n}]\,dt = \int \mathbb{E}\Big[\prod_{y\in\mathbb{Z}} \varphi_\xi(tN_n(y))\Big]\,dt. \tag{4}$$

For t such that $tN_n(y)$ is small, only the behavior of φ_ξ around 0 is relevant. Therefore, for $|t| \le (\sup_y N_n(y))^{-1} \simeq n^{-1+1/\alpha}$,

$$\mathbb{E}\Big[\prod_{y\in\mathbb{Z}} \varphi_\xi(tN_n(y))\Big] \simeq \mathbb{E}\Big[\exp\Big(-|t|^\beta \sum_y N_n(y)^\beta \big(A_1 + iA_2\,\mathrm{sgn}(t)\big)\Big)\Big].$$

Now, $\sum_y N_n(y)^\beta$ is of order $n^{\beta\delta}$, and a change of variable $t \rightsquigarrow n^\delta t$ leads to the dominant part in the integral (4).

For $t \ge (\sup_y N_n(y))^{-1} \simeq n^{-1+1/\alpha}$, the behavior of φ_ξ away from 0 comes into play. In the strongly nonlattice case, one can find $\epsilon_0 > 0$ and $\rho \in (0, 1)$ such that $|\varphi_\xi(t)| \le \rho$ for $|t| \ge \epsilon_0$, so that for $|t| \ge n^{-1+1/\alpha}$,

$$\Big|\prod_{y\in\mathbb{Z}} \varphi_\xi(tN_n(y))\Big| \le \rho^{\#\{y;N_n(y)\ge\epsilon_0/t\}} \le \rho^{\#\{y;N_n(y)\ge\epsilon_0 n^{1-1/\alpha}\}}.$$

It is easily seen that there is a large number of points visited at least $n^{1-1/\alpha}$ times, leading to the result.

The lattice case is more delicate, since in this case $|\varphi_\xi(t)| = 1$ for $t \in \frac{2\pi}{d}\mathbb{Z}$, so that the inequality $|\varphi_\xi(tN_n(y))| \le \rho$ is only valid for the y such that $d(tN_n(y); \frac{2\pi}{d}\mathbb{Z}) \ge \epsilon_0$. Thus, the main difficulty is to show that for $|t| \ge n^{1-1/\alpha}$, there are a lot of such sites. This is done by a surgery on the trajectories of the random walk.

Let us briefly describe now the organization of the paper. In the next section, we prove Theorem 1. In Sections 3 and 4, we sketch the proofs of Theorem 2 and Theorem 3 which are easier and follow the same lines. In Section 5, the local limit theorem for random walks evolving on randomly oriented lattices is obtained by using similar techniques as for the proof of Theorem 1. Finally, in the Appendix, we prove some auxiliary results on the range of the random walk S, that we should need, but which could also be of independent interest.

2. Lattice case, $\alpha > 1$: Proof of Theorem 1.

2.1. *Finiteness of $C(x)$.*

LEMMA 4. *For all $x \in \mathbb{R}$, $0 < C(x) < +\infty$.*

PROOF. Let $x \in \mathbb{R}$. Since $\int_{\mathbb{R}} L_1(y)\,dy = 1$ and $\beta \le 2$, we have a.s.

$$\int_{\mathbb{R}} L_1^\beta(y)\,dy \le 1 + \sup_y L_1(y)^{(\beta-1)_+}.$$

Hence, $\int_{\mathbb{R}} L_1^\beta(y)\,dy$ is a.s. finite. So $C(x) > 0$.

Let us prove now that $C(x)$ is finite. First, we have

$$C(x) \le \|f_\beta\|_\infty \mathbb{E}[|L|_\beta^{-1}].$$

Let us assume now that $\beta > 1$. By Hölder's inequality,

$$1 = \int_{\mathbb{R}} L_1(y)\,dy \le |L|_\beta \biggl(\int_{\mathbb{R}} \mathbf{1}\bigl(L_1(y) > 0\bigr)\,dy\biggr)^{1-1/\beta}.$$

Thus, by using Jensen's inequality, we get

$$\begin{aligned} C(x) &\le \|f_\beta\|_\infty \mathbb{E}\biggl[\biggl(\int_{\mathbb{R}} \mathbf{1}\bigl(L_1(y) > 0\bigr)\,dy\biggr)^{1-1/\beta}\biggr] \\ &\le \|f_\beta\|_\infty \biggl(\mathbb{E}\biggl[\biggl(\int_{\mathbb{R}} \mathbf{1}\bigl(L_1(y) > 0\bigr)\,dy\biggr)\biggr]\biggr)^{1-1/\beta} \\ &= \|f_\beta\|_\infty \bigl(\mathbb{E}\bigl[\lambda(U([0,1]))\bigr]\bigr)^{1-1/\beta}, \end{aligned}$$

where λ denotes the Lebesgue measure on $\mathbb{R}$ and $U([0,1])$ the set of points visited by U before time 1. This finishes the proof in the case $\beta > 1$, since the last quantity is finite (see, e.g., [28], page 703).

Next, if $\beta = 1$, then $|L|_\beta = 1$ and $C(x) = f_\beta(x) < +\infty$.

Assume finally that $\beta < 1$. Then

$$1 = \int_{\mathbb{R}} L_1(y)\,dy \le |L|_\beta^\beta \Big(\sup_x L_1(x)\Big)^{1-\beta},$$

so that

$$\mathbb{E}[|L|_\beta^{-1}] \le \mathbb{E}\Big[\Big(\sup_x L_1(x)\Big)^{(1-\beta)/\beta}\Big] = \frac{1-\beta}{\beta}\int_0^{+\infty} t^{1/\beta-2}\mathbb{P}\Big[\sup_x L_1(x) \ge t\Big]\,dt.$$

Therefore, it suffices to prove that there exists a constant $c > 0$ such that

$$\mathbb{P}\Big[\sup_x L_1(x) \ge t\Big] \le 2\exp(-ct) \qquad \text{for all } t > 0. \tag{5}$$

This follows from stronger results proved in [26], but for sake of completeness, let us give a soft argument here. For $a > 0$, let $\tau_a := \inf\{t : \sup_x L_t(x) \ge a\}$. The random variable τ_a is a stopping time, and by continuity of $t \mapsto \sup_x L_t(x)$, $\sup_x L_{\tau_a}(x) = a$ on $\{\tau_a < \infty\}$. It follows then from the inequality

$$\sup_x L_{t+s}(x) \le \sup_x L_t(x) + \sup_x\big(L_{t+s}(x) - L_t(x)\big)$$

and from the strong Markov property that for any $a > 0$ and $b > 0$,

$$\begin{aligned}\mathbb{P}\Big[\sup_x L_1(x) \ge a+b\Big] &= \mathbb{P}\Big[\tau_a \le 1; \sup_x L_1(x) \ge a+b\Big]\\ &\le \mathbb{E}\Big[\mathbf{1}_{\{\tau_a \le 1\}}\mathbb{P}_{U_{\tau_a}}\Big[\sup_x L_1(x) \ge b\Big]\Big],\end{aligned}$$

where for any v, $\mathbb{P}_v$ denotes the law of the process U starting from v. By translation invariance, the law of $\sup_x L_1(x)$ does not depend on the starting point of U. Therefore, for any $a > 0$ and $b > 0$,

$$\begin{aligned}\mathbb{P}\Big[\sup_x L_1(x) \ge a+b\Big] &\le \mathbb{P}[\tau_a \le 1]\mathbb{P}\Big[\sup_x L_1(x) \ge b\Big]\\ &= \mathbb{P}\Big[\sup_x L_1(x) \ge a\Big]\mathbb{P}\Big[\sup_x L_1(x) \ge b\Big].\end{aligned} \tag{6}$$

Let $M > 0$ be a median of $\sup_x L_1(x)$. By (6), for all $t > 0$,

$$\mathbb{P}\Big[\sup_x L_1(x) \ge t\Big] \le \mathbb{P}\Big[\sup_x L_1(x) \ge M\Big]^{\lfloor t/M \rfloor} \le \Big(\frac{1}{2}\Big)^{\lfloor t/M \rfloor},$$

which ends the proof of (5). $\square$

2.2. *A first reduction.*

LEMMA 5. *Let $n \geq 1$ and $x \in \mathbb{Z}$ be given.*
- *If $\mathbb{P}[n\xi_1 - x \notin d\mathbb{Z}] = 1$, then $\mathbb{P}(Z_n = x) = 0$.*
- *If $\mathbb{P}[n\xi_1 - x \in d\mathbb{Z}] = 1$, then*

$$\mathbb{P}(Z_n = x) = \frac{d}{2\pi} \int_{-\pi/d}^{\pi/d} \exp(-itx) \mathbb{E}\Big[\prod_{y \in \mathbb{Z}} \varphi_\xi(tN_n(y))\Big] dt.$$

PROOF. This kind of result is relatively general (see, e.g., [20], Chapter 4), but we give a proof for reader's convenience. We have

$$\mathbb{P}(Z_n = x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-itx) \varphi_n(t)\, dt,$$

where φ_n is the characteristic function of Z_n given by

$$\varphi_n(t) := \mathbb{E}\Big[\prod_{y \in \mathbb{Z}} \varphi_\xi(tN_n(y))\Big] \qquad \text{for all } t \in \mathbb{R}.$$

Notice that $e^{2i\pi\xi_1/d} = \mathbb{E}[e^{2i\pi\xi_1/d}]$ almost surely. Hence, for any integer $m \geq 0$ and any $u \in \mathbb{R}$,

$$\varphi_\xi\left(\frac{2m\pi}{d} + u\right) = \varphi_\xi\left(\frac{2\pi}{d}\right)^m \varphi_\xi(u).$$

Therefore,

$$\begin{aligned}
\mathbb{P}(Z_n = x) &= \frac{1}{2\pi} \sum_{k=0}^{d-1} \int_{-\pi/d}^{\pi/d} \exp\left(-i\left(t + \frac{2k\pi}{d}\right)x\right) \varphi_n\left(\frac{2k\pi}{d} + t\right) dt \\
&= \frac{1}{2\pi} \sum_{k=0}^{d-1} \int_{-\pi/d}^{\pi/d} \exp(-itx) \exp\left(-i\frac{2k\pi}{d}x\right) \\
&\qquad \times \mathbb{E}\left[\prod_y \left\{\varphi_\xi\left(\frac{2\pi}{d}\right)^{kN_n(y)} \varphi_\xi(tN_n(y))\right\}\right] dt \\
&= \frac{1}{2\pi} \left(\sum_{k=0}^{d-1} \exp\left(-i\frac{2k\pi}{d}x\right) \varphi_\xi\left(\frac{2\pi}{d}\right)^{kn}\right) \int_{-\pi/d}^{\pi/d} \exp(-itx) \varphi_n(t)\, dt,
\end{aligned}$$

since $\sum_y N_n(y) = n$. Moreover, $[e^{-i(2\pi/d)x} \varphi_\xi(\frac{2\pi}{d})^n]^d = e^{-i2\pi x} e^{2i\pi n\xi_1} = 1$, thus $e^{-i(2\pi/d)x} \varphi_\xi(\frac{2\pi}{d})^n$ is a dth root of the unity. Hence,

$$\sum_{k=0}^{d-1} e^{-i(2k\pi/d)x} \varphi_\xi\left(\frac{2\pi}{d}\right)^{kn} = \begin{cases} d, & \text{if } \varphi_\xi\left(\frac{2\pi}{d}\right)^n e^{-i(2\pi/d)x} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\varphi_\xi(\frac{2\pi}{d}) = e^{2i\pi\xi_1/d}$ a.s., the lemma follows. □

2.3. *The event* Ω_n. Set

$$N_n^* := \sup_y N_n(y) \quad \text{and} \quad R_n := \#\{y : N_n(y) > 0\}.$$

LEMMA 6. *For every* $n \ge 1$ *and* $\gamma > 0$, *set*

$$\Omega_n = \Omega_n(\gamma) := \left\{ R_n \le n^{1/\alpha+\gamma} \text{ and } \sup_{y \ne z} \frac{|N_n(y) - N_n(z)|}{|y-z|^{(\alpha-1)/2}} \le n^{(1-1/\alpha+\gamma)/2} \right\}.$$

Then $\mathbb{P}(\Omega_n) = 1 - o(n^{-\delta})$. *Moreover, given* $\eta \ge \gamma \max(\alpha/2, 2(\beta-1)/\beta)$, *the following also holds on* Ω_n:

$$N_n^* \le n^{1-1/\alpha+\eta} \quad \text{and} \quad V_n := \sum_z N_n^\beta(z) \ge \begin{cases} n^{\delta\beta-\eta\beta/2}, & \text{if } \beta > 1, \\ n^{\delta\beta-\eta(1-\beta)}, & \text{if } \beta \le 1. \end{cases} \tag{7}$$

PROOF. We prove in the Appendix that for every $\gamma > 0$, there exists $C > 0$ such that

$$\mathbb{P}(R_n \le \mathbb{E}[R_n] n^\gamma) = 1 - \mathcal{O}(e^{-Cn^\gamma}).$$

Since there exists $c > 0$ such that $\mathbb{E}[R_n] \sim cn^{1/\alpha}$ (see [32], page 36), we conclude that

$$\mathbb{P}(R_n \le n^{1/\alpha+\gamma}) = 1 - o(n^{-\delta}).$$

Now let us prove that

$$\mathbb{P}\left(\sup_{y \ne z} \frac{|N_n(y) - N_n(z)|}{|y-z|^{(\alpha-1)/2}} \ge \sqrt{n^{1-1/\alpha+\gamma}} \right) = o(n^{-\delta}).$$

According to the proof of Proposition 5.4 in [28], we have: $\mathbb{E}[|S_n|^p] = \mathcal{O}(n^{p/\alpha})$, for all $p \in (1, \alpha)$. Then Doob's inequality gives that, for all $\delta' > \delta/p$,

$$\mathbb{P}\left(\sup_{k=1,\dots,n} |S_k| \ge n^{1/\alpha+\delta'} \right) = \mathcal{O}(n^{-p\delta'}) = o(n^{-\delta}).$$

So we can restrict ourselves to the set $A_n := \{\sup_{k=1,\dots,n} |S_k| < n^{1/\alpha+\delta'}\}$. But on A_n, if $N_n(z) > 0$ then necessarily $z \in (-n^{1/\alpha+\delta'}, n^{1/\alpha+\delta'})$. Thus,

$$\begin{aligned} &\mathbb{P}\left(\sup_{y,z} \frac{|N_n(y) - N_n(z)|}{|y-z|^{(\alpha-1)/2}} \ge \sqrt{n^{1-1/\alpha+\gamma}}; A_n \right) \\ &\quad \le 5n^{2/\alpha+2\delta'} \sup_{y \ne z} \mathbb{P}\left(\frac{|N_n(y) - N_n(z)|}{|y-z|^{(\alpha-1)/2}} \ge \sqrt{n^{1-1/\alpha+\gamma}} \right). \end{aligned} \tag{8}$$

Moreover, the Markov inequality gives for all $m \ge 1$

$$\begin{aligned} &\mathbb{P}\left(\frac{|N_n(y) - N_n(z)|}{|y-z|^{(\alpha-1)/2}} \ge \sqrt{n^{1-1/\alpha+\gamma}} \right) \le \frac{\mathbb{E}[|N_n(y) - N_n(z)|^{2m}]}{|y-z|^{(\alpha-1)m} n^{(1-1/\alpha+\gamma)m}} \\ &\qquad \text{for all } y \ne z. \end{aligned} \tag{9}$$

In addition, according to [23] [see the formula in the middle of page 77, with $m = \mathcal{O}(n)$, $a_m^{-1} = \mathcal{O}(n^{-1/\alpha})$ and $Q(z)^{-1} = \mathcal{O}(z^\alpha)$], we have for all $m \geq 1$,

$$\sup_{y\neq z} \frac{\mathbb{E}[|N_n(y) - N_n(z)|^{2m}]}{|y-z|^{(\alpha-1)m}} = \mathcal{O}\big(n^{(1-1/\alpha)m}\big). \tag{10}$$

Thus, if we take $m > (\delta + 2/\alpha + 2\delta')/\gamma$, then by using (8), (9) and (10), we get

$$\mathbb{P}\biggl(\sup_{y\neq z} \frac{|N_n(y) - N_n(z)|}{|y-z|^{(\alpha-1)/2}} \geq \sqrt{n^{1-1/\alpha+\gamma}}\biggr) = \mathcal{O}\biggl(\frac{n^{2/\alpha+2\delta'}}{n^{\gamma m}}\biggr) = o(n^{-\delta}).$$

We now prove (7), starting with the upper bound for N_n^*. For this, let y_0 be such that $N_n(y_0) = N_n^*$, and let z_0 be the closest point to y_0 such that $N_n(z_0) = 0$. Then on Ω_n,

$$|y_0 - z_0| \leq R_n \leq n^{1/\alpha+\gamma},$$

and thus

$$\begin{aligned} N_n(y_0) &\leq \sqrt{|y_0 - z_0|^{\alpha-1} n^{1-1/\alpha+\gamma}} \leq \sqrt{n^{(1/\alpha+\gamma)(\alpha-1)} n^{1-1/\alpha+\gamma}} \\ &= n^{1-1/\alpha+\alpha\gamma/2}. \end{aligned} \tag{11}$$

The desired upper bound for N_n^* follows if $\eta \geq \alpha\gamma/2$.

To prove the lower bound for V_n, we use the fact that $n = \sum_y N_n(y)$. When $\beta > 1$, this gives by using Hölder's inequality

$$n \leq \biggl(\sum_z N_n^\beta(z)\biggr)^{1/\beta} R_n^{1-1/\beta} \leq (V_n)^{1/\beta} n^{(1/\alpha+\gamma)(1-1/\beta)}.$$

Hence $V_n^{1/\beta} \geq n^{\delta-\gamma(\beta-1)/\beta}$, and the desired lower bound for V_n follows if $2(\beta - 1)\gamma \leq \eta\beta$. When $\beta \leq 1$, we write

$$n = \sum_y N_n(y) \leq V_n (N_n^*)^{1-\beta},$$

and the desired lower bound follows from the upper bound for N_n^* proved just above. $\square$

2.4. *Scheme of the proof.* Let $\eta > 0$. Set $\gamma := \eta\beta/2$. We observe that $\gamma \leq \eta$ and that (7) holds with this choice of (η, γ). We also set

$$\overline{\eta} := \begin{cases} \eta, & \text{if } \beta \geq 1, \\ \eta/\beta, & \text{if } \beta < 1. \end{cases}$$

By Lemmas 5 and 6, we have to estimate

$$\frac{d}{2\pi} \int_{-\pi/d}^{\pi/d} e^{-it\lfloor n^\delta x\rfloor} \mathbb{E}\biggl[\prod_y \varphi_\xi(tN_n(y))\mathbf{1}_{\Omega_n}\biggr] dt.$$

This is done in several steps presented in the following propositions.

PROPOSITION 7. *Let* $\eta \in (0, \frac{1}{2\alpha(\beta+1)})$. *Then, we have*

$$\frac{d}{2\pi}\int_{|t|\le n^{-\delta+\overline{\eta}}} e^{-it\lfloor n^\delta x\rfloor}\mathbb{E}\Big[\prod_y \varphi_\xi(tN_n(y))\mathbf{1}_{\Omega_n}\Big]\,dt = d\frac{C(x)}{n^\delta} + o(n^{-\delta}),$$

uniformly in $x \in \mathbb{R}$.

Recall next that the characteristic function ϕ of the stable distribution $\mathcal{S}_\beta$ has the following form:

$$\phi(u) = e^{-|u|^\beta(A_1 + iA_2\,\mathrm{sgn}(u))},$$

for some $0 < A_1 < \infty$, $|A_1^{-1}A_2| \le |\tan(\pi\beta/2)|$. It follows (see [22], Theorem 2.6.5) that the characteristic function φ_ξ of ξ_1 satisfies

$$(12)\qquad 1 - \varphi_\xi(u) \sim |u|^\beta\big(A_1 + iA_2\,\mathrm{sgn}(u)\big) \qquad \text{when } u \to 0.$$

Therefore, there exist constants $\varepsilon_0 > 0$ and $\sigma > 0$ such that

$$(13)\qquad \max(|\phi(u)|, |\varphi_\xi(u)|) \le \exp(-\sigma|u|^\beta) \qquad \text{for all } u \in [-\varepsilon_0, \varepsilon_0].$$

Since $\overline{\varphi_\xi(t)} = \varphi_\xi(-t)$ for every $t \ge 0$, the following propositions achieve the proof of Theorem 1:

PROPOSITION 8. *Let* η *be as in Proposition* 7. *Then there exists* $c > 0$ *such that*

$$\int_{n^{-\delta+\overline{\eta}}}^{\varepsilon_0 n^{-1+1/\alpha-\eta}} \mathbb{E}\Big[\prod_y |\varphi_\xi(tN_n(y))|\mathbf{1}_{\Omega_n}\Big]\,dt = o(e^{-n^c}).$$

PROPOSITION 9. *Let* η *be as in Proposition* 7 *and let* $\varepsilon \in (\eta, \frac{\alpha-1}{\alpha(3+2\beta(\alpha-1))})$ *be given. Then there exists* $c > 0$ *such that*

$$\int_{\varepsilon_0 n^{-1+1/\alpha-\eta}}^{n^{-1+1/\alpha+\varepsilon}} \mathbb{E}\Big[\prod_y |\varphi_\xi(tN_n(y))|\mathbf{1}_{\Omega_n}\Big]\,dt = o(e^{-n^c}).$$

PROPOSITION 10. *Let* η *be such that* $\gamma < \min(\frac{1}{2\alpha^2}, \frac{1}{2}\frac{\alpha-1}{\alpha})$ *and let* $\varepsilon \in ((\frac{2\alpha}{\beta} + 1)\gamma, 1 - \frac{1}{\alpha})$ *be given. Then there exists* $c > 0$ *such that*

$$\int_{n^{-1+1/\alpha+\varepsilon}}^{\pi/d} \mathbb{E}\Big[\prod_y |\varphi_\xi(tN_n(y))|\mathbf{1}_{\Omega_n}\Big]\,dt = o(e^{-n^c}).$$

To end the proof of Theorem 1, we observe that there exists (η, ε) satisfying all the hypotheses of these propositions (by taking $\eta > 0$ small enough and $\varepsilon < \frac{\alpha-1}{\alpha(3+2\beta(\alpha-1))}$ large enough).

2.5. *Proof of Proposition 7*. Remember that $V_n = \sum_{z\in\mathbb{Z}} N_n^\beta(z)$. We start with a preliminary lemma.

LEMMA 11. *If $\beta > 1$, then*

$$\sup_n \mathbb{E}\left[\left(\frac{n^\delta}{V_n^{1/\beta}}\right)^{\beta/(\beta-1)}\right] < +\infty.$$

If $\beta \le 1$, then for all $p \ge 1$,

$$\sup_n \mathbb{E}\left[\left(\frac{n^\delta}{V_n^{1/\beta}}\right)^{p}\right] < +\infty.$$

A direct consequence of this lemma is that the sequence $(n^\delta V_n^{-1/\beta}, n \ge 1)$ is uniformly integrable.

PROOF OF LEMMA 11. We start with the case $\beta > 1$. We already observed in the proof of Lemma 6 that for every $n \ge 1$,

$$n \le V_n^{1/\beta} R_n^{1-1/\beta}.$$

But it is proved in [28], equation (7.a), that $\mathbb{E}[R_n] = \mathcal{O}(n^{1/\alpha})$. The result follows. We suppose now that $\beta \le 1$. Since we have

$$n = \sum_x N_n(x) \le V_n (N_n^*)^{1-\beta}, \tag{14}$$

we get

$$\frac{n^\delta}{V_n^{1/\beta}} \le \left(\frac{N_n^*}{n^{1-1/\alpha}}\right)^{1/\beta - 1}. \tag{15}$$

We use next the fact that N_n^* is a subadditive functional

$$N_{n+m}^* \le N_n^* + N_m^* \circ \theta_n, \tag{16}$$

where

$$\begin{aligned} N_m^* \circ \theta_n &:= \sup_x \sum_{k=0}^{m-1} \mathbf{1}_{\{S_{n+k}=x\}} \\ &= \sup_x \sum_{k=0}^{m-1} \mathbf{1}_{\{S_{n+k}-S_n=x\}}, \end{aligned}$$

is independent of $\sigma(S_0, \dots, S_{n-1})$. Moreover, $0 \le N_{n+1}^* - N_n^* \le 1$. Therefore, we can prove in exactly the same way as for the range [see (46) in the Appendix], that

$$\mathbb{P}(N_n^* \ge a+b) \le \mathbb{P}(N_n^* \ge a)\mathbb{P}(N_n^* \ge b) \qquad \text{for all } a, b \in \mathbb{N}. \tag{17}$$

Now it is known (see, e.g., [5]) that $N_n^*/n^{1-1/\alpha}$ converges in distribution toward $\sup_x L_1(x)$. Let $t>0$, be such that $\mathbb{P}[\sup_x L_1(x)\ge t]\le 1/2$. Since

$$\lim_{n\to\infty}\mathbb{P}(N_n^*\ge \lfloor tn^{1-1/\alpha}\rfloor)\le \mathbb{P}\Big(\sup_x L_1(x)\ge t\Big)\le 1/2,$$

we obtain that for n large enough, $\mathbb{P}(N_n^*\ge\lfloor tn^{1-1/\alpha}\rfloor)\le 2/3$. Hence for n large enough, and all $p\ge 1$,

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\frac{N_n^*}{n^{1-1/\alpha}}\bigg)^p\bigg] &= p\int_0^\infty x^{p-1}\mathbb{P}(N_n^*\ge xn^{1-1/\alpha})\,dx\\ &\le pt^p\int_0^\infty u^{p-1}\mathbb{P}(N_n^*\ge tn^{1-1/\alpha}u)\,du\\ &\le pt^p\int_0^\infty u^{p-1}\mathbb{P}(N_n^*\ge\lfloor tn^{1-1/\alpha}\rfloor)^{\lfloor u\rfloor}\,du\\ &\le pt^p\int_0^\infty u^{p-1}\bigg(\frac{2}{3}\bigg)^{\lfloor u\rfloor}du,\end{aligned}\tag{18}$$

where the first inequality in (18) comes from (17). Thus, for all $p\ge 1$,

$$\sup_n\mathbb{E}\bigg[\bigg(\frac{N_n^*}{n^{1-1/\alpha}}\bigg)^p\bigg]<\infty.\tag{19}$$

The lemma now follows from (15). □

The next step is the following lemma.

LEMMA 12. *Under the hypotheses of Proposition 7, we have*

$$\int_{|t|\le n^{-\delta+\overline{\eta}}}e^{-it\lfloor n^\delta x\rfloor}\mathbb{E}\bigg[\bigg\{\prod_y\varphi_\xi(tN_n(y))-e^{-|t|^\beta V_n(A_1+iA_2\,\mathrm{sgn}(t))}\bigg\}\mathbf{1}_{\Omega_n}\bigg]\,dt=o(n^{-\delta}),$$

uniformly in $x\in\mathbb{R}$, where A_1 and A_2 are the constants appearing in (12).

PROOF. It suffices to prove that

$$\int_{|t|\le n^{-\delta+\overline{\eta}}}\bigg|\mathbb{E}\bigg[\prod_y\varphi_\xi(tN_n(y))\mathbf{1}_{\Omega_n}\bigg]-\mathbb{E}\big[e^{-|t|^\beta V_n(A_1+iA_2\,\mathrm{sgn}(t))}\mathbf{1}_{\Omega_n}\big]\bigg|\,dt=o(n^{-\delta}).$$

Set

$$E_n(t):=\prod_y\varphi_\xi(tN_n(y))-\prod_y\exp\big(-|t|^\beta N_n^\beta(y)\big(A_1+iA_2\,\mathrm{sgn}(t)\big)\big).$$

Observe that

$$\begin{aligned}E_n(t)=\sum_y\bigg(\prod_{z<y}\varphi_\xi(tN_n(z))\bigg)&\big(\varphi_\xi(tN_n(y))-e^{-|t|^\beta N_n^\beta(y)(A_1+iA_2\,\mathrm{sgn}(t))}\big)\\ &\times\bigg(\prod_{z>y}e^{-|t|^\beta N_n^\beta(z)(A_1+iA_2\,\mathrm{sgn}(t))}\bigg).\end{aligned}$$

But on Ω_n, if $|t| \le n^{-\delta+\overline{\eta}}$, then

$$|t| N_n(z) \le n^{\eta+\overline{\eta}-1/(\alpha\beta)}. \tag{20}$$

This implies in particular that $|t|N_n(z) < \varepsilon_0$ for n large enough, since the hypothesis on η implies $\eta + \overline{\eta} < 1/(\alpha\beta)$. Thus, by using (13), we get

$$|E_n(t)| \le \sum_y \bigl|\varphi_\xi(tN_n(y)) - \exp\bigl(-|t|^\beta N_n^\beta(y)\bigl(A_1 + iA_2 \operatorname{sgn}(t)\bigr)\bigr)\bigr| \\ \times \exp\biggl(-\sigma|t|^\beta \sum_{z\neq y} N_n^\beta(z)\biggr),$$

for n large enough. Observe next that (12) implies

$$\bigl|\varphi_\xi(u) - \exp\bigl(-|u|^\beta\bigl(A_1 + iA_2 \operatorname{sgn}(u)\bigr)\bigr)\bigr| \le |u|^\beta h(|u|) \qquad \text{for all } u \in \mathbb{R},$$

with h a continuous and monotone function on $[0, +\infty)$ vanishing in 0. Therefore, by using (20), we get

$$|E_n(t)| \le |t|^\beta h\bigl(n^{\eta+\overline{\eta}-1/(\alpha\beta)}\bigr) \sum_y N_n^\beta(y) \exp\biggl(-\sigma|t|^\beta \sum_{z\neq y} N_n^\beta(z)\biggr).$$

Now on Ω_n, according to (7) and the hypothesis on η, if n is large enough,

$$\sum_{z\neq y} N_n^\beta(z) \ge V_n/2 \qquad \text{for all } y \in \mathbb{Z}.$$

By using this and the change of variables $v = tV_n^{1/\beta}$, we get

$$\int_{|t|\le n^{-\delta+\overline{\eta}}} \mathbb{E}[|E_n(t)|\mathbf{1}_{\Omega_n}]\,dt \le h\bigl(n^{\eta+\overline{\eta}-1/(\alpha\beta)}\bigr)\mathbb{E}[V_n^{-1/\beta}] \int_{\mathbb{R}} |v|^\beta \exp(-\sigma|v|^\beta/2)\,dv \\ = o(\mathbb{E}[V_n^{-1/\beta}]),$$

which proves the result according to Lemma 11. □

Finally, Proposition 7 follows from the following lemma.

LEMMA 13. *Under the hypotheses of Proposition 7, we have*

$$\frac{d}{2\pi} \int_{|t|\le n^{-\delta+\overline{\eta}}} e^{-it\lfloor n^\delta x\rfloor} \mathbb{E}\bigl[e^{-|t|^\beta V_n(A_1+iA_2 \operatorname{sgn}(t))}\mathbf{1}_{\Omega_n}\bigr]\,dt = d\frac{C(x)}{n^\delta} + o(n^{-\delta}),$$

uniformly in $x \in \mathbb{R}$.

PROOF. Set

$$I_{n,x} := \int_{|t|\le n^{-\delta+\overline{\eta}}} e^{-it\lfloor n^\delta x\rfloor} e^{-|t|^\beta V_n(A_1+iA_2 \operatorname{sgn}(t))}\,dt.$$

Since $|\lfloor n^\delta x\rfloor - n^\delta x| \le 1$, for all n and x, it is immediate that

$$I_{n,x} = \int_{|t|\le n^{-\delta+\overline{\eta}}} e^{-itn^\delta x} e^{-|t|^\beta V_n (A_1 + i A_2 \operatorname{sgn}(t))}\, dt + \mathcal{O}(n^{-2\delta+2\overline{\eta}}).$$

But $2\overline{\eta} < 1/(\alpha\beta) < \delta$ by hypothesis, so actually

$$I_{n,x} = \int_{|t|\le n^{-\delta+\overline{\eta}}} e^{-itn^\delta x} e^{-|t|^\beta V_n (A_1 + i A_2 \operatorname{sgn}(t))}\, dt + o(n^{-\delta}).$$

Next, after some changes of variables, we get

$$\begin{aligned}(21)\qquad &\int_{|t|\le n^{-\delta+\overline{\eta}}} e^{-itn^\delta x} e^{-|t|^\beta V_n (A_1 + i A_2 \operatorname{sgn}(t))}\, dt \\ &\quad = n^{-\delta}\left\{2\pi \frac{n^\delta}{V_n^{1/\beta}} f_\beta\left(\frac{n^\delta x}{V_n^{1/\beta}}\right) - J_{n,x}\right\},\end{aligned}$$

where

$$J_{n,x} := \int_{|v|\ge n^{\overline{\eta}}} e^{-ivx} e^{-|v|^\beta (V_n/n^{\beta\delta})(A_1 + i A_2 \operatorname{sgn}(v))}\, dv.$$

Now it is known that $W_n := n^\delta V_n^{-1/\beta}$ converges in distribution, as $n \to \infty$, toward $W := |L|_\beta^{-1}$ (see [10], Lemma 14 or [24], Lemma 6). Then by Skorohod's representation theorem, we can find a sequence $(\widetilde{W}_n, n \ge 1)$ and $\widetilde{W}$ distributed, respectively, as $(W_n, n \ge 1)$ and W such that $\widetilde{W}_n$ converges almost surely toward $\widetilde{W}$. Moreover, Lemma 11 ensures that the sequence $(\widetilde{W}_n, n \ge 1)$ is uniformly integrable, so actually the convergence holds in $\mathbb{L}^1$. Let us deduce that

$$(22)\qquad \mathbb{E}[g_x(W_n)] = \mathbb{E}[g_x(W)] + o(1),$$

where $g_x : z \mapsto z f_\beta(xz)$ and the $o(1)$ is uniform in x. First,

$$\begin{aligned}\bigl|\mathbb{E}[g_x(W_n)] - \mathbb{E}[g_x(W)]\bigr| &\le \sup_{x,z\in\mathbb{R}} |(g_x)'(z)| \mathbb{E}[|\widetilde{W}_n - \widetilde{W}|] \\ &\le \sup_u |f_\beta(u) + u f_\beta'(u)| \mathbb{E}[|\widetilde{W}_n - \widetilde{W}|].\end{aligned}$$

But remember that

$$f_\beta(u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itu} e^{-|t|^\beta (A_1 + i A_2 \operatorname{sgn}(t))}\, dt.$$

So after differentiation under the integral sign and integration by parts, we get

$$u f_\beta'(u) = -\frac{1}{2\pi} \int_{\mathbb{R}} e^{itu} \bigl(1 - \beta \operatorname{sgn}(t) |t|^\beta \bigl(A_1 + i A_2 \operatorname{sgn}(t)\bigr)\bigr) e^{-|t|^\beta (A_1 + i A_2 \operatorname{sgn}(t))}\, dt.$$

In particular, $\sup_u |f_\beta(u) + u f_\beta'(u)|$ is finite, and this proves (22).

In view of (21), it only remains to prove that $\mathbb{E}[J_{n,x} \mathbf{1}_{\Omega_n}] = o(1)$. But this follows from the basic inequality

$$\mathbb{E}[|J_{n,x} \mathbf{1}_{\Omega_n}|] \le \int_{|v|\ge n^{\overline{\eta}}} \mathbb{E}\bigl[e^{-A_1 |v|^\beta V_n/n^{\beta\delta}} \mathbf{1}_{\Omega_n}\bigr]\, dv$$

and from the lower bound for V_n given in (7). □

2.6. *Proof of Proposition* 8. Recall that on Ω_n, $N_n(y) \le n^{1-1/\alpha+\eta}$, for all $y \in \mathbb{Z}$. Hence, by (13),

$$\int_{n^{-\delta+\overline{\eta}}}^{\varepsilon_0 n^{-1+1/\alpha-\eta}} \mathbb{E}\left[\prod_y |\varphi_\xi(tN_n(y))| \mathbf{1}_{\Omega_n}\right] dt \le \int_{n^{-\delta+\overline{\eta}}}^{\varepsilon_0 n^{-1+1/\alpha-\eta}} \mathbb{E}[\exp(-\sigma t^\beta V_n)\mathbf{1}_{\Omega_n}]\, dt.$$

But on Ω_n, we can also use the lower bound for V_n given in (7), which implies that

$$\int_{n^{-\delta+\overline{\eta}}}^{\varepsilon_0 n^{-1+1/\alpha-\eta}} \mathbb{E}\left[\prod_y |\varphi_\xi(tN_n(y))| \mathbf{1}_{\Omega_n}\right] dt \le e^{-\sigma n^{c\eta}},$$

for some constant $c > 0$, depending on β. This proves the proposition.

2.7. *Proof of Proposition* 9. First note that by using again (13), we get

$$\prod_y |\varphi_\xi(tN_n(y))| \le \exp\left(-\sigma t^\beta \sum_{z\,:\, N_n(z) \le \varepsilon_0 n^{1-1/\alpha-\varepsilon}} N_n^\beta(z)\right) \qquad \text{for all } t \le n^{-1+1/\alpha+\varepsilon}. \tag{23}$$

The proof will then be a consequence of the following.

LEMMA 14. *Under the hypotheses of Proposition* 9, *for n large enough and on Ω_n, we have*

$$\#\left\{z : \frac{\varepsilon_0}{10} n^{1-1/\alpha-\varepsilon} \le N_n(z) \le \varepsilon_0 n^{1-1/\alpha-\varepsilon}\right\} \ge \left(\frac{\varepsilon_0}{10}\right)^{2/(\alpha-1)} n^{1/\alpha-(2\varepsilon+\gamma)/(\alpha-1)}.$$

Indeed according to this lemma and (23), we get for n large enough and on Ω_n,

$$\begin{aligned}\prod_y |\varphi_\xi(tN_n(y))| &\le \exp\left(-\sigma' n^{-\beta(1-1/\alpha+\eta)} n^{1/\alpha-(2\varepsilon+\gamma)/(\alpha-1)} n^{\beta(1-1/\alpha-\varepsilon)}\right)\\ &\le \exp\left(-\sigma' n^{1/\alpha-\beta(\eta+\varepsilon)-(2\varepsilon+\gamma)/(\alpha-1)}\right)\\ &\qquad \text{for all } \varepsilon_0 n^{-1+1/\alpha-\eta} \le t \le n^{-1+1/\alpha+\varepsilon},\end{aligned}$$

for some constant $\sigma' > 0$. This proves Proposition 9, since the hypothesis on ε and γ implies that

$$\frac{1}{\alpha} - \beta(\eta+\varepsilon) - \frac{2\varepsilon+\gamma}{\alpha-1} > \frac{1}{\alpha} - 2\beta\varepsilon - \frac{3\varepsilon}{\alpha-1} > 0.$$

PROOF OF LEMMA 14. Let y_1 be such that $N_n(y_1) = N_n^* = \sup_z N_n(z)$. Since $n = \sum_z N_n(z) \le N_n^* R_n$, we have $N_n(y_1) \ge n^{1-1/\alpha-\gamma}$, on Ω_n. Set

$$y_0 := \min\left\{y \ge y_1 : N_n(y) \le \frac{\varepsilon_0}{2} n^{1-1/\alpha-\varepsilon}\right\}.$$

Observe that $y_0 > y_1$ for n large enough, since $\varepsilon > \gamma$ by hypothesis. In particular,

$$N_n(y_0 - 1) > \frac{\varepsilon_0}{2} n^{1-1/\alpha-\varepsilon} \geq N_n(y_0).$$

But on Ω_n,

$$N_n(y_0 - 1) - N_n(y_0) \leq n^{(1-1/\alpha+\gamma)/2}.$$

Moreover, the hypotheses made on γ and ε imply that $\gamma < (1 - 1/\alpha)/3$ and $\varepsilon < (1 - 1/\alpha)/3$. Thus $\varepsilon < (1 - 1/\alpha - \gamma)/2$, or equivalently $(1 - 1/\alpha + \gamma)/2 < 1 - 1/\alpha - \varepsilon$. Therefore,

$$\frac{\varepsilon_0}{4} n^{1-1/\alpha-\varepsilon} \leq N_n(y_0) \leq \frac{\varepsilon_0}{2} n^{1-1/\alpha-\varepsilon}, \tag{24}$$

for n large enough. Next if $|y_0 - z| \leq (\frac{\varepsilon_0}{10})^{2/(\alpha-1)} n^{1/\alpha-(2\varepsilon+\gamma)/(\alpha-1)}$, then on Ω_n,

$$|N_n(z) - N_n(y_0)| \leq \sqrt{|y_0 - z|^{\alpha-1} n^{1-1/\alpha+\gamma}} \leq \frac{\varepsilon_0}{10} n^{1-1/\alpha-\varepsilon}.$$

Together with (24), this proves the lemma. □

2.8. *Proof of Proposition* 10. Let M and N be two positive integers such that $\mathbb{P}(X_1 = N) > 0$ and $\mathbb{P}(X_1 = -M) > 0$. We denote by $\mathcal{C}^+$ the $(M + N)$-tuple $(N, \dots, N, -M, \dots, -M)$ in which N is repeated M times and then $-M$ is repeated N times. We denote by $\mathcal{C}^-$ the "symmetric" $(M + N)$-tuple $(-M, \dots, -M, N, \dots, N)$ in which $-M$ is repeated N times and then N is repeated M times. Set $T := M + N$ and observe that

$$p := \mathbb{P}\big((X_1, \dots, X_T) = \mathcal{C}^+\big) = \mathbb{P}\big((X_1, \dots, X_T) = \mathcal{C}^-\big) > 0.$$

Let us notice that $(X_1, \dots, X_T) = \mathcal{C}^+$ corresponds to a trajectory going up to MN (in M steps) and then coming back down to 0 (in N steps). Analogously, $(X_1, \dots, X_T) = \mathcal{C}^-$ corresponds to a trajectory that goes down to $-MN$ (in N steps) and comes back up to 0 (in M steps).

We introduce now the event

$$\mathcal{D}_n := \left\{ C_n > \frac{np}{2T} \right\},$$

where

$$C_n := \#\left\{ k = 0, \dots, \left\lfloor \frac{n}{T} \right\rfloor - 1 : \big(X_{kT+1}, \dots, X_{(k+1)T}\big) = \mathcal{C}^{\pm} \right\}.$$

Since the sequences $(X_{kT+1}, \dots, X_{(k+1)T})$, for $k \geq 0$, are independent of each other, Chernoff's inequality implies that there exists $c > 0$ such that

$$\mathbb{P}(\mathcal{D}_n) = 1 - o(e^{-cn}).$$

We introduce now the notion of "peak." We say that there is a peak based on y at time n if $S_n = y$ and $(X_{n+1}, \dots, X_{n+T}) = \mathcal{C}^{\pm}$. We will see (in Lemma 15 below) that, on $\Omega_n \cap \mathcal{D}_n$, there is a large number of $y \in \mathbb{Z}$ on which are based a large number of peaks. For any $y \in \mathbb{Z}$, let

$$C_n(y) := \#\left\{k = 0, \dots, \left\lfloor \frac{n}{T} \right\rfloor - 1 : S_{kT} = y \text{ and } (X_{kT+1}, \dots, X_{(k+1)T}) = \mathcal{C}^{\pm}\right\},$$

be the number of peaks based on y before time n (and at times which are multiple of T), and let

$$p_n := \#\{y \in \mathbb{Z} : C_n(y) \geq n^{1-1/\alpha-2\gamma}\}$$

be the number of sites $y \in \mathbb{Z}$ on which at least $n^{1-1/\alpha-2\gamma}$ peaks are based.

LEMMA 15. *On $\Omega_n \cap \mathcal{D}_n$, we have $p_n \geq 3NMn^{1/\alpha-\alpha\gamma}$, for n large enough.*

PROOF. Note that $C_n(y) \leq N_n(y)$ for all $y \in \mathbb{Z}$. Thus, on $\Omega_n \cap \mathcal{D}_n$,

$$\begin{aligned} \frac{np}{2T} &\leq \sum_{y \in \mathbb{Z} : C_n(y) < n^{1-1/\alpha-2\gamma}} C_n(y) + \sum_{y \in \mathbb{Z} : C_n(y) \geq n^{1-1/\alpha-2\gamma}} C_n(y) \\ &\leq n^{1-1/\alpha-2\gamma} R_n + N_n^* p_n \\ &\leq n^{1-\gamma} + p_n n^{1-1/\alpha+\alpha\gamma/2}, \end{aligned}$$

according to (11). This proves the lemma. $\square$

We have proved that, if n is large enough, the event $\Omega_n \cap \mathcal{D}_n$ is contained in the event

$$\mathcal{E}_n := \{p_n \geq 3NMn^{1/\alpha-\alpha\gamma}\}.$$

Now, on $\mathcal{E}_n$, we define Y_i for $i = 1, \dots, \lfloor n^{1/\alpha-\alpha\gamma} \rfloor$, by

$$Y_1 := \min\{y \in \mathbb{Z} : C_n(y) \geq n^{1-1/\alpha-2\gamma}\}$$

and

$$Y_{i+1} := \min\{y \geq Y_i + 3NM : C_n(y) \geq n^{1-1/\alpha-2\gamma}\} \qquad \text{for } i \geq 1.$$

The Y_i's are sites on which at least $n^{1-1/\alpha-2\gamma}$ peaks are based and are such that $|Y_i - Y_j| \geq 3NM$, if $i \neq j$. For every $i = 1, \dots, \lfloor n^{1/\alpha-\alpha\gamma} \rfloor$, let $t_i^1, \dots, t_i^{\lfloor n^{1-1/\alpha-2\gamma} \rfloor}$ be the $\lfloor n^{1-1/\alpha-2\gamma} \rfloor$ first times (which are multiples of T) when a peak is based on the site Y_i. We also define $N_n^0(Y_i + NM)$ as the number of visits of S before time n to $Y_i + NM$, which do not occur during the time intervals $[t_i^j, t_i^j + T]$, for $j \leq \lfloor n^{1-1/\alpha-2\gamma} \rfloor$.

LEMMA 16. *Conditionally to the event $\mathcal{E}_n$, $((N_n(Y_i + MN) - N_n^0(Y_i + MN), i \geq 1)$ is a sequence of independent identically distributed random variables with binomial distribution $\mathcal{B}(\lfloor n^{1-1/\alpha-2\gamma} \rfloor; \frac{1}{2})$. Moreover, this sequence is independent of $((N_n^0(Y_i + MN), i \geq 1)$.*

PROOF. On $\mathcal{E}_n$, we have

$$N_n(Y_i+MN)-N_n^0(Y_i+MN)=\sum_{j=1}^{\lfloor n^{1-1/\alpha-2\gamma}\rfloor}\mathbf{1}_{\{(X_{t_i^j+1},\dots,X_{t_i^j+T})\in\mathcal{C}^+\}},$$

since the peaks based on the other Y_k's cannot pass through Y_i+MN. But conditionally to $\mathcal{E}_n$, the sequence $(\mathbf{1}_{\{(X_{t_i^j+1},\dots,X_{t_i^j+T})\in\mathcal{C}^+\}})_{i,j}$ is a sequence of independent Bernoulli random variables with parameter $1/2$, which is independent of $(X_k, k\notin\bigcup_{i,j}[t_i^j,\dots,t_i^j+T])$. Since $N_n^0(Y_i+MN)$ only depends on the values of the X_k's for $k\notin\bigcup_{i,j}[t_i^j,\dots,t_i^j+T]$, the result follows. $\square$

Let now $\rho:=\sup\{|\varphi_\xi(u)|: d(u,\frac{2\pi}{d}\mathbb{Z})\ge\varepsilon_0\}$. According to (13),

$$\begin{aligned}|\varphi_\xi(u)|&\le\rho\mathbf{1}_{\{d(u,(2\pi/d)\mathbb{Z})\ge\epsilon_0\}}+\exp\left(-\sigma d\left(u,\frac{2\pi}{d}\mathbb{Z}\right)^\beta\right)\mathbf{1}_{\{d(u,(2\pi/d)\mathbb{Z})<\epsilon_0\}}\\&\le\exp(-\sigma n^{-1/\alpha+2\alpha\gamma}),\end{aligned}$$

as soon as $d(u,\frac{2\pi}{d}\mathbb{Z})\ge n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}$ and $\rho\le\exp(-\sigma n^{-1/\alpha+2\alpha\gamma})$. But recall that $\rho<1$ and $2\alpha^2\gamma<1$. Therefore, for n large enough,

$$\begin{aligned}&\prod_z|\varphi_\xi(tN_n(z))|\\&\quad\le\exp\left(-\sigma n^{-1/\alpha+2\alpha\gamma}\#\left\{z:d\left(tN_n(z),\frac{2\pi}{d}\mathbb{Z}\right)\ge n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}\right\}\right).\end{aligned}\tag{25}$$

Then notice that

$$d\left(tN_n(z),\frac{2\pi\mathbb{Z}}{d}\right)\ge n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}\quad\iff\quad N_n(z)\in\mathcal{I}:=\bigcup_{k\in\mathbb{Z}}I_k,\tag{26}$$

where for all $k\in\mathbb{Z}$,

$$I_k:=\left[\frac{2k\pi}{dt}+\frac{n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}}{t},\frac{2(k+1)\pi}{dt}-\frac{n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}}{t}\right].$$

In particular, $\mathbb{R}\setminus\mathcal{I}=\bigcup_{k\in\mathbb{Z}}J_k$, where for all $k\in\mathbb{Z}$,

$$J_k:=\left(\frac{2k\pi}{dt}-\frac{n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}}{t},\frac{2k\pi}{dt}+\frac{n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}}{t}\right).$$

LEMMA 17. *Under the hypotheses of Proposition 10, for every $i\le\lfloor n^{1/\alpha-\alpha\gamma}\rfloor$, $t\in(n^{-1+1/\alpha+\varepsilon},\pi/d)$ and n large enough,*

$$\mathbb{P}(N_n(Y_i+MN)\in\mathcal{I}|\mathcal{E}_n,\ N_n^0(Y_i+MN))\ge\tfrac{1}{3}\qquad\textit{almost surely.}$$

Assume for a moment that this lemma holds true and let us finish now the proof of Proposition 10. Lemmas 16 and 17 ensure that conditionally to $\mathcal{E}_n$ and $((N_n^0(Y_i + MN), i \geq 1)$, the events $\{N_n(Y_i + MN) \in \mathcal{I}\}$, $i \geq 1$, are independent of each other, and all happen with probability at least $1/3$. Therefore, since $\Omega_n \cap \mathcal{D}_n \subseteq \mathcal{E}_n$, there exists $c > 0$, such that

$$\mathbb{P}\biggl(\Omega_n \cap \mathcal{D}_n, \#\{i : N_n(Y_i + MN) \in \mathcal{I}\} \leq \frac{n^{1/\alpha-\alpha\gamma}}{4}\biggr) \leq \mathbb{P}\biggl(B_n \leq \frac{n^{1/\alpha-\alpha\gamma}}{4}\biggr) = o(\exp(-cn)),$$

where for all $n \geq 1$, B_n has binomial distribution $\mathcal{B}(\lfloor n^{1/\alpha-\alpha\gamma}\rfloor; \frac{1}{3})$.

But if $\#\{z : N_n(z) \in \mathcal{I}\} \geq n^{1/\alpha-\alpha\gamma}/4$, then by (25) and (26) there exists a constant $c > 0$, such that

$$\prod_z |\varphi_\xi(tN_n(z))| \leq \exp(-cn^{1/\alpha-\alpha\gamma} n^{-1/\alpha+2\alpha\gamma}),$$

which proves Proposition 10.

PROOF OF LEMMA 17. First notice that by Lemma 16, for any $H \geq 0$,

$$\mathbb{P}\bigl(N_n(Y_i + MN) \in \mathcal{I} | \mathcal{E}_n,\ N_n^0(Y_i + MN) = H\bigr) = \mathbb{P}(H + b_n \in \mathcal{I}), \tag{27}$$

where b_n is a random variable with binomial distribution $\mathcal{B}(\lfloor n^{1-1/\alpha-2\gamma}\rfloor; \frac{1}{2})$. We will use the following result whose proof is postponed.

LEMMA 18. *Under the hypotheses of Proposition* 10, *for every* $t \in (n^{-1+1/\alpha+\varepsilon}, \pi/d)$ *and for* n *large enough*, *the following hold*:

(i) *For any integer* k *such that all the elements of* $I_k - H$ *are smaller than* $\frac{1}{2}\lfloor n^{1-1/\alpha-2\gamma}\rfloor$,

$$\mathbb{P}\bigl(b_n \in (I_k - H)\bigr) \geq \mathbb{P}\bigl(b_n \in (J_k - H)\bigr).$$

(ii) *For any integer* k *such that all the elements of* $I_k - H$ *are larger than* $\frac{1}{2}\lfloor n^{1-1/\alpha-2\gamma}\rfloor$,

$$\mathbb{P}\bigl(b_n \in (I_k - H)\bigr) \geq \mathbb{P}\bigl(b_n \in (J_{k+1} - H)\bigr).$$

Now call k_0 the largest integer satisfying the condition appearing in (i) and k_1 the smallest integer satisfying the condition appearing in (ii). We have $k_1 = k_0 + 1$ or $k_1 = k_0 + 2$. According to Lemma 18, we have

$$\begin{aligned}\mathbb{P}(H + b_n \in \mathcal{I}) &\geq \sum_{k \leq k_0} \mathbb{P}(H + b_n \in I_k) + \sum_{k \geq k_1} \mathbb{P}(H + b_n \in I_k) \\ &\geq \sum_{k \leq k_0} \mathbb{P}(H + b_n \in J_k) + \sum_{k \geq k_1} \mathbb{P}(H + b_n \in J_{k+1}) \\ &= \mathbb{P}(H + b_n \notin \mathcal{I}) - \mathbb{P}(H + b_n \in J_{k_0+1} \cup J_{k_1}).\end{aligned}$$

Hence,

$$\mathbb{P}(H + b_n \in \mathcal{I}) \geq \tfrac{1}{2}[1 - \mathbb{P}(H + b_n \in J_{k_0+1} \cup J_{k_1})].$$

Let $\bar{b}_n := 2(b_n - \frac{1}{2}\lfloor n^{1-1/\alpha-2\gamma}\rfloor)\lfloor n^{1-1/\alpha-2\gamma}\rfloor^{-1/2}$, so that $\bar{b}_n$ converges in distribution to a standard normal variable, whose distribution function is denoted by Φ. The interval J_{k_1} being of length $2n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}/t$,

$$\begin{aligned}
&\mathbb{P}(H + b_n \in J_{k_1})\\
&\quad = \mathbb{P}(\bar{b}_n \in [m_n, M_n]), \qquad \text{with } M_n - m_n = 4\frac{n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}}{t\sqrt{\lfloor n^{1-1/\alpha-2\gamma}\rfloor}}\\
&\quad \leq \Phi(M_n) - \Phi(m_n) + \frac{C}{\sqrt{n^{1-1/\alpha-2\gamma}}} \qquad \text{(by the Berry–Esseen inequality)}\\
&\quad \leq \frac{M_n - m_n}{\sqrt{2\pi}} + \frac{C}{\sqrt{n^{1-1/\alpha-2\gamma}}}\\
&\quad \leq C' n^{1/2+1/(2\alpha)+\gamma+2\alpha\gamma/\beta-1/(\alpha\beta)-1/\alpha-\varepsilon} + \frac{C}{\sqrt{n^{1-1/\alpha-2\gamma}}},
\end{aligned}$$

for $t \geq n^{-1+1/\alpha+\varepsilon}$, and some constants $C > 0$ and $C' > 0$. Since $\alpha \leq 2$, $\beta \leq 2$, $\gamma < \frac{1}{2}\frac{\alpha-1}{\alpha}$ and $\varepsilon > 2\alpha\gamma/\beta + \gamma$ by hypothesis, we conclude that $\mathbb{P}(H + b_n \in J_{k_1}) = o(1)$. The same holds for $\mathbb{P}(H + b_n \in J_{k_0+1})$, so that for n large enough,

$$\mathbb{P}(H + b_n \in \mathcal{I}) \geq \tfrac{1}{2}[1 - o(1)] \geq \tfrac{1}{3}.$$

Together with (27), this concludes the proof of Lemma 17. $\square$

PROOF OF LEMMA 18. We only prove (i), since (ii) is similar. So let k be an integer such that all the elements of $I_k - H$ are smaller than $\frac{1}{2}\lfloor n^{1-1/\alpha-2\gamma}\rfloor$. Assume that $(J_k - H) \cap \mathbb{Z}$ contains at least one nonnegative integer [otherwise $\mathbb{P}(b_n \in (J_k - H)) = 0$ and there is nothing to prove]. Let z_k denote the greatest integer in $J_k - H$, so that by our assumption $\mathbb{P}(b_n = z_k) > 0$ (remind that $0 \leq z_k < \frac{1}{2}\lfloor n^{1-1/\alpha-2\gamma}\rfloor$). By monotonicity of the function $z \mapsto \mathbb{P}(b_n = z)$, for $z \leq \frac{1}{2}\lfloor n^{1-1/\alpha-2\gamma}\rfloor$, we get

$$\mathbb{P}(b_n \in J_k - H) \leq \mathbb{P}(b_n = z_k)\#\big((J_k - H) \cap \mathbb{Z}\big) \leq \mathbb{P}(b_n = z_k)\left\lceil \frac{2n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}}{t} \right\rceil.$$

In the same way,

$$\begin{aligned}
\mathbb{P}(b_n \in I_k - H) &\geq \mathbb{P}(b_n = z_k)\#\big((I_k - H) \cap \mathbb{Z}\big)\\
&\geq \mathbb{P}(b_n = z_k)\left\lfloor \frac{2\pi}{dt} - \frac{2n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}}{t} \right\rfloor.
\end{aligned}$$

Hence,

$$\mathbb{P}(b_n \in I_k - H) \geq \frac{\lfloor (2\pi/(dt)) - 2n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}/t \rfloor}{\lceil 2n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}/t \rceil} \mathbb{P}(b_n \in J_k - H).$$

But $\pi/(dt) \geq 1$ and $2\alpha^2\gamma < 1$ by hypothesis. It follows immediately that for n large enough, we have $2n^{-1/(\alpha\beta)+2\alpha\gamma/\beta} < \pi/(2d)$, and so

$$\begin{aligned}\left\lfloor \frac{2\pi}{dt} - \frac{2n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}}{t} \right\rfloor \geq \left\lfloor \frac{3\pi}{2dt} \right\rfloor \geq 1 + \left\lfloor \frac{\pi}{2dt} \right\rfloor \geq \left\lceil \frac{\pi}{2dt} \right\rceil \\ \geq \left\lceil \frac{2n^{-1/(\alpha\beta)+2\alpha\gamma/\beta}}{t} \right\rceil.\end{aligned}$$

This concludes the proof of the lemma. □

3. Lattice case, $\alpha < 1$: Proof of Theorem 2. We only sketch the proof, since it is very similar and simpler than in the case $\alpha > 1$. In particular we keep the same notation, for instance for N_n^*, R_n, V_n, $\varepsilon_0, \ldots$

We first introduce the analogue Ω_n' of Ω_n:

$$\Omega_n' = \Omega_n'(\varepsilon) := \{N_n^* \leq n^\varepsilon\},$$

which is well defined for any ε. Note that on Ω_n', we have

$$V_n \geq R_n \geq n^{1-\varepsilon}. \tag{28}$$

Since $N_n^* = \sup_{k=0}^{n-1}[N_n(S_k) - N_k(S_k)]$, we obtain that

$$\mathbb{P}(N_n^* \geq n^\varepsilon) \leq n\mathbb{P}\Big(\sup_m N_m(0) \geq n^\varepsilon\Big) \leq n p_0^{n^\varepsilon - 1},$$

where $p_0 := \mathbb{P}(\exists k \geq 1 \ : \ S_k = 0)$. Since $\alpha < 1$, the random walk S is transient and $p_0 < 1$. It follows that $\mathbb{P}(\Omega_n') = 1 - o(\exp(-n^c))$, for some constant $c > 0$, and we can restrict our study to this set. Moreover, it is known (see, for instance, the Introduction in [24] for an argument) that

$$\frac{1}{n} V_n = \frac{1}{n} \sum_{y \in \mathbb{Z}} N_n^\beta(y) \underset{n \to +\infty}{\longrightarrow} \mathbb{E}[\widetilde{N}_\infty^{\beta-1}(0)] = r^{-\beta} \qquad \text{a.s.}$$

We claim now that $(n^{1/\beta} V_n^{-1/\beta}, n \geq 1)$ is uniformly integrable. Indeed, if $\beta \geq 1$, this comes from the fact that V_n is larger than n, and when $\beta < 1$, this follows from the following.

LEMMA 19. *If $\beta < 1$, there exists $\gamma > 0$ such that*

$$\sup_n \mathbb{E}\Big[\exp\Big(\gamma \frac{n}{V_n}\Big)\Big] < \infty. \tag{29}$$

PROOF. Since $n = \sum_x N_n(x)$, Hölder's inequality gives

$$\frac{n}{V_n} \le \frac{\sum_x N_n(x)^2}{n}.$$

Since

$$\frac{1}{n}\sum_x N_n(x)^2 = \frac{1}{n}\sum_{k=0}^{n-1} N_n(S_k),$$

Jensen's inequality gives

$$\exp\left(\gamma \frac{\sum_x N_n(x)^2}{n}\right) \le \frac{1}{n}\sum_{k=0}^{n-1} \exp(\gamma N_n(S_k)).$$

Hence,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\gamma \frac{\sum_x N_n(x)^2}{n}\right)\right] &\le \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}[\exp(\gamma N_n(S_k))] \\ &\le \mathbb{E}[\exp(\gamma \widetilde{N}_\infty(0))].\end{aligned}$$

Then, (29) directly follows from the fact that $\widetilde{N}_\infty(0)$ is equal to 1 plus the sum of two independent geometric variables with positive parameter, and thus has finite exponential moments. □

Let $\varepsilon > 0$ and $\eta > 0$ be such that $\eta + \varepsilon < 1/\beta$ and $\varepsilon < \eta\beta < 1/2$. As in the proof of Proposition 7, we deduce that

$$\frac{d}{2\pi}\int_{|t|\le n^{-1/\beta+\eta}} e^{-it\lfloor n^{1/\beta}x\rfloor}\mathbb{E}\left[\prod_y \varphi_\xi(tN_n(y))\right]dt = \frac{D(x)}{n^{1/\beta}} + o(n^{-1/\beta}),$$

where the $o(n^{-1/\beta})$ is uniform in x. It remains to prove that

$$\frac{d}{2\pi}\int_{n^{-1/\beta+\eta}}^{\pi/d}\left|\mathbb{E}\left[\prod_y \varphi_\xi(tN_n(y))\mathbf{1}_{\Omega_n'}\right]\right|dt = o(n^{-1/\beta}). \tag{30}$$

As in the proof of Proposition 10 [see the beginning of Section 2.8 for the definitions of $\mathcal{D}_n, C_n, C_n(y), \ldots$], we are led to prove that

$$\frac{d}{2\pi}\int_{n^{-1/\beta+\eta}}^{\pi/d}\left|\mathbb{E}\left[\prod_y \varphi_\xi(tN_n(y))\mathbf{1}_{\Omega_n'\cap\mathcal{D}_n}\right]\right|dt = o(n^{-1/\beta}).$$

Let $p_n' := \#\{y : C_n(y) \ge 1\}$ be the random variable equal to the number of sites of $\mathbb{Z}$ on which at least one peak is based. Let us notice that on $\Omega_n' \cap \mathcal{D}_n$, we have

$$\frac{np}{2T} \le C_n = \sum_y C_n(y) \le \sum_{y : C_n(y)\ge 1} N_n(y) \le p_n' n^\varepsilon.$$

Thus, $\Omega'_n \cap \mathcal{D}_n \subseteq \mathcal{E}'_n$, where $\mathcal{E}'_n := \{p'_n \geq c_0 n^{1-\varepsilon}\}$, for a well chosen constant $c_0 > 0$. As in the proof of Proposition 10, we construct $(Y_i)_i$ such that $C_n(Y_i) \geq 1$ and $Y_{i+1} - Y_i > MN$. For every i, we define $N_n^0(Y_i + MN)$ as the number of visits to the site $Y_i + MN$ without taking into account the possible visit during the first peak based on Y_i. Next, we see that on $\mathcal{E}'_n$, $(N_n(Y_i + MN) - N_n^0(Y_i + MN), i \leq c_0 n^{1-\varepsilon})$ is a sequence of i.i.d. random variables with Bernoulli distribution with parameter $1/2$.

Let $t \in [n^{-1/\beta+\eta}, \frac{\pi}{d}]$. We define the good and bad intervals, respectively, by

$$I'_k := \left[\frac{2k\pi}{dt} + \frac{1}{2}, \frac{2(k+1)\pi}{dt} - \frac{1}{2}\right]$$

and

$$J'_k := \left(\frac{2k\pi}{dt} - \frac{1}{2}, \frac{2k\pi}{dt} + \frac{1}{2}\right).$$

Set also $\mathcal{I}' := \bigcup_{k\in\mathbb{Z}} I'_k$. We observe that J'_k is an open interval of length 1 and I'_k is a closed interval of length $2\pi/(dt) - 1 \geq 1$ (since $t \leq \pi/d$). Hence, if $N_n^0(Y_i + MN)$ is not in $\mathcal{I}'$, then $N_n^0(Y_i + MN) + 1$ is in $\mathcal{I}'$. This ensures that, on $\mathcal{E}'_n$, $N_n(Y_i + MN)$ belongs to $\mathcal{I}'$ with probability at least $1/2$. Therefore, as after Lemma 17, we get

$$\mathbb{P}\left(\Omega_n \cap \mathcal{D}_n; \#\{i : N_n(Y_i + MN) \in \mathcal{I}'\} < \frac{c_0 n^{1-\varepsilon}}{3}\right) = o(n^{-1/\beta}).$$

Hence, we just have to prove that

$$\frac{d}{2\pi} \int_{n^{-1/\beta+\eta}}^{\pi/d} \left|\mathbb{E}\left[\prod_y \varphi_\xi(tN_n(y)) \mathbf{1}_{\mathcal{H}_{n,t}}\right]\right| dt = o(n^{-1/\beta}),$$

with $\mathcal{H}_{n,t} := \{\#\{y : N_n(y) \in \mathcal{I}'\} \geq \frac{c_0 n^{1-\varepsilon}}{3}\}$. As after Lemma 16, we notice that, if n is large enough, we have

$$d\left(u, \frac{2\pi}{d}\mathbb{Z}\right) \geq \frac{n^{-1/\beta+\eta}}{2} \quad \Rightarrow \quad |\varphi_\xi(u)| \leq \exp\left(-\frac{\sigma}{2^\beta} n^{-1+\beta\eta}\right).$$

We notice also that if $N_n(y) \in \mathcal{I}'$, then $d(tN_n(y), \frac{2\pi}{d}\mathbb{Z}) \geq t/2$, and thus $d(tN_n(y), \frac{2\pi}{d}\mathbb{Z}) \geq n^{-1/\beta+\eta}/2$. Now, on $\mathcal{H}_{n,t}$, we know that at least $c_0 n^{1-\varepsilon}/3$ sites y satisfy this property. Therefore,

$$\left|\mathbb{E}\left[\prod_y \varphi_\xi(tN_n(y)) \mathbf{1}_{\mathcal{H}_{n,t}}\right]\right| \leq \exp\left(-\frac{c_0\sigma}{2^\beta 3} n^{1-\varepsilon} n^{-1+\beta\eta}\right) = o(n^{-1/\beta}),$$

since $\varepsilon < \beta\eta$. This gives (30) and achieves the proof of Theorem 2.

4. The strongly nonlattice case: Proof of Theorem 3. We assume here that ξ is strongly nonlattice. In that case, there exist[2] $\varepsilon_0 > 0$, $\sigma > 0$ and $\rho < 1$ such that

$$|\varphi_\xi(u)| \le \rho \qquad \text{if } |u| \ge \varepsilon_0,$$
$$|\varphi_\xi(u)| \le \exp(-\sigma|u|^\beta) \qquad \text{if } |u| < \varepsilon_0.$$

Case $\alpha > 1$. We use here the notation of Section 2 with the hypotheses on γ, η, $\bar\eta$ and ε of Propositions 7–10. Let h_0 be the density of Polya's distribution

$$h_0(y) = \frac{1}{\pi}\frac{1-\cos(y)}{y^2}.$$

Its Fourier transform is $\hat h_0(t) = (1-|t|)_+$. For $\theta \in \mathbb{R}$, let $h_\theta(y) = \exp(i\theta y)h_0(y)$ with the Fourier transform $\hat h_\theta(t) = \hat h_0(t+\theta)$. As proved in [16] (see the proof of Theorem 5.4, page 114), it is enough to show that for all $\theta \in \mathbb{R}$,

$$\lim_{n\to\infty} n^\delta \mathbb{E}[h_\theta(Z_n - n^\delta x)] = C(x)\hat h_\theta(0). \tag{31}$$

By the Fourier inverse transform,

$$n^\delta \mathbb{E}[h_\theta(Z_n - n^\delta x)] = \frac{n^\delta}{2\pi}\int_{\mathbb{R}} e^{-iun^\delta x}\mathbb{E}\Big[\prod_{x\in\mathbb{Z}} \varphi_\xi(uN_n(x))\Big]\hat h_\theta(u)\,du.$$

Since $\hat h_\theta \in L^1$, we can restrict our study to the event Ω_n of Lemma 6. The part of the integral corresponding to $|u| \le n^{-\delta+\bar\eta}$ is treated exactly as in Proposition 7. The only change is that we have to check that

$$\lim_{n\to\infty} n^\delta \int_{|u|\le n^{-\delta+\bar\eta}} \mathbb{E}\big[e^{-|u|^\beta V_n(A_1+iA_2\,\mathrm{sgn}(u))}\mathbf{1}_{\Omega_n}\big]\big(\hat h_\theta(u) - \hat h_\theta(0)\big)\,du = 0,$$

which is obviously the case since $2\bar\eta < \delta$, using the fact that $\hat h_\theta$ is a Lipschitz function.

Now since $\hat h_\theta$ is bounded, the part corresponding to $n^{-\delta+\bar\eta} \le |u| \le n^{-1+1/\alpha+\varepsilon}$ does not need any additional treatment. Actually, the proofs of Propositions 8 and 9 only use the behavior of φ_ξ around 0, which is the same in the lattice or nonlattice case.

We now turn our attention to the part of the integral corresponding to $|u| \ge n^{-1+1/\alpha+\varepsilon}$ and prove that

$$\lim_{n\to\infty} n^\delta \int_{|u|\ge n^{-1+1/\alpha+\varepsilon}} e^{-iun^\delta x}\mathbb{E}\Big[\prod_x \varphi_\xi(uN_n(x))\mathbf{1}_{\Omega_n}\Big]\hat h_\theta(u)\,du = 0. \tag{32}$$

---

[2]The existence of ε_0 satisfying the second assertion comes from (12). Now the hypothesis ξ strongly nonlattice ensures that there exist $M > 0$ and $\rho_1 < 1$ such that $|\varphi_\xi(u)| < \rho_1$ if $|u| > M$. Moreover there cannot exist $u \in [\varepsilon_0, M]$ such that $|\varphi_\xi(u)| = 1$. Otherwise, we would have $|\varphi_\xi(nu)| = 1$ for every integer n and this would contradict the nonlattice hypothesis. Since φ_ξ is continuous, this ensures the existence of $\rho_2 < 1$ such that $|\varphi_\xi(u)| < \rho_2$ if $\varepsilon_0 \le |u| \le M$. Finally, we get the first assertion with $\rho := \max(\rho_1, \rho_2)$.

To this end, note that

$$\left|\mathbb{E}\left[\prod_x \varphi_\xi(uN_n(x))\mathbf{1}_{\Omega_n}\right]\right| \le \mathbb{E}\left[\rho^{\#\{x\,:\,|uN_n(x)|\ge\varepsilon_0\}}\mathbf{1}_{\Omega_n}\right]$$

and that on Ω_n, for $|u| \ge n^{-1+1/\alpha+\varepsilon}$,

$$\begin{aligned} n = \sum_x N_n(x) &\le \frac{\varepsilon_0}{|u|} R_n + N_n^* \#\{x : |uN_n(x)| \ge \varepsilon_0\} \\ &\le \varepsilon_0 n^{1-\varepsilon+\eta\beta/2} + n^{1-1/\alpha+\eta}\#\{x : |uN_n(x)| \ge \varepsilon_0\}. \end{aligned}$$

Hence, since $\varepsilon > \eta\beta/2$, for n large enough, on Ω_n, and for $|u| \ge n^{-1+1/\alpha+\varepsilon}$,

$$\#\{x : |uN_n(x)| \ge \varepsilon_0\} \ge \tfrac{1}{2} n^{1/\alpha-\eta}.$$

Therefore, for n large enough,

$$\begin{aligned} &\left| n^\delta \int_{|u|\ge n^{-1+1/\alpha+\varepsilon}} e^{-iun^\delta x}\mathbb{E}\left[\prod_x \varphi_\xi(uN_n(x))\mathbf{1}_{\Omega_n}\right]\hat{h}_\theta(u)\,du\right| \\ &\qquad \le n^\delta \rho^{(1/2)n^{1/\alpha-\eta}} \int_{\mathbb{R}} \hat{h}_\theta(u)\,du, \end{aligned}$$

which tends to zero since $\eta < 1/\alpha$.

Case $\alpha < 1$. Using the notation and hypotheses on $\varepsilon, \eta, \gamma$ of Section 3, one has to prove that for all $\theta \in \mathbb{R}$ and all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} n^{1/\beta} \int_{\mathbb{R}} e^{-iun^{1/\beta}x}\mathbb{E}\left[\prod_x \varphi_\xi(uN_n(x))\mathbf{1}_{\Omega'_n}\right]\hat{h}_\theta(u)\,du = D(x)\hat{h}_\theta(0). \tag{33}$$

Again, the only change in the proof concerns the part of the integral corresponding to $|u| \ge n^{-1/\beta+\eta}$. We use here the bound

$$\begin{aligned} |\varphi_\xi(uN_n(x))| &\le \exp(-\sigma|u|^\beta N_n^\beta(x))\mathbf{1}_{\{|uN_n(x)|\le\varepsilon_0\}} + \rho\mathbf{1}_{\{|uN_n(x)|\ge\varepsilon_0\}} \\ &\le \exp(-\sigma n^{-1+\eta\beta} N_n^\beta(x))\mathbf{1}_{\{|uN_n(x)|\le\varepsilon_0\}} + \rho\mathbf{1}_{\{|uN_n(x)|\ge\varepsilon_0\}}. \end{aligned}$$

If $\eta < 1/\beta$, then for n large enough, $\rho \le \exp(-\sigma n^{-1+\eta\beta})$. Therefore, if n is large enough, then for all x and u such that $N_n(x) \ge 1$ and $|u| \ge n^{-1/\beta+\eta}$, we have

$$|\varphi_\xi(uN_n(x))| \le \exp(-\sigma n^{-1+\eta\beta}).$$

Hence,

$$\left|\mathbb{E}\left[\prod_x \varphi_\xi(uN_n(x))\mathbf{1}_{\Omega'_n}\right]\right| \le \mathbb{E}[\exp(-\sigma n^{-1+\eta\beta}R_n)\mathbf{1}_{\Omega'_n}] \le \exp(-\sigma n^{\eta\beta-\varepsilon}).$$

Therefore, since $\varepsilon < \eta\beta$,

$$\lim_{n\to\infty} n^{1/\beta} \int_{|u|\ge n^{-1/\beta+\eta}} e^{-iun^{1/\beta}x}\mathbb{E}\left[\prod_x \varphi_\xi(uN_n(x))\mathbf{1}_{\Omega'_n}\right]\hat{h}_\theta(u)\,du = 0.$$

This concludes the proof of Theorem 3.

5. Random walks on randomly oriented lattices.

5.1. *Model and result.* We consider parallel moving pavements with different fixed speeds, independently chosen at the beginning with the same distribution. We study the random walk $(M_n, n \geq 0)$ representing the position of a walker who at each time stays on the same moving pavement with probability $p \in (0, 1)$, or jumps to another one with probability $1 - p$.

Let us define $(M_n, n \geq 0)$ more precisely. Let μ_X be a distribution on $\mathbb{Z}$ in the normal domain of attraction of a centered stable distribution with index $1 < \alpha \leq 2$ and density function denoted by $f_\alpha(\cdot)$. Let also $\xi := (\xi_y, y \in \mathbb{Z})$ be a sequence of independent centered $\mathbb{Z}$-valued random variables with distribution μ_ξ belonging to the normal domain of attraction of a stable distribution with index $1 < \beta \leq 2$ and density function denoted by $f_\beta(\cdot)$. For each $y \in \mathbb{Z}$, ξ_y will be the only horizontal displacement allowed on line y. Let $p \in (0, 1)$. Given ξ, the random walk $(M_n = (M_n^{(1)}, M_n^{(2)}), n \geq 0)$ is a Markov chain starting from $M_0 := (0, 0)$ and such that at time $n + 1$, it moves either horizontally of $\xi_{M_n^{(2)}}$ (with probability p) or makes a vertical jump with distribution μ_X [with probability $(1 - p)$], that is,

$$\mathbb{P}\big(M_{n+1} - M_n = (\xi_{M_n^{(2)}}, 0) | \xi,\ M_1, \dots, M_n\big) = p \qquad \text{if } \xi_{M_n^{(2)}} \neq 0,$$
$$\mathbb{P}\big(M_{n+1} - M_n = (0, x) | \xi,\ M_1, \dots, M_n\big) = (1 - p)\mu_X(x) \qquad \text{if } x \neq 0$$

and

$$\mathbb{P}(M_{n+1} = M_n | \xi,\ M_1, \dots, M_n) = p + (1 - p)\mu_X(0) \qquad \text{if } \xi_{M_n^{(2)}} = 0.$$

These random walks were first introduced by Campanino and Pétritis in [6] in the particular case when $p = 1/3$ and when μ_X and μ_ξ are Rademacher distributions, that is, take values ± 1 with probability $1/2$. They proved the transience of M as well as a law of large numbers. In [18], Guillotin-Plantard and Le Ny established the link between the Campanino and Pétritis random walk and the random walk in random scenery and proved a functional limit theorem for the first one. It was also conjectured there that the probability of return to the origin of the Campanino and Pétritis random walk is equivalent to a constant times $n^{-5/4}$. We prove this result here, as well as a generalization to the case of the random walks M considered above.

To state our result, we will use the following representation of M:

Let $X := (X_n, n \geq 1)$ be a sequence of independent random variables with distribution μ_X. The random variable X_n corresponds to the vertical move at time n which will be chosen with probability $1 - p$. Let also $(\epsilon_n, n \geq 0)$ be a sequence of independent Bernoulli random variables with parameter p, that is, such that $\mathbb{P}(\epsilon_1 = 1) = 1 - \mathbb{P}(\epsilon_1 = 0) = p$, and independent of X. If $\epsilon_n = 1$, the particle M moves horizontally at time n, otherwise it moves vertically. We then first define S

by $S_0 := 0$ and

$$S_n := \sum_{k=1}^{n} Y_k \qquad \text{for } n \geq 1,$$

where $Y_k := X_k(1-\epsilon_k)$. We next define $\widetilde{Z}$ by $\widetilde{Z}_0 := 0$ and

$$\widetilde{Z}_n := \sum_{k=1}^{n} \xi_{S_{k-1}} \epsilon_k = \sum_{y\in\mathbb{Z}} \xi_y \widetilde{N}_n(y) \qquad \text{for } n \geq 1,$$

where

$$\widetilde{N}_n(y) := \#\{k = 1, \dots, n : S_{k-1} = y \text{ and } \epsilon_k = 1\}.$$

Then it is straightforward that the couple $(\widetilde{Z}, S)$ has the same distribution as M.

We just notice that the process S in this section is not exactly the same as in the previous sections (it is the same if we replace X by Y). However, we use the same notation just for convenience.

Now it is known that $(n^{-1/\alpha} S_{[nt]}, t \geq 0)$ converges in distribution, when $n \to \infty$, to a Lévy process $\widetilde{U} = (\widetilde{U}_t, t \geq 0)$ where $\widetilde{U} = (1-p)^{1/\alpha} U$ and U is the process introduced in the Introduction. We will use the fact that $(n^{-1/\alpha} S_{[nt]}, t \geq 0 \mid S_n = 0)$ converges in distribution to $\widetilde{U}^0 = (\widetilde{U}_t^0, t \geq 0)$ the associated bridge, that is, the process $\widetilde{U}$ starting from 0 and conditioned by $\widetilde{U}_1 = 0$. Let $(L_t^0(x), t \in [0,1], x \in \mathbb{R})$ be the local time process of $\widetilde{U}^0$ and set $|L^0|_\beta := (\int_{\mathbb{R}} (L_1^0(x))^\beta \, dx)^{1/\beta}$.

Let φ_ξ be the characteristic function of ξ_1. Recall that d is the positive integer such that $\{u : |\varphi_\xi(u)| = 1\} = (2\pi/d)\mathbb{Z}$. Let d_0 be the smallest positive integer m such that $\varphi_\xi(2\pi/d)^m = 1$ and let d_1 be the greatest common divisor of $\{m \geq 1 : \mathbb{P}(X_1 + \cdots + X_m = 0) > 0\}$.

THEOREM 20. *Assume that d_1 is a multiple of d_0, and let $E = dp^{-1} f_\alpha(0) \times f_\beta(0)\mathbb{E}(|L^0|_\beta^{-1})$. Then*

$$\mathbb{P}(M_n = (0,0)) = \begin{cases} E \times n^{-1-1/(\alpha\beta)} + o(n^{-1-1/\alpha\beta}), & \textit{if } n \textit{ is a multiple of } d_0, \\ 0, & \textit{otherwise}. \end{cases}$$

REMARK 21. In the case of the Campanino and Pétritis random walk, $d_0 = d_1 = 2$. So the hypothesis of the theorem is satisfied.

REMARK 22. A corollary of our result is that the processes M considered here are transient, this can be seen by using Borel–Cantelli lemma.

REMARK 23. It is most likely that an analogue result can be proved when $\alpha < 1$ or $\beta \leq 1$. We leave the details to the interested reader. In the same way, one could certainly obtain similar estimates for the probabilities of return in $([n^\delta x], [n^{1/\alpha} y])$, with a constant E depending on x and y.

REMARK 24. An analogue result holds true for the couple (Z_n, S_n) and can be proved similarly.

5.2. *The event* $\widetilde{\Omega}_n$. Let $(N_n(y), y \in \mathbb{Z})$ and R_n denote, respectively, the local time process and the range of S at time n:

$$N_n(y) := \#\{k = 0, \dots, n-1 : S_k = y\} \quad \text{and} \quad R_n := \#\{y : N_n(y) > 0\}.$$

For $\gamma > 0$, set $\widetilde{\Omega}_n = \widetilde{\Omega}_n(\gamma) := \mathcal{A}_n \cap \mathcal{B}_n \cap \mathcal{C}_n$, where

$$\mathcal{A}_n := \Big\{R_n \le n^{1/\alpha+\gamma} \text{ and } \sup_y N_n(y) \le n^{1-1/\alpha+\gamma}\Big\},$$

$$\mathcal{B}_n := \left\{\sum_{k=1}^n \epsilon_k \ge \frac{np}{2}\right\}$$

and

$$\mathcal{C}_n := \left\{\sup_{y \neq z} \frac{|\widetilde{N}_n(y) - \widetilde{N}_n(z)|}{|y-z|^{(\alpha-1)/2}} \le n^{(1-1/\alpha+\gamma)/2}\right\}.$$

LEMMA 25. *For all* $\gamma > 0$, $\mathbb{P}(\widetilde{\Omega}_n) = 1 - o(n^{-1-1/(\alpha\beta)})$.

PROOF. According to the proof of Lemma 6, $\mathbb{P}(R_n \le n^{1/\alpha+\gamma}) = 1 - o(n^{-1-1/(\alpha\beta)})$. Moreover, according to the proof of Lemma 11 [see (19)], we have for all $\nu \ge 1$,

$$\mathbb{E}\Big[\sup_y N_n^\nu(y)\Big] = \mathcal{O}\big(n^{\nu(1-1/\alpha)}\big). \tag{34}$$

Hence, by the use of the Markov inequality, we get

$$\mathbb{P}\Big(\sup_{y \in \mathbb{Z}} N_n(y) \ge n^{1-1/\alpha+\gamma}\Big) = o\big(n^{-1-1/(\alpha\beta)}\big).$$

It follows that $\mathbb{P}(\mathcal{A}_n) = 1 - o(n^{-1-1/(\alpha\beta)})$.

Next, it is well known that $\mathbb{P}(\mathcal{B}_n) = 1 - o(n^{-1-1/(\alpha\beta)})$.

Finally, as in the proof of Lemma 6, the estimate of $\mathbb{P}(\mathcal{C}_n)$ comes from the following lemma.

LEMMA 26. *For any integer* $\nu \ge 1$, *there exists a constant* $C_\nu > 0$ *such that, for every* $n \ge 1$ *and every* $x, y \in \mathbb{Z}$

$$\mathbb{E}\big[\big(\widetilde{N}_n(x) - \widetilde{N}_n(y)\big)^{2\nu}\big] \le C_\nu |x-y|^{\nu(\alpha-1)} n^{\nu(1-1/\alpha)}.$$

PROOF. Let x and y be fixed, with $x \neq y$ (otherwise, there is nothing to prove). We have

$$\widetilde{N}_n(x) = pN_n(x) + \sum_{k=1}^n \mathbf{1}_{\{S_{k-1}=x\}} \bar{\epsilon}_k, \tag{35}$$

where $\bar{\epsilon}_k = \mathbf{1}_{\{\epsilon_k=1\}} - p$. Set $H_n(x) := \sum_{k=1}^n \mathbf{1}_{\{S_{k-1}=x\}}\bar{\epsilon}_k$. For all $x \in \mathbb{Z}$, $(H_n(x), n \geq 1)$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_k, \epsilon_k, k \leq n)$. Hence, $(H_n(x) - H_n(y), n \geq 1)$ is a martingale as well. According to Burkholder's inequality (see [21], Theorem 2.11, page 23), for all integer $\nu \geq 1$, there exists a constant $C = C(\nu)$ such that for all $n \geq 1$,

$$\begin{aligned}&\mathbb{E}\big[\big(H_n(x) - H_n(y)\big)^{2\nu}\big]^{1/(2\nu)}\\ &\quad\leq C\left\{\mathbb{E}\left[\left(\sum_{k=1}^n \mathbb{E}(d_k^2(x,y)|\mathcal{F}_{k-1})\right)^\nu\right]^{1/(2\nu)} + \mathbb{E}\Big[\sup_{k=1,\dots,n} |d_k(x,y)|^{2\nu}\Big]^{1/(2\nu)}\right\},\end{aligned}$$

where $d_k(x,y)$ is the martingale increment

$$d_k(x,y) := H_k(x) - H_{k-1}(x) - H_k(y) + H_{k-1}(y) = \big(\mathbf{1}_{\{S_{k-1}=x\}} - \mathbf{1}_{\{S_{k-1}=y\}}\big)\bar{\epsilon}_k.$$

Note that for all $k \geq 1$, and all $x, y \in \mathbb{Z}$, $|d_k(x,y)| \leq 1$, and that

$$\begin{aligned}\sum_{k=1}^n \mathbb{E}(d_k^2(x,y)|\mathcal{F}_{k-1}) &= \operatorname{Var}(\epsilon_1)\sum_{k=1}^n \big(\mathbf{1}_{\{S_{k-1}=x\}} - \mathbf{1}_{\{S_{k-1}=y\}}\big)^2\\ &= \operatorname{Var}(\epsilon_1)\big(N_n(y) + N_n(x)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\big[\big(H_n(x) - H_n(y)\big)^{2\nu}\big]^{1/(2\nu)} &\leq C\{1 + \mathbb{E}[N_n^\nu(y)]^{1/(2\nu)} + \mathbb{E}[N_n^\nu(x)]^{1/(2\nu)}\}\\ &\leq C\big(1 + 2n^{(1-1/\alpha)/2}\big) \qquad \text{[by using (34)]}\\ &\leq 3Cn^{(1-1/\alpha)/2}|x-y|^{(\alpha-1)/2},\end{aligned}$$

since $|x - y| \geq 1$ and $n \geq 1$. Hence, according to [23] [see equation (10)],

$$\begin{aligned}\mathbb{E}\big\{\big(\widetilde{N}_n(x) - \widetilde{N}_n(y)\big)^{2\nu}\big\}^{1/(2\nu)} &\leq p\mathbb{E}\big\{\big(N_n(x) - N_n(y)\big)^{2\nu}\big\}^{1/(2\nu)}\\ &\quad + \mathbb{E}\big\{\big(H_n(x) - H_n(y)\big)^{2\nu}\big\}^{1/(2\nu)}\\ &\leq C_\nu n^{(1-1/\alpha)/2}|x-y|^{(\alpha-1)/2},\end{aligned}$$

for some constant $C_\nu > 0$. This proves Lemma 26. □

This concludes also the proof of Lemma 25. □

5.3. *Expression of the return probability by an integral.* According to the result of the previous subsection, we are led to the study of $\mathbb{P}(\widetilde{Z}_n = 0, S_n = 0, \widetilde{\Omega}_n)$. As in Lemma 5, we have

$$\begin{aligned}\mathbb{P}\big(M_n = (0,0), \widetilde{\Omega}_n\big) &= \mathbb{P}(\widetilde{Z}_n = 0, S_n = 0, \widetilde{\Omega}_n)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{E}\Big[\prod_{y\in\mathbb{Z}} \varphi_\xi(t\widetilde{N}_n(y))\mathbf{1}_{\{S_n=0\}}\mathbf{1}_{\widetilde{\Omega}_n}\Big]\,dt.\end{aligned}$$

By following the proof of Lemma 5 [note that a priori $\sum_y \widetilde{N}_n(y)$ is not equal to n here], we get

$$\begin{aligned}(36)\qquad &\mathbb{P}(\widetilde{Z}_n = 0, S_n = 0, \widetilde{\Omega}_n)\\ &= \frac{d}{2\pi}\int_{-\pi/d}^{\pi/d} \mathbb{E}\biggl[\prod_{y\in\mathbb{Z}} \varphi_\xi(t\widetilde{N}_n(y))\mathbf{1}_{\{\sum_y \widetilde{N}_n(y)\in d_0\mathbb{Z}\}}\mathbf{1}_{\{S_n=0\}}\mathbf{1}_{\widetilde{\Omega}_n}\biggr]\,dt.\end{aligned}$$

In the sequel, we consider η, γ and ε satisfying all the hypotheses of Section 2.4 and $\gamma < (\alpha-1)/(4\alpha)$.

5.4. *Estimate of the integral away from the origin.* The following is very similar to the case of RWRS.

LEMMA 27. *We have*

$$\int_{n^{-\delta+\eta}}^{\pi/d} \mathbb{E}\biggl[\prod_{y\in\mathbb{Z}} |\varphi_\xi(t\widetilde{N}_n(y))|\mathbf{1}_{\widetilde{\Omega}_n}\biggr]\,dt = o\bigl(n^{-1-1/(\alpha\beta)}\bigr).$$

PROOF. First, set

$$\widetilde{V}_n := \sum_{y\in\mathbb{Z}} \widetilde{N}_n(y)^\beta.$$

Since on $\widetilde{\Omega}_n$, $\sum_y \widetilde{N}_n(y) = \sum_{k=1}^n \epsilon_k \ge np/2$ and $\widetilde{N}_n(y) \le N_n(y) \le n^{1-1/\alpha+\gamma}$, by following the proof of Lemma 6, we get on $\widetilde{\Omega}_n$

$$\widetilde{V}_n \ge cn^{\delta\beta-\gamma},$$

for some constant $c>0$. Let now ε be as in Proposition 9. Then the proofs of Propositions 8 and 9 lead to

$$\int_{n^{-\delta+\eta}}^{n^{-1+1/\alpha+\varepsilon}} \mathbb{E}\biggl[\prod_{y\in\mathbb{Z}} |\varphi_\xi(t\widetilde{N}_n(y))|\mathbf{1}_{\widetilde{\Omega}_n}\biggr]\,dt = o\bigl(n^{-1-1/(\alpha\beta)}\bigr).$$

But we can also easily adapt the proof of Proposition 10 to obtain

$$\int_{n^{-1+1/\alpha+\varepsilon}}^{\pi/d} \mathbb{E}\biggl[\prod_{y\in\mathbb{Z}} |\varphi_\xi(t\widetilde{N}_n(y))|\mathbf{1}_{\widetilde{\Omega}_n}\biggr]\,dt = o\bigl(n^{-1-1/(\alpha\beta)}\bigr).$$

Indeed, we just need to use "flat peaks" instead of peaks. These "flat peaks" are defined as follows. Let M and N be such that $\mathbb{P}(Y_1 = N) > 0$ and $\mathbb{P}(Y_1 = -M) > 0$. Then an "upper flat peak" is a sequence of the type

$$\begin{aligned}&(Y_{H+1},\dots,Y_{H+M},\epsilon_{H+M+1},Y_{H+M+2},\dots,Y_{H+M+N+1})\\ &\quad = (N,\dots,N,1,-M,\dots,-M),\end{aligned}$$

where H is any multiple of $M+N+1$, and one can define analogously a "lower flat peak." We leave to the reader to check that we can then follow the proof of Proposition 10 simply by replacing everywhere peaks by flat peaks. This concludes the proof of Lemma 27. □

5.5. *Estimate of the integral near the origin.* We turn now to the estimate of the integral in (36) on the interval $[-n^{-\delta+\eta}, n^{-\delta+\eta}]$. For this, we will roughly follow the same lines as for the proof of Proposition 7. However, the technical details are more involved here, since we have to make all calculus conditionally to $\{S_n = 0\}$. The first step is the following lemma.

LEMMA 28. *We have*

$$\sup_n \mathbb{E}\biggl[\biggl(\frac{n^{\delta\beta}}{\widetilde{V}_n}\biggr)^{1/(\beta-1)} \mathbf{1}_{\widetilde{\Omega}_n} \Big| S_n = 0\biggr] < +\infty. \tag{37}$$

PROOF. Remind that on $\widetilde{\Omega}_n$, $np/2 \le \sum_y \widetilde{N}_n(y) \le \widetilde{V}_n^{1/\beta} R_n^{1-1/\beta}$. Observe on the other hand that $\delta\beta/(\beta-1) = \beta/(\beta-1) - 1/\alpha$. Thus, there is a constant $C > 0$ such that for all $n \ge 1$, on $\widetilde{\Omega}_n$,

$$\biggl(\frac{n^{\delta\beta}}{\widetilde{V}_n}\biggr)^{1/(\beta-1)} \le C \frac{R_n}{n^{1/\alpha}}.$$

It follows from the above inequality that

$$\mathbb{E}\biggl[\biggl(\frac{n^{\delta\beta}}{\widetilde{V}_n}\biggr)^{1/(\beta-1)} \mathbf{1}_{\widetilde{\Omega}_n} \mathbf{1}_{\{S_n=0\}}\biggr] \le C\mathbb{E}\biggl[\frac{R_n}{n^{1/\alpha}} \mathbf{1}_{\{S_n=0\}}\biggr].$$

Set $m := \lfloor n/2 \rfloor$ and $m' := \lceil n/2 \rceil$. By using that $R_n \le R_{m'} + \#\{S_{m'+1}, \dots, S_n\} = R_{m'} + \#\{S_{m'+1} - S_n, \dots, S_{n-1} - S_n, 0\}$ and Markov property [resp., on the sequences $(S_k, k \ge 0)$ and $(S_n - S_{n-k}, 0 \le k \le n)$], we get

$$\begin{aligned}\mathbb{E}\biggl[\biggl(\frac{n^{\delta\beta}}{\widetilde{V}_n}\biggr)^{1/(\beta-1)} \mathbf{1}_{\widetilde{\Omega}_n} \mathbf{1}_{\{S_n=0\}}\biggr] &\le C\mathbb{E}\biggl[\frac{R_{m'}}{n^{1/\alpha}}\biggr] \times \sup_x \mathbb{P}_x(S_m = 0) \\ &= \mathcal{O}(n^{-1/\alpha}),\end{aligned}$$

since $\sup_x \mathbb{P}_x(S_m = 0) = \mathcal{O}(n^{-1/\alpha})$ and $\mathbb{E}(R_{m'}) = \mathcal{O}(n^{1/\alpha})$ (see [28], equation (7.a), page 703). We next divide all terms by $\mathbb{P}(S_n = 0)$ which is of order $n^{-1/\alpha}$ and this proves the lemma. □

We deduce the following lemma.

LEMMA 29. *We have*

$$\begin{aligned}\mathbb{P}(\widetilde{Z}_n = 0, S_n = 0, \widetilde{\Omega}_n) = n^{-1-1/(\alpha\beta)} d\mathbb{E}\biggl[\frac{n^{\delta}}{\widetilde{V}_n^{1/\beta}} \mathbf{1}_{\{\sum_y \widetilde{N}_n(y) \in d_0\mathbb{Z}\}} \mathbf{1}_{\widetilde{\Omega}_n} \Big| S_n = 0\biggr] \\ \times f_\alpha(0) f_\beta(0) + o\bigl(n^{-1-1/(\alpha\beta)}\bigr).\end{aligned}$$

PROOF. By following the proof of Lemma 12, we see that, uniformly on $\widetilde{\Omega}_n$, we have

$$\int_{|t|\le n^{-\delta+\eta}} \left|\prod_y \varphi_\xi(t\widetilde{N}_n(y)) - e^{-|t|^\beta \widetilde{V}_n(A_1+iA_2\,\mathrm{sgn}(t))}\right| dt = o(\widetilde{V}_n^{-1/\beta}).$$

By using Lemma 28, we get

$$\begin{aligned}
&\int_{|t|\le n^{-\delta+\eta}} \mathbb{E}\left[\left|\prod_y \varphi_\xi(t\widetilde{N}_n(y)) - e^{-|t|^\beta \widetilde{V}_n(A_1+iA_2\,\mathrm{sgn}(t))}\right| \mathbf{1}_{\widetilde{\Omega}_n} \mathbf{1}_{\{S_n=0\}}\right] dt \\
&\quad = o(1) \times \mathbb{E}\big[\widetilde{V}_n^{-1/\beta} \mathbf{1}_{\widetilde{\Omega}_n} \mathbf{1}_{\{S_n=0\}}\big] \\
&\quad = o(n^{-\delta-1/\alpha}) \times \mathbb{E}\big[(n^{\delta\beta}\widetilde{V}_n^{-1})^{1/(\beta-1)} \mathbf{1}_{\widetilde{\Omega}_n} | S_n = 0\big]^{(\beta-1)/\beta} \\
&\qquad [\text{since } \mathbb{P}(S_n=0) = \mathcal{O}(n^{-1/\alpha})] \\
&\quad = o\big(n^{-1-1/(\alpha\beta)}\big).
\end{aligned}$$

By using (36) and Lemma 27, we see that it remains to estimate

$$\int_{|t|\le n^{-\delta+\eta}} \mathbb{E}\big[e^{-|t|^\beta \widetilde{V}_n(A_1+iA_2\,\mathrm{sgn}(t))} \mathbf{1}_{\{\sum_y \widetilde{N}_n(y)\in d_0\mathbb{Z}\}} \mathbf{1}_{\{S_n=0\}} \mathbf{1}_{\widetilde{\Omega}_n}\big] dt.$$

But, as in the proof of Lemma 13, we have

$$\int_{|t|\le n^{-\delta+\eta}} e^{-|t|^\beta \widetilde{V}_n(A_1+iA_2\,\mathrm{sgn}(t))}\, dt = n^{-\delta}\left\{2\pi \frac{n^\delta}{\widetilde{V}_n^{1/\beta}} f_\beta(0)\right\} + o(n^{-\delta}),$$

uniformly on $\widetilde{\Omega}_n$. We next take the expectation in both sides and we conclude the proof by using that $\mathbb{P}(S_n=0) \sim f_\alpha(0) n^{-1/\alpha}$. □

The following lemma allows us to get rid of $\mathbf{1}_{\{\sum_y \widetilde{N}_n(y)\in d_0\mathbb{Z}\}}$.

LEMMA 30. *Assume that d_1 is a multiple of d_0. On $\{S_n=0\}$, we have*

$$\sum_y \widetilde{N}_n(y) \in d_0\mathbb{Z} \quad \Leftrightarrow \quad n \in d_0\mathbb{Z}.$$

PROOF. Let $m_n := \sum_y \widetilde{N}_n(y) = \sum_{k=1}^n \varepsilon_k$ be the number of horizontal moves before time n.

If $S_n = 0$, the number $n - m_n$ of vertical moves is necessarily in $d_1\mathbb{Z}$ and so in $d_0\mathbb{Z}$, since d_1 is a multiple of d_0 by hypothesis. Hence, m_n is in $d_0\mathbb{Z}$ if and only if n is in $d_0\mathbb{Z}$. □

We will need the following estimate.

LEMMA 31. *Let $V_n := \sum_{x\in\mathbb{Z}} N_n(x)^\beta$. Then*

$$\mathbb{E}[|\widetilde{V}_n - p^\beta V_n| | S_n = 0] = \mathcal{O}\big(n^{\delta\beta-(\alpha-1)/(2\alpha)}\big).$$

PROOF. Set again $m = \lfloor n/2 \rfloor$ and $m' = \lceil n/2 \rceil$. By using the inequality $|a^\beta - b^\beta| \le \beta |a - b|(a^{\beta-1} + b^{\beta-1})$ and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\mathbb{E}[|\widetilde{V}_n - p^\beta V_n||S_n = 0] \\
&\quad \le \beta \mathbb{E}\Big[\sum_{x\in\mathbb{Z}} (\widetilde{N}_n(x)^{\beta-1} + p^{\beta-1} N_n(x)^{\beta-1})^2 | S_n = 0\Big]^{1/2} \qquad (38)\\
&\qquad \times \mathbb{E}\Big[\sum_{x\in\mathbb{Z}} (\widetilde{N}_n(x) - pN_n(x))^2 | S_n = 0\Big]^{1/2}.
\end{aligned}$$

We now estimate both expectations in the right-hand side of the above inequality. First, note that $N_n(x) = N_m(x) + (N_n(x) - N_m(x))$ and that the sequence $((N_n(x) - N_m(x), x \in \mathbb{Z})|S_n = 0)$ has the same distribution as $((N_{m'+1}(-x) - N_1(-x), x \in \mathbb{Z})|S_n = 0)$. Thus, the Markov property gives

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{x\in\mathbb{Z}} (\widetilde{N}_n(x)^{\beta-1} + p^{\beta-1} N_n(x)^{\beta-1})^2 | S_n = 0\Big] \\
&\quad \le 4 \sum_{x\in\mathbb{Z}} \mathbb{E}[N_n(x)^{2(\beta-1)} | S_n = 0] \\
&\quad \le C\Big\{\sum_{x\in\mathbb{Z}} \sum_{M\in\mathbb{Z}} \mathbb{E}[N_m(x)^{2(\beta-1)} \mathbf{1}_{\{S_m = M\}}] \frac{\mathbb{P}(S_{m'} = -M)}{\mathbb{P}(S_n = 0)} \\
&\qquad + \sum_{x\in\mathbb{Z}} \sum_{M\in\mathbb{Z}} \mathbb{E}[(N_{m'}(x))^{2(\beta-1)} \mathbf{1}_{\{S_{m'} = -M\}}] \frac{\mathbb{P}(S_m = M)}{\mathbb{P}(S_n = 0)}\Big\},
\end{aligned}$$

for some constant $C > 0$. Since $\sup_M \mathbb{P}(S_{m'} = -M)/\mathbb{P}(S_n = 0) < +\infty$ and $\sup_M \mathbb{P}(S_m = M)/\mathbb{P}(S_n = 0) < +\infty$, we get

$$\mathbb{E}\Big[\sum_{x\in\mathbb{Z}} (\widetilde{N}_n(x)^{\beta-1} + p^{\beta-1} N_n(x)^{\beta-1})^2 | S_n = 0\Big] \le C \sum_{x\in\mathbb{Z}} \mathbb{E}[N_{m'}(x)^{2(\beta-1)}].$$

Then the Markov property again and (34) show that

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{x\in\mathbb{Z}} (\widetilde{N}_n(x)^{\beta-1} + p^{\beta-1} N_n(x)^{\beta-1})^2 | S_n = 0\Big] \\
&\quad \le C\mathbb{E}[R_{m'}] \times \mathbb{E}[N_{m'}(0)^{2(\beta-1)}] \qquad (39)\\
&\quad = \mathcal{O}(n^{2(\beta-1)(1-1/\alpha)+1/\alpha}).
\end{aligned}$$

The same argument gives

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}} \mathbb{E}[(\widetilde{N}_n(x) - pN_n(x))^2 | S_n = 0] \\
&\quad \le C\Big\{\sum_{x\in\mathbb{Z}} \mathbb{E}[(\widetilde{N}_m(x) - pN_m(x))^2] + \sum_{x\in\mathbb{Z}} \mathbb{E}[(\widetilde{N}_{m'}(x) - pN_{m'}(x))^2]\Big\},
\end{aligned}$$

for some constant $C > 0$. Then by using (35) [note that $\bar{\varepsilon}_k$ is centered and independent of $(S_{\ell-1}, \varepsilon_\ell, S_{k-1})$ if $\ell < k$], we get

$$\sum_{x\in\mathbb{Z}} \mathbb{E}\big[\big(\widetilde{N}_n(x) - pN_n(x)\big)^2 | S_n = 0\big] = \mathcal{O}(n). \tag{40}$$

The lemma now follows by combining (38), (39) and (40) since $(\beta - 1)(1 - \frac{1}{\alpha}) + \frac{1}{2\alpha} + \frac{1}{2} = \delta\beta - \frac{\alpha-1}{2\alpha}$. □

LEMMA 32. *Conditionally to the event $\{S_n = 0\}$, the sequence $(V_n/n^{\delta\beta}, n \geq 0)$ converges in distribution to the random variable $\int_{\mathbb{R}} (L_1^0(x))^\beta \, dx$.*

PROOF. According to [15], the lemma will essentially follow from the two following statements:

(RW1) The sequence of conditioned processes $((n^{-1/\alpha} S_{\lfloor nt\rfloor} \mid S_n = 0), t \in [0, 1])$ converges in distribution to the bridge $(\widetilde{U}_t^0, t \in [0, 1])$, as $n \to \infty$.

(RW2) (i)

$$\sup_y \mathbb{E}[N_n(y)^2 | S_n = 0] = \mathcal{O}(n^{2-2/\alpha}).$$

(ii) There exists a constant $C > 0$ such that for every $x, y \in \mathbb{R}$,

$$\mathbb{E}\big[\big(N_n(\lfloor n^{1/\alpha} x\rfloor) - N_n(\lfloor n^{1/\alpha} y\rfloor)\big)^2 | S_n = 0\big] \leq C n^{2-2/\alpha} |x - y|^{\alpha-1}.$$

Part (RW1) is proved in [29].

We prove now (RW2) starting with part (i). By using the same argument as in the proof of Lemma 31, we get

$$\mathbb{E}[N_n(y)^2 | S_n = 0] \leq C\big(\mathbb{E}[N_m(y)^2] + \mathbb{E}[N_{m'+1}(-y)^2]\big),$$

for some constant $C > 0$, with m and m' as in the previous lemma. The desired result now follows from Lemma 1 in [24].

For part (ii), set $N_n(x, y) := N_n(x) - N_n(y)$. Then use again the argument of the previous lemma, which gives

$$\mathbb{E}[N_n(x, y)^2 | S_n = 0] \leq C\big(\mathbb{E}[N_m(x, y)^2] + \mathbb{E}[N_{m'+1}(-x, -y)^2] + 1\big),$$

for some constant $C > 0$. The result then follows from Lemma 3 in [24].

We can now apply Theorem 4.1 in [15] in the case when the random scenery is a sequence of i.i.d. random variables with β-stable distribution and with characteristic function of the form $\theta \mapsto \exp(-c|\theta|^\beta)$. We deduce that conditionally to $\{S_n = 0\}$,

$$n^{-\delta} \sum_{k=1}^{n} \xi_{S_k} \xrightarrow[n\to\infty]{\mathcal{L}} \int_{\mathbb{R}} L_1^0(x) \, dY_x,$$

where $(Y_x, x \in \mathbb{R})$ is a two-sided β-stable Lévy process independent of $\widetilde{U}^0$ and limit in distribution of $(n^{-1/\beta}\sum_{k=0}^{\lfloor nx\rfloor}\xi_k, x \in \mathbb{R})$, when $n \to \infty$. Therefore (see, for instance, Lemma 5 in [24]), for every $\theta \in \mathbb{R}$,

$$\mathbb{E}\big(\exp(-c|\theta|^\beta n^{-\delta\beta} V_n)|S_n = 0\big) \to \mathbb{E}\big(e^{-c|\theta|^\beta \int_{\mathbb{R}}(L_1^0(x))^\beta\,dx}\big) \qquad \text{when } n \to \infty,$$

which proves the lemma. □

LEMMA 33. *Conditionally to the event $\{S_n = 0\}$, the sequence $(n^{\delta\beta}\widetilde{V}_n^{-1}\mathbf{1}_{\widetilde{\Omega}_n}, n \ge 0)$ converges in distribution to the random variable $(p|L^0|_\beta)^{-\beta}$.*

PROOF. By Lemma 32, the sequence $(n^{\delta\beta}V_n^{-1}, n \ge 0)$ converges in distribution to $|L^0|_\beta^{-\beta}$, conditionally to $\{S_n = 0\}$. On the other hand, Lemma 25 implies that the sequence $(\mathbf{1}_{\widetilde{\Omega}_n}, n \ge 0)$ converges in distribution to the constant 1, conditionally to $\{S_n = 0\}$. Hence, the sequence $(n^{\delta\beta}V_n^{-1}\mathbf{1}_{\widetilde{\Omega}_n}, n \ge 0)$ converges in distribution to $|L^0|_\beta^{-\beta}$, conditionally to $\{S_n = 0\}$. Next, recall that on $\widetilde{\Omega}_n$, $V_n \ge \widetilde{V}_n \ge cn^{\delta\beta-\gamma}$, for some constant $c > 0$. Thus, Lemma 31 gives

$$\begin{aligned}\mathbb{E}\left[\left|\frac{n^{\delta\beta}}{\widetilde{V}_n} - \frac{n^{\delta\beta}}{p^\beta V_n}\right|\mathbf{1}_{\widetilde{\Omega}_n}\middle| S_n = 0\right] &= \mathcal{O}\big(n^{-2\delta\beta+2\gamma+2\delta\beta-(\alpha-1)/2\alpha}\big)\\ &= \mathcal{O}\big(n^{2\gamma-(\alpha-1)/(2\alpha)}\big).\end{aligned}$$

Therefore, since $\gamma < (\alpha-1)/(4\alpha)$, the left-hand side in the above equation converges to 0 when $n \to \infty$. The lemma follows. □

We finally obtain the following proof.

PROOF OF THEOREM 20. The uniform integrability of the sequence $(n^\delta \times \widetilde{V}_n^{-1/\beta}\mathbf{1}_{\widetilde{\Omega}_n}, n \ge 0)$ conditionally to $\{S_n = 0\}$ is deduced from Lemma 28 (since it is bounded in $L^{\beta/(\beta-1)}$). It then follows from Lemma 33 that

$$\mathbb{E}\left[\frac{n^\delta}{\widetilde{V}_n^{1/\beta}}\mathbf{1}_{\widetilde{\Omega}_n}\middle| S_n = 0\right] \to p^{-1}\mathbb{E}[|L^0|_\beta^{-1}] \qquad \text{when } n \to \infty.$$

The theorem now follows from Lemmas 29 and 30. □

APPENDIX: CONTROL OF THE RANGE

We first gather some known facts about the range R_n of the random walk $(S_n, n \ge 0)$. First of all, this walk is transient if, and only if, $\alpha < 1$. Moreover, there exists a constant $c > 0$ such that

$$\mathbb{E}[R_n] \sim c\begin{cases} n, & \text{if } \alpha < 1 \text{ (see [32], page 36)},\\ \dfrac{n}{\log(n)}, & \text{if } \alpha = 1 \text{ (see [28], Theorem 6.9, page 698)},\\ n^{1/\alpha}, & \text{if } \alpha > 1 \text{ (see [28], equation (7.a), page 703)}.\end{cases} \tag{41}$$

In addition, if $\alpha \le 1$ (see [32], pages 38–40, for $\alpha < 1$, and [28], Theorem 6.9, for $\alpha = 1$), then

$$\frac{R_n}{\mathbb{E}[R_n]} \to 1 \qquad \text{a.s.} \tag{42}$$

If $\alpha > 1$, it is proved in [28], Theorem 7.1, page 703, that

$$\frac{R_n}{n^{1/\alpha}} \to \lambda(U([0,1])) \qquad \text{in distribution,}$$

where λ denotes the Lebesgue measure, and $(U(s), s \in [0,1])$ is an α-stable process. In this case, it is also proved in [28] that the constant c appearing in (41) is $\mathbb{E}[\lambda(U([0,1]))]$, so that

$$\frac{R_n}{\mathbb{E}[R_n]} \to \frac{\lambda(U([0,1]))}{\mathbb{E}[\lambda(U([0,1]))]} \qquad \text{in distribution.} \tag{43}$$

Our aim in this Appendix is to prove the following result.

LEMMA 34. *Assume that* $\alpha \in (0,2]$. *Let* $\gamma \in (0, 1/\alpha)$ *and set*

$$\mathcal{R}_n := \{\mathbb{E}[R_n]n^{-\gamma} \le R_n \le \mathbb{E}[R_n]n^{\gamma}\}.$$

Then there exists a constant $C > 0$, *such that*

$$\mathbb{P}(\mathcal{R}_n) = 1 - \mathcal{O}(\exp(-Cn^{\gamma})). \tag{44}$$

PROOF. We first prove that for n large enough,

$$\mathbb{P}\big[R_n \ge \mathbb{E}[R_n]n^{\gamma}\big] \le \exp(-Cn^{\gamma}). \tag{45}$$

Let us recall that for every $a, b \in \mathbb{N}$, we have

$$\mathbb{P}(R_n \ge a + b) \le \mathbb{P}(R_n \ge a)\mathbb{P}(R_n \ge b). \tag{46}$$

The proof is given for instance in [9] and goes as follows. Let $\tau := \inf\{k : R_k \ge a\}$. Note that τ is a stopping time, and that $R_\tau = a$ on $\{\tau < \infty\}$. Moreover,

$$\begin{aligned}\mathbb{P}(R_n \ge a + b) &= \mathbb{P}(\tau \le n; R_n \ge a + b)\\ &= \sum_{j=1}^{n} \mathbb{P}(\tau = j; R_n \ge R_j + b).\end{aligned}$$

Now, for $j \le n$, $R_n \le R_j + \#\{S_{j+1}, \dots, S_n\} = R_j + \#\{S_{j+1} - S_j, \dots, S_n - S_j\}$. By independence, we have

$$\begin{aligned}\mathbb{P}(R_n \ge a + b) &\le \sum_{j=1}^{n} \mathbb{P}(\tau = j)\mathbb{P}(R_{n-j} \ge b)\\ &\le \mathbb{P}(R_n \ge b)\mathbb{P}(\tau \le n),\end{aligned}$$

proving (46). Hence,

$$\mathbb{P}(R_n \geq \mathbb{E}[R_n]n^\gamma) \leq \mathbb{P}\biggl(R_n \geq \lfloor 3\mathbb{E}[R_n]\rfloor \biggl\lfloor \frac{n^\gamma}{3} \biggr\rfloor\biggr) \leq \mathbb{P}(R_n \geq \lfloor 3\mathbb{E}[R_n]\rfloor)^{\lfloor n^\gamma/3\rfloor}$$
$$\leq \biggl(\frac{\mathbb{E}[R_n]}{\lfloor 3\mathbb{E}[R_n]\rfloor}\biggr)^{\lfloor n^\gamma/3\rfloor} \leq \biggl(\frac{\mathbb{E}[R_n]}{3\mathbb{E}[R_n]-1}\biggr)^{\lfloor n^\gamma/3\rfloor} \leq \biggl(\frac{1}{2}\biggr)^{\lfloor n^\gamma/3\rfloor}.$$

This finishes the proof of (45). It remains now to prove that for n large enough,

$$\mathbb{P}(R_n \leq \mathbb{E}[R_n]n^{-\gamma}) \leq \exp(-Cn^\gamma). \tag{47}$$

To this end, let $I_1, \dots, I_N$ be disjoint subsequent intervals of $\{0, \dots, n\}$, of the same length l_n depending on n, so that $l_n \gg 1$ and $N = \lfloor n/l_n \rfloor$. Note that

$$R_n \geq \max_{j=1}^{N}(\#\{S_k, k \in I_j\}),$$

and that the random variables $(\#\{S_k, k \in I_j\}, 1 \leq j \leq N)$ are i.i.d. with the same law as R_{l_n}. Hence,

$$\begin{aligned}\mathbb{P}(R_n \leq \mathbb{E}[R_n]n^{-\gamma}) &\leq \mathbb{P}\Bigl(\max_{j=1}^{N}(\#\{S_k, k \in I_j\}) \leq E(R_n)n^{-\gamma}\Bigr)\\ &= \mathbb{P}(R_{l_n} \leq \mathbb{E}[R_n]n^{-\gamma})^N.\end{aligned} \tag{48}$$

Choose now l_n such that $\mathbb{E}[R_{l_n}] \sim 3\mathbb{E}[R_n]n^{-\gamma}$. By (41), this gives

$$l_n \sim \begin{cases} 3n^{1-\gamma}, & \text{if } \alpha < 1,\\ 3(1-\gamma)n^{1-\gamma}, & \text{if } \alpha = 1,\\ 3^\alpha n^{1-\alpha\gamma}, & \text{if } \alpha > 1,\end{cases}$$

so that

$$N \sim \begin{cases} \dfrac{1}{3}n^\gamma, & \text{if } \alpha < 1,\\ \dfrac{1}{3(1-\gamma)}n^\gamma, & \text{if } \alpha = 1,\\ \dfrac{1}{3^\alpha}n^{\alpha\gamma}, & \text{if } \alpha > 1.\end{cases} \tag{49}$$

For n large enough, $\mathbb{E}[R_{l_n}] \geq 2\mathbb{E}[R_n]n^{-\gamma}$, and it follows from (48) that

$$\mathbb{P}(R_n \leq \mathbb{E}[R_n]n^{-\gamma}) \leq \mathbb{P}\biggl(R_{l_n} \leq \frac{\mathbb{E}[R_{l_n}]}{2}\biggr)^N. \tag{50}$$

For $\alpha \leq 1$, $\mathbb{P}(R_{l_n} \leq \frac{\mathbb{E}[R_{l_n}]}{2})$ tends to zero by (42). By (43), for $\alpha > 1$, we have

$$\limsup_{n\to\infty} \mathbb{P}\biggl(R_{l_n} \leq \frac{\mathbb{E}[R_{l_n}]}{2}\biggr) \leq \mathbb{P}\biggl[(\lambda(U([0,1]))) \leq \frac{1}{2}\mathbb{E}[\lambda(U([0,1]))]\biggr] < 1,$$

since a.s. $\lambda(U([0,1])) > 0$. In any case, there exists $p < 1$, such that for all $\gamma \in (0, 1/\alpha)$, and for n large enough,

$$\mathbb{P}\left(R_{l_n} \le \frac{\mathbb{E}[R_{l_n}]}{2}\right) \le p.$$

Together with (50) and (49), this proves (47) and the lemma. □

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F. CASTELL
LATP, UMR CNRS 6632
CENTRE DE MATHÉMATIQUES ET INFORMATIQUE
UNIVERSITÉ AIX-MARSEILLE I
39, RUE JOLIOT CURIE
13 453 MARSEILLE CEDEX 13
FRANCE
E-MAIL: Fabienne.Castell@cmi.univ-mrs.fr

N. GUILLOTIN-PLANTARD
INSTITUT CAMILLE JORDAN
CNRS UMR 5208
UNIVERSITÉ DE LYON
UNIVERSITÉ LYON 1
43, BOULEVARD DU 11 NOVEMBRE 1918
69622 VILLEURBANNE
FRANCE
E-MAIL: nadine.guillotin@univ-lyon1.fr

F. PÈNE
UNIVERSITÉ EUROPÉENNE DE BRETAGNE
UNIVERSITÉ DE BREST
LABORATOIRE DE MATHÉMATIQUES
UMR CNRS 6205
29238 BREST CEDEX
FRANCE
E-MAIL: francoise.pene@univ-brest.fr

B. SCHAPIRA
DÉPARTEMENT DE MATHÉMATIQUES
CNRS UMR 8628, BÂT. 425
UNIVERSITÉ PARIS-SUD 11
F-91405 ORSAY CEDEX
FRANCE
E-MAIL: bruno.schapira@math.u-psud.fr