Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2010, Vol. 46, No. 4, 1178–1194
DOI: 10.1214/09-AIHP353

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Limit theorem for random walk in weakly dependent random scenery

Nadine Guillotin-Plantard[a] and Clémentine Prieur[b]

[a] *Université de Lyon, Institut Camille Jordan, Bâtiment Braconnier, 43 avenue du 11 novembre 1918, 69622 Villeurbanne Cedex, France. E-mail: nadine.guillotin@univ-lyon1.fr*

[b] *Université Joseph Fourier, Tour IRMA, MOISE-LJK, B.P. 53, 38041 Grenoble Cedex 9, France. E-mail: Clementine.Prieur@imag.fr*

Received 12 July 2009; revised 14 August 2009; accepted 10 December 2009

Abstract. Let $S = (S_k)_{k\geq 0}$ be a random walk on $\mathbb{Z}$ and $\xi = (\xi_i)_{i\in\mathbb{Z}}$ a stationary random sequence of centered random variables, independent of S. We consider a random walk in random scenery that is the sequence of random variables $(U_n)_{n\geq 0}$, where

$$U_n = \sum_{k=0}^{n} \xi_{S_k}, \quad n \in \mathbb{N}.$$

Under a weak dependence assumption on the scenery ξ we prove a functional limit theorem generalizing Kesten and Spitzer's [*Z. Wahrsch. Verw. Gebiete* **50** (1979) 5–25] theorem.

Résumé. Soit $S = (S_k)_{k\geq 0}$ une marche aléatoire sur $\mathbb{Z}$ et $\xi = (\xi_i)_{i\in\mathbb{Z}}$ une suite stationnaire de variables aléatoires centrées, indépendante de S. Nous considérons une marche aléatoire en scène aléatoire définie par la suite de variables aléatoires $(U_n)_{n\geq 0} = (\sum_{k=0}^{n} \xi_{S_k})_{n\geq 0}$. Sous une hypothèse de dépendance faible portant sur la scène ξ, nous montrons un théorème de la limite centrale fonctionnel généralisant le théorème de Kesten et Spitzer [*Z. Wahrsch. Verw. Gebiete* **50** (1979) 5–25].

MSC: Primary 60F05; 60G50; 62D05; secondary 37C30; 37E05

Keywords: Random walks; Random scenery; Weak dependence; Limit theorem; Local time

1. Introduction

Let $X = (X_i)_{i\geq 1}$ be a sequence of independent and identically distributed random vectors with values in $\mathbb{Z}^d$. We write

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i \quad \text{for } n \geq 1,$$

for the $\mathbb{Z}^d$-random walk $S = (S_n)_{n\in\mathbb{N}}$ generated by the family X. Let $\xi = (\xi_x)_{x\in\mathbb{Z}^d}$ be a family of real random variables, independent of S. The sequence ξ plays the role of the random scenery. The random walk in random scenery (RWRS) is the process defined by

$$U_n = \sum_{k=0}^{n} \xi_{S_k}, \quad n \in \mathbb{N}.$$

RWRS was first introduced in dimension one by Kesten and Spitzer [19] and Borodin [5,6] in order to construct new self-similar stochastic processes. Functional limit theorems for RWRS were obtained under the assumption that the random variables $\xi_x, x \in \mathbb{Z}^d$, are independent and identically distributed. For $d = 1$, Kesten and Spitzer [19] proved that when X and ξ belong to the domains of attraction of different stable laws of indices $1 < \alpha \le 2$ and $0 < \beta \le 2$, respectively, then there exists $\delta > \frac{1}{2}$ such that $(n^{-\delta} U_{[nt]})_{t\ge 0}$ converges weakly as $n \to \infty$ to a self-similar process with stationary increments, δ being related to α and β by $\delta = 1 - \alpha^{-1} + (\alpha\beta)^{-1}$. The case $0 < \alpha < 1$ and β arbitrary is easier; they showed then that $(n^{-1/\beta} U_{[nt]})_{t\ge 0}$ converges weakly, as $n \to \infty$, to a stable process with index β. Bolthausen [4] gave a method to solve the case $\alpha = 1$ and $\beta = 2$ and especially, he proved that when $(S_n)_{n\in\mathbb{N}}$ is a recurrent $\mathbb{Z}^2$-random walk, $((n \log n)^{-1/2} U_{[nt]})_{t\ge 0}$ satisfies a functional central limit theorem. For an arbitrary transient $\mathbb{Z}^d$-random walk, $n^{-1/2} U_n$ is asymptotically normal (see Spitzer [27], p. 53). Maejima [23] generalized the result of Kesten and Spitzer [19] to the case where $(\xi_x)_{x\in\mathbb{Z}}$ are i.i.d. $\mathbb{R}^d$-valued random variables which belong to the domain of attraction of an operator stable random vector with exponent B. If we denote by D the linear operator on $\mathbb{R}^d$ defined by $D = (1 - \frac{1}{\alpha})I + \frac{1}{\alpha}B$, he proved that $(n^{-D} U_{[nt]})_{t\ge 0}$ converges weakly to an operator self similar with exponent D and having stationary increments.

One-dimensional random walks in random scenery recently arose in the study of random walks evolving on oriented versions of $\mathbb{Z}^2$ (see Guillotin-Plantard and Le Ny [15,16]) as well as in the context of charged polymers (see Chen and Khoshnevisan [8]). The understanding of these models in the case where the orientations or the charges are not independently distributed requires functional limit theorems for $\mathbb{Z}$-random walk in correlated random scenery. To our knowledge, only the case of strongly correlated stationary random sceneries has been studied by Lang and Xanh [21]. In their paper, the increments of the random walk S are assumed to belong to the domain of attraction of a non-degenerate stable law of index α, $0 < \alpha \le 2$. They further suppose that the scenery ξ satisfies the non-central limit theorem of Dobrushin and Major [12] with a scaling factor $n^{-d+(\beta k)/2}$, $\beta k < d$. Under the assumption $\beta k < \alpha$, it is proved that $(n^{-1+\beta k/(2\alpha)} U_{[nt]})_{t\ge 0}$ converges weakly as $n \to +\infty$ to a self-similar process with stationary increments, which can be represented as a multiple Wiener–Itô integral of a random function. Our aim is to study the intermediary case of a stationary random scenery ξ which satisfies a weak dependence condition introduced in Dedecker et al. [11] and to prove Kesten and Spitzer's theorem under this new assumption. In Guillotin-Plantard and Prieur [17] the case of a transient $\mathbb{Z}$-random walk was considered and a central limit theorem for the sequence $(U_n)_{n\in\mathbb{N}}$ was proved. In this paper the one-dimensional random walk will be assumed to be recurrent.

Our paper is organized as follows: In Section 2, we introduce the dependence setting under which we work in the sequel. In Section 3 we introduce in details our model and give the main result. In Section 4 properties of the local time of the random walk are given as well as the ones of the intersection local time. Models for which we can compute bounds for our dependence coefficients are presented in Section 5. Finally, the proof of our theorem is given in the last section.

2. Weak dependence conditions

In this section, we recall the definition of the dependence coefficients which we will use in the sequel. They have first been introduced in Dedecker et al. [11]. Our weak dependence condition will be less restrictive than the mixing one. The reader interested in this question would find more details in Guillotin-Plantard and Prieur [17].

On the Euclidean space $\mathbb{R}^m$, we define the metric

$$d_1(x, y) = \sum_{i=1}^{m} |x_i - y_i|.$$

Let $\Lambda = \bigcup_{m\in\mathbb{N}^*} \Lambda_m$, where Λ_m is the set of Lipschitz functions $f : \mathbb{R}^m \to \mathbb{R}$ with respect to the metric d_1. If $f \in \Lambda_m$, we denote by $\mathrm{Lip}(f) := \sup_{x\ne y} \frac{|f(x)-f(y)|}{d_1(x,y)}$ the Lipschitz modulus of f. The set of functions $f \in \Lambda$ such that $\mathrm{Lip}(f) \le 1$ is denoted by $\tilde{\Lambda}$.

Definition 2.1. *Let ξ be a $\mathbb{R}^m$-valued random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, assumed to be square integrable. For any σ-algebra $\mathcal{M}$ of $\mathcal{A}$, we define the θ_2-dependence coefficient*

$$\theta_2(\mathcal{M}, \xi) = \sup\{ \| \mathbb{E}(f(\xi)|\mathcal{M}) - \mathbb{E}(f(\xi)) \|_2, f \in \tilde{\Lambda} \}. \tag{2.1}$$

Here, $\|X\|_2 = [\mathbb{E}(X^2)]^{1/2}$ *denotes the norm in* L^2.

We now define the coefficient $\theta_{k,2}$ for a sequence of σ-algebras and a sequence of $\mathbb{R}$-valued random variables.

Definition 2.2. *Let* $(\xi_i)_{i\in\mathbb{Z}}$ *be a sequence of square integrable random variables valued in* $\mathbb{R}$. *Let* $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ *be a sequence of* σ*-algebras of* $\mathcal{A}$. *For any* $k \in \mathbb{N}^* \cup \{\infty\}$ *and* $n \in \mathbb{N}$, *we define*

$$\theta_{k,2}(n) = \max_{1\le l\le k} \frac{1}{l} \sup\bigl\{\theta_2\bigl(\mathcal{M}_p, (\xi_{j_1}, \dots, \xi_{j_l})\bigr), p + n \le j_1 < \cdots < j_l\bigr\}$$

and

$$\theta_2(n) = \theta_{\infty,2}(n) = \sup_{k\in\mathbb{N}^*} \theta_{k,2}(n).$$

Definition 2.3. *Let* $(\xi_i)_{i\in\mathbb{Z}}$ *be a sequence of square integrable random variables valued in* $\mathbb{R}$. *Let* $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ *be a sequence of* σ*-algebras of* $\mathcal{A}$. *The sequence* $(\xi_i)_{i\in\mathbb{Z}}$ *is said to be* θ_2*-weakly dependent with respect to* $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ *if* $\theta_2(n) \longrightarrow_{n\to+\infty} 0$.

Remark. *Replacing the* $\|\cdot\|_2$ *norm in* (2.1) *by the* $\|\cdot\|_1$ *norm, we get the* θ_1 *dependence coefficient first introduced by Doukhan and Louhichi* [14].

3. Model and results

Let $S = (S_k)_{k\ge0}$ be a $\mathbb{Z}$-random walk ($S_0 = 0$) whose increments $(X_i)_{i\ge1}$ are centered and square integrable. We denote by $\mathbf{P}_{X_1}$ the law of the random variable X_1. For any $q \in \mathbb{N}^*$ such that $\mathbf{P}(X_1 \in [-q, q])$ is non zero, we define the probability measure on $\mathbb{Z}$

$$\mathbf{P}_q = \frac{\mathbf{P}_{X_1}|_{[-q,q]}}{\mathbf{P}(X_1 \in [-q, q])}.$$

The random walk S is said to satisfy the property (**P**) if there exists $q \in \mathbb{N}^*$ such that

$$\bigl\{x \in \mathbb{Z}; \exists n, \mathbf{P}_q^{(\star n)}(x) > 0\bigr\} = \mathbb{Z}.$$

In particular, if there exists some $q \in \mathbb{N}^*$ such that the random walk associated to $\mathbf{P}_q$ is centered and aperiodic then S satisfies the property (**P**). For instance, the simple random walk on $\mathbb{Z}$ verifies (**P**). Let $\xi = (\xi_i)_{i\in\mathbb{Z}}$ be a sequence of centered real random variables. The sequences S and ξ are defined on a same probability space denoted by $(\Omega, \mathcal{F}, \mathbf{P})$ and are assumed to be independent. We are interested in the asymptotic behaviour of the following sum

$$U_n = \sum_{k=0}^{n} \xi_{S_k}.$$

The case where the ξ_i's are independent and identically distributed random variables with positive variance has been considered by Kesten and Spitzer [19] and Borodin [5,6]. They studied the weak convergence of the sequence of stochastic processes

$$n^{-3/4} U_{nt}, \quad t \ge 0, n \ge 1,$$

where U_s is defined as the linear interpolation

$$U_s = U_n + (s - n)(U_{n+1} - U_n) \quad \text{when } n \le s \le n + 1.$$

Consider a standard Brownian motion $(B_t)_{t\geq 0}$, denote by $(L_t(x))_{t\geq 0}$ its corresponding local time at $x\in\mathbb{R}$ and introduce a pair of independent Brownian motions $(Z_+(x), Z_-(x))$, $x\geq 0$ defined on the same probability space as $(B_t)_{t\geq 0}$ and independent of him. The following process is well-defined for all $t\geq 0$:

$$\Delta_t = \int_0^\infty L_t(x)\,\mathrm{d}Z_+(x) + \int_0^\infty L_t(-x)\,\mathrm{d}Z_-(x). \tag{3.1}$$

It was proved by Kesten and Spitzer [19] that this process has a self-similar continuous version of index $\frac{3}{4}$, with stationary increments. We denote by $\stackrel{\mathcal{C}}{\Longrightarrow}$ the convergence in the space of continuous functions $\mathcal{C}([0,\infty),\mathbb{R})$ endowed with the uniform topology.

Theorem 3.1 (Kesten and Spitzer [19]). *Assume that the ξ_i's are independent and identically distributed with positive variance $\sigma^2>0$. Then,*

$$\left(\frac{1}{n^{3/4}}U_{nt}\right)_{t\geq 0} \stackrel{\mathcal{C}}{\Longrightarrow} (\sigma\Delta_t)_{t\geq 0}. \tag{3.2}$$

A simple proof of this theorem was proposed by Cadre [7], Section 2.5.a., using a weak limit theorem for stochastic integrals (Theorem 1.1. in Kurtz and Protter [20]). Applying Cadre's method, Theorem 3.1 can be extended to any scenery given by a stationary and ergodic sequence of square integrable martingale differences. Then, a natural idea is to generalize the result to any stationary and ergodic sequence ξ of square integrable random variables as it was done for the central limit theorem. Under suitable assumptions on the sequence, for instance the convergence of the series $\sum_{k=0}^{\infty}\mathbb{E}(\xi_k|\mathcal{M}_0)$ in $\mathbb{L}^2$, the scenery ξ is equal to a martingale differences sequence modulo a coboundary term and satisfies a Donsker theorem. However, the RWRS associated to the coboundary term (if it is non zero) is not negligible. It can be proved that the $\mathbb{L}^2$-norm of this sum correctly normalized by $n^{3/4}$ converges to some positive constant.

In order to weaken the assumptions on the field ξ we introduce a sequence $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ of σ-algebras of $\mathcal{F}$ defined by

$$\mathcal{M}_i = \sigma(\xi_j, j\leq i), \quad i\in\mathbb{Z}.$$

In the sequel, the dependence coefficients will be defined with respect to the sequence of σ-algebras $(\mathcal{M}_i)_{i\in\mathbb{Z}}$.

Theorem 3.2. *Assume that the following conditions are satisfied*:

(A_0) *The random walk S satisfies the property* (**P**).
(A_1) *$\xi=\{\xi_i\}_{i\in\mathbb{Z}}$ is a stationary sequence of square integrable random variables.*
(A_2) *$\theta_2^{\xi}(\cdot)$ is bounded above by a non-negative function $g(\cdot)$ such that*:

- *$x\mapsto x^{3/2}g(x)$ is non-increasing,*
- *$\exists 0<\varepsilon<1$, $\sum_{i=0}^{\infty}2^{3i/2}g(2^{i\varepsilon})<\infty$.*

Then, as n tends to infinity,

$$\left(\frac{1}{n^{3/4}}U_{nt}\right)_{t\geq 0} \stackrel{\mathcal{C}}{\Longrightarrow} \sqrt{\sum_{i\in\mathbb{Z}}\mathbb{E}(\xi_0\xi_i)}(\Delta_t)_{t\geq 0}. \tag{3.3}$$

Remark. *Assumptions* (A_1) *and* (A_2) *imply that*

$$\forall\lambda\in[0,1/2[,\quad \sum_{k\in\mathbb{Z}}|k|^{\lambda}\left|\mathbb{E}(\xi_0\xi_k)\right|<+\infty. \tag{3.4}$$

Indeed, this sum is equal to

$$\mathbb{E}(\xi_0^2)+2\sum_{k=1}^{\infty}|k|^{\lambda}\left|\mathbb{E}(\xi_0\xi_k)\right|$$

and for any $k \geq 1$, *from Cauchy–Schwarz inequality, we get*

$$\begin{aligned}\left|\mathbb{E}(\xi_0\xi_k)\right| &= \left|\mathbb{E}\big(\xi_0\mathbb{E}(\xi_k|\mathcal{M}_0)\big)\right| \\ &\leq \|\xi_0\|_2\theta_2^{\xi}(k) \\ &\leq \|\xi_0\|_2 g(k).\end{aligned}$$

The result (3.4) *follows by remarking that*

$$\sum_{k=1}^{\infty}|k|^{\lambda}g(k) \leq g(1)\sum_{k=1}^{\infty}\frac{1}{k^{3/2-\lambda}}$$

which is finite for every $\lambda \in [0, 1/2[$.

Remark. *If* $\theta_2^{\xi}(n) = \mathcal{O}(n^{-a})$ *for some positive* a, *condition* (A_2) *holds for* $a > 3/2$.

4. Properties of the occupation times of the random walk

The random walk $S = (S_k)_{k\geq 0}$ is defined as in the previous section and is assumed to verify the property (**P**). The local time of the random walk is defined for every $i \in \mathbb{Z}$ by

$$N_n(i) = \sum_{k=0}^{n}\mathbf{1}_{\{S_k=i\}}.$$

The local time of self-intersection at point i of the random walk $(S_n)_{n\geq 0}$ is defined by

$$\alpha(n,i) = \sum_{k,l=0}^{n}\mathbf{1}_{\{S_k-S_l=i\}}.$$

The stochastic properties of the sequences $(N_n(i))_{n\in\mathbb{N},i\in\mathbb{Z}}$ and $(\alpha(n,i))_{n\in\mathbb{N},i\in\mathbb{Z}}$ are well-known when the random walk S is strongly aperiodic. A random walk who satisfies the property (**P**) is not strongly aperiodic in general. However, a local limit theorem for the random walks satisfing (**P**) was proved by Cadre [7] (see Lemma 2.4.5, p. 70), then it is not difficult to adapt the proofs of the strongly aperiodic case to our setting: for assertion (i) see Lemma 4 in Kesten and Spitzer [19], for (ii)(a) see Lemma 3.1 in Dombry and Guillotin-Plantard [13]. Result (ii)(b) is obtained from Lemma 6 in Kesten and Spitzer [19]; details are omitted. Assertion (iii) is an adaptation to dimension one of Lemma 2.3.2 of Cadre's [7] thesis.

Proposition 4.1. (i) *The sequence* $n^{-3/4}\max_{i\in\mathbb{Z}}N_n(i)$ *converges in probability to* 0.

(ii) (a) *For any* $p \in [1,+\infty)$, *there exists some constant* C *such that for all* $n \geq 1$,

$$\mathbb{E}\big(\alpha(n,0)^p\big) \leq Cn^{3p/2}.$$

(b) *For any* $m \geq 1$, *for any real* $\theta_1,\dots,\theta_m$, *for any* $0 \leq t_1 \leq \cdots \leq t_m$, *the sequence*

$$n^{-3/2}\sum_{i\in\mathbb{Z}}\left(\sum_{k=1}^{m}\theta_k N_{[nt_k]}(i)\right)^2$$

converges in distribution to

$$\int_{\mathbb{R}}\left(\sum_{k=1}^{m}\theta_k L_{t_k}(x)\right)^2 \mathrm{d}x,$$

where $(L_t(x))_{t\geq 0;x\in\mathbb{R}}$ *is the local time of the real Brownian motion* $(B_t)_{t\geq 0}$.

(iii) *For every* $\lambda \in]0,1[$, *there exists a constant* C *such that for any* $i, j \in \mathbb{Z}$,

$$\left\|\alpha(n,i)-\alpha(n,j)\right\|_2 \le Cn^{(3-\lambda)/2}|i-j|^{\lambda}.$$

5. Examples

In this section, we present examples for which we can compute upper bounds for $\theta_2(n)$ for any $n \ge 1$. We refer to Chapter 3 in Dedecker et al. [11] and references therein for more details.

5.1. *Example* 1: *Causal functions of stationary sequences*

Let $(E, \mathcal{E}, \mathbb{Q})$ be a probability space. Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a stationary sequence of random variables with values in a measurable space $\mathcal{S}$. Assume that there exists a real valued function H defined on a subset of $\mathcal{S}^{\mathbb{N}}$, such that $H(\varepsilon_0, \varepsilon_{-1}, \varepsilon_{-2}, \ldots)$ is defined almost surely. The stationary sequence $(\xi_n)_{n\in\mathbb{Z}}$ defined by $\xi_n = H(\varepsilon_n, \varepsilon_{n-1}, \varepsilon_{n-2}, \ldots)$ is called a causal function of $(\varepsilon_i)_{i\in\mathbb{Z}}$.

Assume that there exists a stationary sequence $(\varepsilon'_i)_{i\in\mathbb{Z}}$ distributed as $(\varepsilon_i)_{i\in\mathbb{Z}}$ and independent of $(\varepsilon_i)_{i\le 0}$. Define $\xi^*_n = H(\varepsilon'_n, \varepsilon'_{n-1}, \varepsilon'_{n-2}, \ldots)$. Clearly, ξ^*_n is independent of $\mathcal{M}_0 = \sigma(\xi_i, i \le 0)$ and distributed as ξ_n. Let $(\delta_2(i))_{i>0}$ be a non increasing sequence such that

$$\left\|\mathbb{E}\left(\left|\xi_i - \xi^*_i\right| | \mathcal{M}_0\right)\right\|_2 \le \delta_2(i). \tag{5.1}$$

Then the coefficient θ_2 of the sequence $(\xi_n)_{n\ge 0}$ satisfies

$$\theta_2(i) \le \delta_2(i). \tag{5.2}$$

Let us consider the particular case where the sequence of innovations $(\varepsilon_i)_{i\in\mathbb{Z}}$ is absolutely regular in the sense of Volkonskii and Rozanov [26]. Then, according to Theorem 4.4.7 in Berbee [2], if E is rich enough, there exists $(\varepsilon'_i)_{i\in\mathbb{Z}}$ distributed as $(\varepsilon_i)_{i\in\mathbb{Z}}$ and independent of $(\varepsilon_i)_{i\le 0}$ such that

$$\mathbb{Q}\left(\varepsilon_i \ne \varepsilon'_i \text{ for some } i \ge k | \mathcal{F}_0\right) = \frac{1}{2}\|\mathbb{Q}_{\tilde{\varepsilon}_k|\mathcal{F}_0} - \mathbb{Q}_{\tilde{\varepsilon}_k}\|_v,$$

where $\tilde{\varepsilon}_k = (\varepsilon_k, \varepsilon_{k+1}, \ldots)$, $\mathcal{F}_0 = \sigma(\varepsilon_i, i \le 0)$ and $\|\cdot\|_v$ is the variation norm. In particular if the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ is idependent and identically distributed, it suffices to take $\varepsilon'_i = \varepsilon_i$ for $i > 0$ and $\varepsilon'_i - \varepsilon''_i$ for $i \le 0$, where $(\varepsilon''_i)_{i\in\mathbb{Z}}$ is an independent copy of $(\varepsilon_i)_{i\in\mathbb{Z}}$.

Application to causal linear processes
In that case, $\xi_n = \sum_{j\ge 0} a_j \varepsilon_{n-j}$, where $(a_j)_{j\ge 0}$ is a sequence of real numbers. We can choose

$$\delta_2(i) \ge \left\|\varepsilon_0 - \varepsilon'_0\right\|_2 \sum_{j\ge i} |a_j| + \sum_{j=0}^{i-1} |a_j| \left\|\varepsilon_{i-j} - \varepsilon'_{i-j}\right\|_2.$$

From Proposition 2.3 in Merlevède and Peligrad [24], we obtain that

$$\delta_2(i) \le \left\|\varepsilon_0 - \varepsilon'_0\right\|_2 \sum_{j\ge i} |a_j| + \sum_{j=0}^{i-1} |a_j| \left(2^2 \int_0^{\beta(\sigma(\varepsilon_k, k\le 0), \sigma(\varepsilon_k, k\ge i-j))} Q^2_{\varepsilon_0}(u)\right)^{1/2} \mathrm{d}u,$$

where Q_{ε_0} is the generalized inverse of the tail function $x \mapsto \mathbb{Q}(|\varepsilon_0| > x)$.

5.2. *Example* 2: *Iterated random functions*

Let $(\xi_n)_{n\geq 0}$ be a real valued stationary Markov chain, such that $\xi_n = F(\xi_{n-1}, \varepsilon_n)$ for some measurable function F and some independent and identically distributed sequence $(\varepsilon_i)_{i>0}$ independent of ξ_0. Let ξ_0^* be a random variable distributed as ξ_0 and independent of $(\xi_0, (\varepsilon_i)_{i>0})$. Define $\xi_n^* = F(\xi_{n-1}^*, \varepsilon_n)$. The sequence $(\xi_n^*)_{n\geq 0}$ is distributed as $(\xi_n)_{n\geq 0}$ and independent of ξ_0. Let $\mathcal{M}_i = \sigma(\xi_j, 0 \leq j \leq i)$. As in Example 1, define the sequence $(\delta_2(i))_{i>0}$ by (5.1). The coefficient θ_2 of the sequence $(\xi_n)_{n\geq 0}$ satisfies the bound (5.2) of Example 1.

Let μ be the distribution of ξ_0 and $(\xi_n^x)_{n\geq 0}$ be the chain starting from $\xi_0^x = x$. With these notations, we can choose $\delta_2(i)$ such that

$$\delta_2(i) \geq \left\|\xi_i - \xi_i^*\right\|_2 = \left(\int\int \left\|\xi_i^x - \xi_i^y\right\|_2^2 \mu(\mathrm{d}x)\mu(\mathrm{d}y)\right)^{1/2}.$$

For instance, if there exists a sequence $(d_2(i))_{i\geq 0}$ of positive numbers such that

$$\left\|\xi_i^x - \xi_i^y\right\|_2 \leq d_2(i)|x-y|,$$

then we can take $\delta_2(i) = d_2(i)\|\xi_0 - \xi_0^*\|_2$. For example, in the usual case where $\|F(x,\varepsilon_0) - F(y,\varepsilon_0)\|_2 \leq \kappa|x-y|$ for some $\kappa < 1$, we can take $d_2(i) = \kappa^i$.

An important example is $\xi_n = f(\xi_{n-1}) + \varepsilon_n$ for some κ-Lipschitz function f. If ξ_0 has a moment of order 2, then $\delta_2(i) \leq \kappa^i\|\xi_0 - \xi_0^*\|_2$.

5.3. *Example* 3: *Dynamical systems on* [0, 1]

Let $I = [0,1]$, T be a map from I to I and define $X_i = T^i$. If μ is invariant by T, the sequence $(X_i)_{i\geq 0}$ of random variables from (I,μ) to I is strictly stationary.

For any finite measure ν on I, we use the notations $\nu(h) = \int_I h(x)\nu(dx)$. For any finite signed measure ν on I, let $\|\nu\| = |\nu|(I)$ be the total variation of ν. Denote by $\|g\|_{1,\lambda}$ the $\mathbb{L}^1$-norm with respect to the Lebesgue measure λ on I.

Covariance inequalities

In many interesting cases, one can prove that, for any BV function h and any k in $\mathbb{L}^1(I,\mu)$,

$$\left|\operatorname{cov}\big(h(X_0), k(X_n)\big)\right| \leq a_n \left\|k(X_n)\right\|_1 \big(\|h\|_{1,\lambda} + \|dh\|\big), \tag{5.3}$$

for some nonincreasing sequence a_n tending to zero as n tends to infinity.

Spectral gap

Define the operator $\mathcal{L}$ from $\mathbb{L}^1(I,\lambda)$ to $\mathbb{L}^1(I,\lambda)$ via the equality

$$\int_0^1 \mathcal{L}(h)(x)k(x)\,\mathrm{d}\lambda(x) = \int_0^1 h(x)(k\circ T)(x)\,\mathrm{d}\lambda(x), \quad \text{where } h \in \mathbb{L}^1(I,\lambda) \text{ and } k \in \mathbb{L}^\infty(I,\lambda).$$

The operator $\mathcal{L}$ is called the Perron–Frobenius operator of T. In many interesting cases, the spectral analysis of $\mathcal{L}$ in the Banach space of BV-functions equiped with the norm $\|h\|_v = \|dh\| + \|h\|_{1,\lambda}$ can be done by using the Theorem of Ionescu-Tulcea and Marinescu (see Lasota and Yorke [22] and Hofbauer and Keller [18]). Assume that 1 is a simple eigenvalue of $\mathcal{L}$ and that the rest of the spectrum is contained in a closed disk of radius strictly smaller than one. Then there exists a unique T-invariant absolutely continuous probability μ whose density f_μ is BV, and

$$\mathcal{L}^n(h) = \lambda(h)f_\mu + \Psi^n(h) \quad \text{with } \left\|\Psi^n(h)\right\|_v \leq K\rho^n\|h\|_v \tag{5.4}$$

for some $0 \leq \rho < 1$ and $K > 0$. Assume moreover that

$$I_* = \{f_\mu \neq 0\} \text{ is an interval, and there exists } \gamma > 0 \text{ such that } f_\mu > \gamma^{-1} \text{ on } I_*. \tag{5.5}$$

Without loss of generality assume that $I_* = I$ (otherwise, take the restriction to I_* in what follows). Define now the Markov kernel associated to T by

$$P(h)(x) = \frac{\mathcal{L}(f_\mu h)(x)}{f_\mu(x)}. \tag{5.6}$$

It is easy to check (see for instance Barbour, Gerrard and Reinert [1]) that $(X_0, X_1, \ldots, X_n)$ has the same distribution as $(Y_n, Y_{n-1}, \ldots, Y_0)$ where $(Y_i)_{i\geq 0}$ is a stationary Markov chain with invariant distribution μ and transition kernel P. Since $\|fg\|_\infty \leq \|fg\|_v \leq 2\|f\|_v\|g\|_v$, we infer that, taking $C = 2K\gamma(\|df_\mu\| + 1)$,

$$P^n(h) = \mu(h) + g_n \quad \text{with } \|g_n\|_\infty \leq C\rho^n\|h\|_v. \tag{5.7}$$

This estimate implies (5.3) with $a_n = C\rho^n$ (see Dedecker and Prieur [10]).

Expanding maps
Let $([a_i, a_{i+1}[)_{1\leq i\leq N}$ be a finite partition of $[0, 1[$. We make the same assumptions on T as in Collet, Martinez and Schmitt [9]:

1. For each $1 \leq j \leq N$, the restriction T_j of T to $]a_j, a_{j+1}[$ is strictly monotonic and can be extented to a function $\overline{T}_j$ belonging to $C^2([a_j, a_{j+1}])$.
2. Let I_n be the set where $(T^n)'$ is defined. There exists $A > 0$ and $s > 1$ such that $\inf_{x\in I_n} |(T^n)'(x)| > As^n$.
3. The map T is topologically mixing: for any two nonempty open sets U, V, there exists $n_0 \geq 1$ such that $T^{-n}(U) \cap V \neq \varnothing$ for all $n \geq n_0$.

If T satisfies 1., 2. and 3., then (5.4) holds. Assume furthermore that (5.5) holds (see Morita [25] for sufficient conditions). Then, arguing as in Example 4 in Section 7 of Dedecker and Prieur [10], we can prove that for the Markov chain $(Y_i)_{i\geq 0}$ and the σ-algebras $\mathcal{M}_i = \sigma(Y_j, j \leq i)$, there exists a positive constant C such that $\theta_2(i) \leq C\rho^i$.

6. Proof of Theorem 3.2

6.1. *Notations and preliminary technical lemmas*

For any $n \in \mathbb{N}$ and any $i \in \mathbb{Z}$, we denote by $X_{n,i}$ the random variable $N_n(i)\xi_i$ where $N_n(i)$ denotes the local time of the random walk at site i.

We denote by $\mathcal{G} = \sigma(S_k, k \geq 0)$ the σ-algebra generated by the random walk S. For any real random variables X, Y, and any $t \in \mathbb{R}$, we define

$$d_t(X, Y) = \left|\mathbb{E}\left(e^{itX}|\mathcal{G}\right) - \mathbb{E}\left(e^{itY}|\mathcal{G}\right)\right|.$$

Let η be a random variable with standard normal distribution, independent of the random walk S. Let $(X_i)_{i=1,\ldots,3}$ be random variables such that for $i = 1, \ldots, 3$, $\mathbb{E}(X_i|\mathcal{G}) = 0$ and $\mathbb{E}(X_i^2|\mathcal{G})$ is bounded. Let Y_1, Y_2 be random variables such that $\mathbb{E}(Y_i|\mathcal{G}) = 0$ and $\mathbb{E}(Y_i^2|\mathcal{G})$ is bounded for $i = 1, 2$. They are assumed to be independent conditionally to the random walk. Let X be a real random variable and let $t \in \mathbb{R}$, we define

$$A_t(X) = d_t\left(X, \eta\sqrt{\mathbb{E}\left(X^2|\mathcal{G}\right)}\right).$$

The following properties of d_t and A_t hold.

Lemma 6.1 (Lemma 4.3 in Utev [28]).

$$A_t(X_1) \leq \frac{2}{3}|t|^3\mathbb{E}\left(|X_1|^3|\mathcal{G}\right),$$

$$A_t(Y_1 + Y_2) \leq A_t(Y_1) + A_t(Y_2),$$

$$d_t(X_2+X_3,X_2)\le\frac{t^2}{2}\big(\mathbb{E}\big(X_3^2|\mathcal{G}\big)+\big(\mathbb{E}\big(X_2^2|\mathcal{G}\big)\mathbb{E}\big(X_3^2|\mathcal{G}\big)\big)^{1/2}\big),$$

$$\forall a,b\in\mathbb{R},\quad d_t(\eta a,\eta b)\le\frac{t^2}{2}\big|a^2-b^2\big|.$$

The following lemma is a variation on Lemma 1.2. in Utev [28]; the proof is omitted.

Lemma 6.2. *For every $\varepsilon\in]0,1[$, denote $\delta_\varepsilon=(1-\varepsilon^2+2\varepsilon)/2$. Let a non-decreasing sequence of non-negative numbers $a(n)$ be specified, such that there exist non-increasing sequences of non-negative numbers $\varepsilon(k)$, $\gamma(k)$ and a sequence of naturals $T(k)$, satisfying conditions*

$$T(k)\le 2^{-1}\big(k+\big[k^{\delta_\varepsilon}\big]\big),$$
$$a(k)\le\max_{k_0\le s\le k}\big(a\big(T(s)\big)+\gamma(s)\big)$$

for any $k\ge k_0$ with an arbitrary $k_0\in\mathbb{N}^*$. Then

$$a(n)\le a(n_0)+2\sum_{k_0\le 2^j\le n}\gamma\big(2^j\big)$$

for any $n\ge k_0$, where one can take $n_0=2^c$ with $c>\frac{2-\delta_\varepsilon}{1-\delta_\varepsilon}$.

For any $M>0$, we define the functions

$$\varphi_M:\begin{cases}\mathbb{R}\to\mathbb{R},\\ x\mapsto\varphi_M(x)=(x\wedge M)\vee(-M)\end{cases}$$

and

$$\varphi^M:\begin{cases}\mathbb{R}\to\mathbb{R},\\ x\mapsto\varphi^M(x)=x-\varphi_M(x).\end{cases}$$

Let $(\varepsilon_n)_{n\ge1}$ be a sequence of positive numbers tending to 0 as n goes to infinity. For any $n\in\mathbb{N}$ and any $i\in\mathbb{Z}$, we denote by $Z_{n,i}$ the random variable

$$\varphi_{\varepsilon_n n^{3/4}}(X_{n,i})-\mathbb{E}\big(\varphi_{\varepsilon_n n^{3/4}}(X_{n,i})|\mathcal{G}\big).$$

The next lemma will be a key point in the proof of Theorem 3.2.

Lemma 6.3. *Let $0<\varepsilon<1$. There exists some positive constant $C(\varepsilon)$ such that for all $a\in\mathbb{Z}$, for all $v\in\mathbb{N}^*$,*

$$A_t\left(n^{-3/4}\sum_{i=a+1}^{a+v}Z_{n,i}\right)$$

is bounded by

$$C(\varepsilon)\left(|t|^3h^{2/\varepsilon}n^{-9/4}\sum_{i=a+1}^{a+v}\mathbb{E}\big(|Z_{n,i}|^3|\mathcal{G}\big)+t^2\left(h^{(\varepsilon-1)/2}+\sum_{j:2^j\ge h^{1/\varepsilon}}2^{3j/2}g\big(2^{j\varepsilon}\big)\right)n^{-3/2}\sum_{i=a+1}^{a+v}N_n(i)^2\right),$$

where h is an arbitrary positive natural number and g is the function introduced in assumption (A$_2$) *of Theorem 3.2.*

Proof. Let $h\in\mathbb{N}^*$. Let $0<\varepsilon<1$. In the following, C, $C(\varepsilon)$ denote constants which may vary from line to line. Let κ_ε be a positive constant greater than 1 which will be precised further.

Case 1: $v < \kappa_\varepsilon h^{1/\varepsilon}$.

By Lemma 6.1, we have

$$A_t\left(n^{-3/4}\sum_{i=a+1}^{a+v} Z_{n,i}\right) \le \frac{2|t|^3}{3n^{9/4}}\mathbb{E}\left(\left|\sum_{i=a+1}^{a+v} Z_{n,i}\right|^3 \Big| \mathcal{G}\right)$$
$$\le \frac{2}{3n^{9/4}}\kappa_\varepsilon^2 |t|^3 h^{2/\varepsilon}\sum_{i=a+1}^{a+v}\mathbb{E}\big(|Z_{n,i}|^3|\mathcal{G}\big) \tag{6.1}$$

since $|x|^3$ is a convex function.

Case 2: $v \ge \kappa_\varepsilon h^{1/\varepsilon}$.

Without loss of generality, assume that $a = 0$. Let $\delta_\varepsilon = (1-\varepsilon^2+2\varepsilon)/2$. Define then

$$m = [v^\varepsilon], \qquad B = \{u \in \mathbb{N}\colon\ 2^{-1}\big(v - [v^{\delta_\varepsilon}]\big) \le um \le 2^{-1}v\},$$
$$A = \left\{u \in \mathbb{N}\colon\ 0 \le u \le v, \sum_{i=um+1}^{(u+1)m} N_n(i)^2 \le (m/v)^\varepsilon \sum_{i=1}^{v} N_n(i)^2\right\}.$$

Following Utev [28] we prove that, for $0 < \varepsilon < 1$, $A \cap B$ is not wide for v greater than κ_ε. We have indeed

$$|A\cap B| = |B| - |\overline{A}\cap B| \ge |B| - |\overline{A}| \ge \frac{v^{(1-\varepsilon^2)/2}}{2}\big(1 - 4v^{-(1-\varepsilon)^2/2}\big) - \frac{3}{2},$$

where $\overline{A}$ denotes the complementary of the set A. We can find κ_ε large enough so that $|A \cap B|$ be positive.

Let $u \in A \cap B$. We start from the following simple identity

$$Q \equiv n^{-3/4}\sum_{i=1}^{v} Z_{n,i}$$
$$= n^{-3/4}\sum_{i=1}^{um} Z_{n,i} + n^{-3/4}\sum_{i=um+1}^{(u+1)m} Z_{n,i} + n^{-3/4}\sum_{i=(u+1)m+1}^{v} Z_{n,i}$$
$$\equiv Q_1 + Q_2 + Q_3. \tag{6.2}$$

For any fixed $n \ge 0$, and any $i \in \mathbb{Z}$ such that $N_n(i) \ne 0$, define

$$W_{n,i} = \varphi_{\varepsilon_n n^{3/4}/N_n(i)}(\xi_i) - \mathbb{E}\big(\varphi_{\varepsilon_n n^{3/4}/N_n(i)}(\xi_i)|\mathcal{G}\big).$$

If $N_n(i) = 0$, let $W_{n,i} = 0$. As for any fixed $n \ge 0$, $i \in \mathbb{Z}$, the function

$$x \mapsto \varphi_{\varepsilon_n\sigma_n/N_n(i)}(x) - \mathbb{E}\big(\varphi_{\varepsilon_n\sigma_n/N_n(i)}(\xi_i)|\mathcal{G}\big)$$

is 1-Lipschitz, we have for any fixed path of the random walk, for all $l \ge 1$, for all $k \ge 1$,

$$\theta_{k,2}^{W_{\cdot,n}}(l) \le \theta_{k,2}^{\xi}(l),$$

where $W_{\cdot,n} = (W_{n,i})_{i\in\mathbb{Z}}$ and $\xi = (\xi_i)_{i\in\mathbb{Z}}$. We now claim that for any fixed n, any $a, b \in \mathbb{N}$,

$$\mathbb{E}\left(\left(\sum_{i=a}^{b} Z_{n,i}\right)^2 \Big| \mathcal{G}\right) \le C\sum_{i=a}^{b} N_n(i)^2 \tag{6.3}$$

with $C = 2\mathbb{E}(\xi_0^2) + 2\sqrt{2}\mathbb{E}(\xi_0^2)^{1/2}\sum_{l=1}^{\infty}\theta_{1,2}(l)$ which is finite from assumptions (A_1) and (A_2). Indeed, remark that $Z_{n,i} = N_n(i)W_{n,i}$, then

$$\begin{aligned}
\mathbb{E}\left(\left(\sum_{i=a}^{b} Z_{n,i}\right)^2 \Big|\mathcal{G}\right) &= \mathbb{E}\left(\left(\sum_{i=a}^{b} N_n(i)W_{n,i}\right)^2 \Big|\mathcal{G}\right) \\
&= \sum_{i=a}^{b} N_n(i)^2\mathbb{E}\big(W_{n,i}^2|\mathcal{G}\big) + \sum_{i=a}^{b}\sum_{j\in\{a,\dots,b\};\, j\neq i} N_n(i)N_n(j)\mathbb{E}(W_{n,i}W_{n,j}|\mathcal{G}) \\
&\le \sum_{i=a}^{b} N_n(i)^2\mathbb{E}\big(W_{n,i}^2|\mathcal{G}\big) + \sum_{i=a}^{b} N_n(i)^2 \sum_{j\in\{a,\dots,b\};\, j\neq i} \big|\mathbb{E}(W_{n,i}W_{n,j}|\mathcal{G})\big|
\end{aligned}$$

by remarking that $N_n(i)N_n(j) \le \frac{1}{2}(N_n(i)^2 + N_n(j)^2)$.

Then for any $j > i$, using Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}
\big|\mathbb{E}(W_{n,i}W_{n,j}|\mathcal{G})\big| &= \big|\mathbb{E}\big(W_{n,i}\mathbb{E}(W_{n,j}|\mathcal{M}_i)|\mathcal{G}\big)\big| \\
&\le \mathbb{E}\big(W_{n,i}^2|\mathcal{G}\big)^{1/2}\mathbb{E}\big(\mathbb{E}(W_{n,j}|\mathcal{M}_i)^2|\mathcal{G}\big)^{1/2} \\
&\le \mathbb{E}\big(W_{n,i}^2|\mathcal{G}\big)^{1/2}\theta_{1,2}^{\xi}(j-i).
\end{aligned}$$

By remarking that $\mathbb{E}(W_{n,i}^2|\mathcal{G}) \le \mathbb{E}(\xi_0^2)$ (since $|\varphi_M(x)| \le |x|$ for any real x), we deduce inequality (6.3).

By Lemma 6.1,

$$d_t(Q, Q_1 + Q_3) = d_t(Q, Q - Q_2) \le \frac{t^2}{2}\big(\mathbb{E}\big(Q_2^2|\mathcal{G}\big) + \mathbb{E}\big(Q_2^2|\mathcal{G}\big)^{1/2}\mathbb{E}\big(Q^2|\mathcal{G}\big)^{1/2}\big). \tag{6.4}$$

Using (6.4) and (6.3), we get

$$d_t(Q, Q_1 + Q_3) \le Ct^2 v^{(\varepsilon-1)\varepsilon/2} n^{-3/2}\sum_{i=1}^{v} N_n(i)^2. \tag{6.5}$$

Now, given the random variables Q_1 and Q_3, we define two random variables g_1 and g_3 which are assumed independent conditionally to the random walk S such that conditionally to the random walk, the distribution of g_i coincides with that of Q_i, $i = 1, 3$. We have

$$\begin{aligned}
&d_t(Q_1 + Q_3, g_1 + g_3) \\
&\quad = \big|\mathbb{E}\big(\big(e^{itQ_1} - 1\big)\big(e^{itQ_3} - 1\big)|\mathcal{G}\big) - \mathbb{E}\big(e^{itQ_1} - 1|\mathcal{G}\big)\mathbb{E}\big(e^{itQ_3} - 1|\mathcal{G}\big)\big| \\
&\quad \le \mathbb{E}\big(\big|e^{itQ_1} - 1\big|^2|\mathcal{G}\big)^{1/2}\mathbb{E}\big(\big|\mathbb{E}\big(e^{itQ_3} - 1 - \mathbb{E}\big(e^{itQ_3} - 1|\mathcal{G}\big)|\mathcal{M}_{um};\mathcal{G}\big)\big|^2|\mathcal{G}\big) \\
&\quad \le 2|t|n^{-3/4}\mathbb{E}\left(\left|\sum_{i=1}^{um} Z_{n,i}\right|^2\Big|\mathcal{G}\right)^{1/2} v|t|n^{-3/4}\left(\sum_{i=(u+1)m+1}^{v} N_n(i)\right)\theta_2^{\xi}(m+1) \\
&\quad \le Ct^2 v^{3/2} n^{-3/2}\left(\sum_{i=1}^{v} N_n(i)^2\right) g\big(v^{\varepsilon}\big)
\end{aligned}$$

by (6.3), Definition 2.2 and assumption (A_2) of Theorem 3.2. Hence,

$$d_t(Q_1 + Q_3, g_1 + g_3) \le Ct^2 f(v) n^{-3/2}\sum_{i=1}^{v} N_n(i)^2, \tag{6.6}$$

where $f(v) = v^{3/2} g(v^{\varepsilon})$ is non-increasing by assumption (A_2) of Theorem 3.2.

We also have by Lemma 6.1

$$A_t(g_1 + g_3) \le A_t(g_1) + A_t(g_3). \tag{6.7}$$

Finally, still by Lemma 6.1, and using Definition 2.2, we have

$$\begin{aligned} & d_t\big(\eta\sqrt{\mathbb{E}\big(Q^2|\mathcal{G}\big)}, \eta\sqrt{\mathbb{E}\big((g_1+g_3)^2|\mathcal{G}\big)}\big) \\ & \quad \le \frac{t^2}{2}\big|\mathbb{E}\big(Q^2|\mathcal{G}\big) - \mathbb{E}\big((g_1+g_3)^2|\mathcal{G}\big)\big| \qquad (6.8) \\ & \quad \le \frac{t^2}{2}\big|\mathbb{E}\big(Q_2^2|\mathcal{G}\big) + 2\mathbb{E}(Q_1Q_2|\mathcal{G}) + 2\mathbb{E}(Q_2Q_3|\mathcal{G}) + 2\mathbb{E}(Q_1Q_3|\mathcal{G})\big| \\ & \quad \le Ct^2\big(v^{(\varepsilon-1)\varepsilon/2} + f(v)\big)n^{-3/2}\sum_{i=1}^{v} N_n(i)^2. \qquad (6.9) \end{aligned}$$

Since we have

$$\begin{aligned} A_t\left(n^{-3/4}\sum_{i=1}^{v} Z_{n,i}\right) &= A_t(Q) \\ &= d_t\big(Q, \eta\sqrt{\mathbb{E}\big(Q^2|\mathcal{G}\big)}\big) \\ &\le d_t(Q, Q_1+Q_3) + d_t(Q_1+Q_3, g_1+g_3) + A_t(g_1+g_3) + d_t\big(g_1+g_3, \eta\sqrt{\mathbb{E}\big(Q^2|\mathcal{G}\big)}\big) \end{aligned}$$

by combining (6.5)–(6.8), we get the following recurrent inequality:

$$\begin{aligned} A_t\left(n^{-3/4}\sum_{i=1}^{v} Z_{n,i}\right) \le{}& A_t\left(n^{-3/4}\sum_{i=1}^{um} Z_{n,i}\right) + A_t\left(n^{-3/4}\sum_{i=(u+1)m+1}^{v} Z_{n,i}\right) \\ & + Ct^2\big(v^{(\varepsilon-1)\varepsilon/2} + f(v)\big)n^{-3/2}\sum_{i=1}^{v} N_n(i)^2 \end{aligned}$$

for $v \ge \kappa_\varepsilon h^{1/\varepsilon} \ge \kappa_\varepsilon$.

We now apply Lemma 6.2 with:

- $k_0 = \kappa_\varepsilon h^{1/\varepsilon}$,
- for $k \ge k_0$, $T(k) = \max\{u_k m_k, k - u_k m_k - m_k\}$ where u_k and m_k are defined from k as u and m from v (see the proof of $A \cap B$ not wide),
- $c < \frac{\ln(\kappa_\varepsilon)}{\ln(2)}$ (we may need to enlarge κ_ε),
- for $s \ge k_0$, $\gamma(s) = Ct^2(s^{\varepsilon(\varepsilon-1)/2} + f(s))$,
- for $s \ge k_0$,

$$a(s) = \sup_{l\in\mathbb{Z}} \max_{k_0\le i\le s} \frac{A_t(n^{-3/4}\sum_{j=l+1}^{l+i} Z_{n,j})}{n^{-3/2}\sum_{j=l+1}^{l+i} N_n(j)^2}.$$

Applying Lemma 6.2 yields the statement of Lemma 6.3. □

6.2. *Proof of Theorem* 3.2

The proof of Theorem 3.2 is decomposed in two parts: first, we prove the convergence of the finite-dimensional distributions of the process $(n^{-3/2}U_{[nt]})_{t\ge0}$, then its tightness in the space $\mathcal{C}([0, +\infty[)$.

Proof of the convergence of the finite-dimensional distributions
Since the random variable U_n can be rewritten as the sum

$$\sum_{i\in\mathbb{Z}} X_{n,i} = \sum_{i\in\mathbb{Z}} N_n(i)\xi_i,$$

it is enough to prove that for every $m \ge 1$, for any real $\theta_1, \dots, \theta_m$, for any $0 \le t_1 \le t_2 \le \dots \le t_m$, the sequence

$$\frac{1}{n^{3/4}} \sum_{k=1}^{m} \theta_k U_{nt_k} = \frac{1}{n^{3/4}} \sum_{i\in\mathbb{Z}} \left(\sum_{k=1}^{m} \theta_k N_{nt_k}(i) \right) \xi_i$$

converges in distribution to the random variable

$$\sqrt{\sum_{i\in\mathbb{Z}} \mathbb{E}(\xi_0\xi_i)} \sum_{k=1}^{m} \theta_k \Delta_{t_k}.$$

Remark that $|U_{nt} - U_{[nt]}| \le |\xi_{S_{[nt]+1}}| \in L^2$, so it is enough to consider the convergence in distribution of the linear combination $n^{-3/4} \sum_{k=1}^{m} \theta_k U_{[nt_k]}$ for any real $\theta_1, \dots, \theta_m$, for any $0 \le t_1 \le t_2 \le \dots \le t_m$. We only prove the convergence of one-dimensional distributions. The general case is obtained by replacing $N_n(i)$ by the linear combination $\sum \theta_k N_{[nt_k]}(i)$ in the computations.

We first prove the following Lindeberg condition: for any $\varepsilon > 0$,

$$n^{-3/2} \sum_{i\in\mathbb{Z}} \mathbb{E}\big(\big(\varphi^{\varepsilon n^{3/4}}(X_{n,i})\big)^2 \big) \underset{n\to+\infty}{\longrightarrow} 0. \tag{6.10}$$

Proof of the Lindeberg condition (6.10)
We have, for $\varepsilon > 0$ fixed, for n large enough,

$$\begin{aligned} &n^{-3/2} \sum_{i\in\mathbb{Z}} \mathbb{E}\big(\big(\varphi^{\varepsilon n^{3/4}}(X_{n,i})\big)^2 | \mathcal{G} \big) \\ &\quad \le \mathbb{E}\big(\xi_0^2 \mathbf{1}_{\{|\xi_0| > \varepsilon n^{3/4}/\max_j N_n(j)\}} | \mathcal{G} \big) \frac{\alpha(n,0)}{n^{3/2}} =: U_1(\varepsilon, n). \end{aligned}$$

Let $\eta > 0$. We decompose the expectation of $U_1(\varepsilon, n)$ as the sum of

$$U_{1,1}(\varepsilon, n) := \mathbb{E}\left(\mathbf{1}_{\{n^{-3/4}\max_j N_n(j) \ge \eta\}} \mathbb{E}\big(\xi_0^2 \mathbf{1}_{\{|\xi_0| > \varepsilon n^{3/4}/\max_j N_n(j)\}} | \mathcal{G} \big) \frac{\alpha(n,0)}{n^{3/2}} \right)$$

and

$$U_{1,2}(\varepsilon, n) := \mathbb{E}\left(\mathbf{1}_{\{n^{-3/4}\max_j N_n(j) < \eta\}} \mathbb{E}\big(\xi_0^2 \mathbf{1}_{\{|\xi_0| > \varepsilon n^{3/4}/\max_j N_n(j)\}} | \mathcal{G} \big) \frac{\alpha(n,0)}{n^{3/2}} \right).$$

Using (ii)(a) of Proposition 4.1, $U_{1,2}(\varepsilon, n)$ is bounded by

$$C\mathbb{E}\big(\xi_0^2 \mathbf{1}_{\{|\xi_0| > \varepsilon/\eta\}} \big).$$

From assumption (A_1) of Theorem 3.2, for any $\kappa > 0$, there exists $\eta_0 > 0$ such that the above term is less than $\kappa/2$ for any $\eta \le \eta_0$.

We now fix η equal to η_0. Using Cauchy–Schwarz inequality,

$$U_{1,1}(\varepsilon, n) \le \mathbb{E}(\xi_0^2) \mathbb{P}\Big(n^{-3/4} \max_j N_n(j) \ge \eta_0 \Big)^{1/2} \mathbb{E}\left(\frac{\alpha(n,0)^2}{n^3} \right)^{1/2}.$$

From assumption (A_1) of Theorem 3.2, (i) and (ii)(a) of Proposition 4.1, it follows that $U_{1,1}(\varepsilon,n)\le\kappa/2$ for n large enough, then (6.10) is proved.

Since $\varepsilon\to\mathbb{E}(U_1(\varepsilon,n))$ is decreasing, we can find a sequence of positive numbers $(\varepsilon_n)_{n\ge1}$ such that $\varepsilon_n\longrightarrow_{n\to+\infty}0$, and

$$\mathbb{E}\big(U_1(\varepsilon_n,n)\big)\underset{n\to+\infty}{\longrightarrow}0. \tag{6.11}$$

Let us now prove that it implies

$$n^{-3/2}\mathbb{E}\bigg(\bigg(\sum_{i\in\mathbb{Z}}\varphi^{\varepsilon_n n^{3/4}}(X_{n,i})-\mathbb{E}\big(\varphi^{\varepsilon_n n^{3/4}}(X_{n,i})|\mathcal{G}\big)\bigg)^2\bigg)\underset{n\to+\infty}{\longrightarrow}0. \tag{6.12}$$

Proof of (6.12)
For any fixed $n\ge0$, and any $i\in\mathbb{Z}$ such that $N_n(i)\ne0$, define

$$V_{n,i}=\varphi^{\varepsilon_n n^{3/4}/N_n(i)}(\xi_i)-\mathbb{E}\big(\varphi^{\varepsilon_n n^{3/4}/N_n(i)}(\xi_i)|\mathcal{G}\big).$$

If $N_n(i)=0$, let $V_{n,i}=0$. As for any fixed $n\ge0$ and any $i\in\mathbb{Z}$, the function

$$x\mapsto\varphi^{\varepsilon_n n^{3/4}/N_n(i)}(x)$$

is 1-Lipschitz, we have for any fixed path of the random walk, for all $l\ge1$, for all $k\ge1$,

$$\theta^{V_{\cdot,n}}_{k,2}(l)\le\theta^{\xi}_{k,2}(l),$$

where $V_{\cdot,n}=(V_{n,i})_{i\in\mathbb{Z}}$ and $\xi=(\xi_i)_{i\in\mathbb{Z}}$.

$$\begin{aligned}&\mathbb{E}\bigg(\bigg(\sum_{j\in\mathbb{Z}}N_n(j)V_{n,j}\bigg)^2\bigg|\mathcal{G}\bigg)\\&\quad=\sum_{j\in\mathbb{Z}}N_n(j)^2\mathbb{E}\big(V_{n,j}^2|\mathcal{G}\big)+\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z};\,j\ne i}N_n(i)N_n(j)\mathbb{E}(V_{n,i}V_{n,j}|\mathcal{G})\\&\quad\le\sum_{j\in\mathbb{Z}}N_n(j)^2\mathbb{E}\big(V_{n,j}^2|\mathcal{G}\big)+\sum_{i\in\mathbb{Z}}N_n(i)^2\sum_{j\in\mathbb{Z};\,j\ne i}\big|\mathbb{E}(V_{n,i}V_{n,j}|\mathcal{G})\big|\end{aligned}$$

by remarking that $N_n(i)N_n(j)\le\frac{1}{2}(N_n(i)^2+N_n(j)^2)$.

Then for any $j>i$, using Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}\big|\mathbb{E}(V_{n,i}V_{n,j}|\mathcal{G})\big|&=\big|\mathbb{E}\big(V_{n,i}\mathbb{E}(V_{n,j}|\mathcal{M}_i)|\mathcal{G}\big)\big|\\&\le\mathbb{E}\big(V_{n,i}^2|\mathcal{G}\big)^{1/2}\mathbb{E}\big(\mathbb{E}(V_{n,j}|\mathcal{M}_i)^2|\mathcal{G}\big)^{1/2}\\&\le\mathbb{E}\big(V_{n,i}^2|\mathcal{G}\big)^{1/2}\theta^{\xi}_{1,2}(j-i).\end{aligned}$$

Moreover, as $N_n(i)V_{n,i}=\varphi^{\varepsilon_n n^{3/4}}(X_{n,i})-\mathbb{E}(\varphi^{\varepsilon_n n^{3/4}}(X_{n,i})|\mathcal{G})$, we get

$$n^{-3/2}\mathbb{E}\bigg(\bigg(\sum_{i\in\mathbb{Z}}\varphi^{\varepsilon_n n^{3/4}}(X_{n,i})-\mathbb{E}\big(\varphi^{\varepsilon_n n^{3/4}}(X_{n,i})|\mathcal{G}\big)\bigg)^2\bigg)\le\mathbb{E}\bigg(C_n\bigg(\frac{\alpha(n,0)}{n^{3/2}}\bigg)\bigg), \tag{6.13}$$

with $C_n=\sup_{i\in\mathbb{Z}}(\mathbb{E}(V_{n,i}^2|\mathcal{G}))+2\sqrt{\sup_{i\in\mathbb{Z}}(\mathbb{E}(V_{n,i}^2|\mathcal{G}))}\sum_{l=1}^{\infty}\theta^{\xi}_{1,2}(l)$. It remains to prove that the right hand term in (6.13) converges to 0 as n goes to infinity.

We have, for n large enough,

$$\mathbb{E}\big(V_{n,i}^2|\mathcal{G}\big) \le \mathbb{E}\big(\xi_i^2 \mathbf{1}_{\{|\xi_i|>\varepsilon_n n^{3/4}/\max_j N_n(j)\}}|\mathcal{G}\big), \tag{6.14}$$

so using Cauchy–Schwarz inequality,

$$\mathbb{E}\left(C_n\left(\frac{\alpha(n,0)}{n^{3/2}}\right)\right) \le \mathbb{E}\big(U_1(\varepsilon_n,n)\big) + 2\left(\sum_{l=1}^{\infty}\theta_{1,2}^{\xi}(l)\right)\mathbb{E}\big(U_1(\varepsilon_n,n)\big)^{1/2}\mathbb{E}\left(\frac{\alpha(n,0)}{n^{3/2}}\right)^{1/2},$$

which tends to 0 from (ii)(a) of Proposition 4.1, assumption (A_2) from Theorem 3.2 and (6.11). Then, (6.12) is proved.

From (6.12), we conclude that to prove the convergence of the finite-dimensional distributions, it is enough to prove it for the truncated sequence $(Z_{n,i})_{n\ge 0, i\in\mathbb{Z}}$, that is to show that

$$\frac{1}{n^{3/4}}\sum_{i\in\mathbb{Z}} Z_{n,i} \xrightarrow[n\to+\infty]{\mathcal{D}} \sqrt{\sum_k \mathbb{E}(\xi_0\xi_k)}\,\Delta_1. \tag{6.15}$$

Let us decompose

$$\mathbb{E}\big(\mathrm{e}^{\mathrm{i}t(1/n^{3/4})\sum_{i\in\mathbb{Z}} Z_{n,i}}\big) - \mathbb{E}\big(\mathrm{e}^{\mathrm{i}t\sqrt{\sum_k \mathbb{E}(\xi_0\xi_k)}\Delta_1}\big) \tag{6.16}$$

as the sum of $I_i(n)$, $i = 1,\dots,4$, where

$$\begin{aligned}
I_1(n) &= \mathbb{E}\big(\mathrm{e}^{\mathrm{i}t(1/n^{3/4})\sum_{i\in\mathbb{Z}} Z_{n,i}}\big) - \mathbb{E}\big(\mathrm{e}^{-(t^2/(2n^{3/2}))\mathbb{E}((\sum_{i\in\mathbb{Z}} Z_{n,i})^2|\mathcal{G})}\big)\\
I_2(n) &= \mathbb{E}\big(\mathrm{e}^{-(t^2/(2n^{3/2}))\mathbb{E}((\sum_{i\in\mathbb{Z}} Z_{n,i})^2|\mathcal{G})}\big) - \mathbb{E}\big(\mathrm{e}^{-(t^2/(2n^{3/2}))\mathbb{E}((\sum_{i\in\mathbb{Z}} X_{n,i})^2|\mathcal{G})}\big)\\
I_3(n) &= \mathbb{E}\big(\mathrm{e}^{-(t^2/(2n^{3/2}))\mathbb{E}((\sum_{i\in\mathbb{Z}} X_{n,i})^2|\mathcal{G})}\big) - \mathbb{E}\big(\mathrm{e}^{-(t^2/(2n^{3/2}))\sum_{i,j\in\mathbb{Z}} N_n(i)^2\mathbb{E}(\xi_i\xi_j)}\big)\\
I_4(n) &= \mathbb{E}\big(\mathrm{e}^{-(t^2/(2n^{3/2}))\sum_{i,j\in\mathbb{Z}} N_n(i)^2\mathbb{E}(\xi_i\xi_j)}\big) - \mathbb{E}\big(\mathrm{e}^{\mathrm{i}t\sqrt{\sum_k \mathbb{E}(\xi_0\xi_k)}\Delta_1}\big).
\end{aligned}$$

To prove (6.15), it is enough to prove that for any $i = 1,\dots,4$, $I_i(n)$ goes to 0 as $n\to+\infty$.

Estimation of $I_1(n)$. Let us denote by M_n the random variable $\max_{k=0,\dots,n}|S_k|$. From Lemma 6.3, we have

$$\big|I_1(n)\big| \le \mathbb{E}\left(A_t\left(n^{-3/4}\sum_{i=-M_n}^{M_n} Z_{n,i}\right)\right) \le C(t,\varepsilon)\left(h^{2/\varepsilon}n^{-9/4}\sum_{i\in\mathbb{Z}}\mathbb{E}\big(|Z_{n,i}|^3\big) + \delta(h)\right)$$

with $\delta(h) = (h^{(\varepsilon-1)/2} + \sum_{j:2^j\ge h^{1/\varepsilon}} 2^{3j/2}g(2^{j\varepsilon}))\mathbb{E}(\frac{\alpha(n,0)}{n^{3/2}})$.

Hence, using assumption (A_2) from Theorem 3.2 and (ii)(a) from Proposition 4.1, we get $\delta(h) \longrightarrow_{h\to+\infty} 0$.

On the other hand, from assumption (A_1) of Theorem 3.2, there exists a constant $C>0$ such that

$$n^{-9/4}\sum_{i\in\mathbb{Z}}\mathbb{E}\big(|Z_{n,i}|^3\big) \le C\varepsilon_n\mathbb{E}\left(\frac{\alpha(n,0)}{n^{3/2}}\right) \tag{6.17}$$

which tends to zero as n tends to infinity, using (ii)(a) from Proposition 4.1 and the fact that $\varepsilon_n \longrightarrow_{n\to+\infty} 0$. Consequently,

$$\inf_{h\ge 1}\left(h^{2/\varepsilon}\sum_{i\in\mathbb{Z}}\mathbb{E}\big(|Z_{n,i}|^3\big) + \delta(h)\right) \longrightarrow_{n\to+\infty} 0.$$

Estimation of $I_2(n)$. Using that for any $x, y \geq 0$, $|e^{-x} - e^{-y}| \leq |x - y|$ and the fact that

$$Z_{n,i} = X_{n,i} - \left[\varphi^{\varepsilon_n n^{3/4}}(X_{n,i}) - \mathbb{E}\left(\varphi^{\varepsilon_n n^{3/4}}(X_{n,i})|\mathcal{G}\right)\right],$$

we deduce that

$$\begin{aligned}\left|I_2(n)\right| \leq \frac{t^2}{n^{3/2}}\Bigg[&\mathbb{E}\left(\left(\sum_{i\in\mathbb{Z}}\varphi^{\varepsilon_n n^{3/4}}(X_{n,i}) - \mathbb{E}\left(\varphi^{\varepsilon_n n^{3/4}}(X_{n,i})|\mathcal{G}\right)\right)^2\right)\\ &+ \mathbb{E}\left(\left(\sum_{i\in\mathbb{Z}}X_{n,i}\right)^2\right)^{1/2}\mathbb{E}\left(\left(\sum_{i\in\mathbb{Z}}\varphi^{\varepsilon_n n^{3/4}}(X_{n,i}) - \mathbb{E}\left(\varphi^{\varepsilon_n n^{3/4}}(X_{n,i})|\mathcal{G}\right)\right)^2\right)^{1/2}\Bigg].\end{aligned}$$

Since we have

$$\begin{aligned}\mathbb{E}\left(\left(\sum_{i\in\mathbb{Z}}X_{n,i}\right)^2\right) &= \sum_{i,j\in\mathbb{Z}}\mathbb{E}\big(N_n(i)N_n(j)\big)\mathbb{E}(\xi_i\xi_j)\\ &= \sum_{i\in\mathbb{Z}}\mathbb{E}\big(\alpha(n,i) - \alpha(n,0)\big)\mathbb{E}(\xi_0\xi_i) + \mathbb{E}\big(\alpha(n,0)\big)\sum_{i\in\mathbb{Z}}\mathbb{E}(\xi_0\xi_i)\\ &\leq Cn^{3/2}\end{aligned}$$

by combining (ii)(a)–(iii) of Proposition 4.1 and (3.4). Then, from (6.12), we deduce that $I_2(n)$ converges to 0.

Estimation of $I_3(n)$. We have

$$\left|I_3(n)\right| \leq \frac{t^2}{2n^{3/2}}\sum_{i\in\mathbb{Z}}\mathbb{E}\big(\left|\alpha(n,i) - \alpha(n,0)\right|\big)\left|\mathbb{E}(\xi_0\xi_i)\right| = o(1)$$

by combining (iii) of Proposition 4.1 and (3.4).

Estimation of $I_4(n)$. From (ii)(b) of Proposition 4.1, we know that the sequence

$$\mathbb{E}\left(e^{-(t^2/(2n^{3/2}))\sum_{i,j\in\mathbb{Z}}N_n(i)^2\mathbb{E}(\xi_i\xi_j)}\right)$$

converges to $\mathbb{E}(e^{-(t^2/2)\sum_{i\in\mathbb{Z}}\mathbb{E}(\xi_0\xi_i)\int_{\mathbb{R}}L_1^2(x)\,dx})$ which is equal to the characteristic function of the random variable $\sqrt{\sum_{i\in\mathbb{Z}}\mathbb{E}(\xi_0\xi_i)}\Delta_1$.

Proof of the tightness

By Theorem 12.3 of Billingsley [3], it is enough to prove that there exists $K > 0$ such that for all $t_1, t_2 \in [0, T]$, $T < \infty$, s.t. $t_1 \leq t_2$, for all $n \geq 1$,

$$\mathbb{E}\left(|U_{nt_2} - U_{nt_1}|^2\right) \leq Kn^{3/2}|t_2 - t_1|^{3/2}. \tag{6.18}$$

Since $ab \leq (a^2 + b^2)/2$ for any real a, b, we have

$$\begin{aligned}\mathbb{E}\left(|U_{nt_2} - U_{nt_1}|^2\right) &\leq \sum_{i\in\mathbb{Z}}\left|\mathbb{E}(\xi_0\xi_i)\right|\sum_{j\in\mathbb{Z}}\mathbb{E}\left(\left(N_{nt_2}(j) - N_{nt_1}(j)\right)^2\right)\\ &= C\mathbb{E}\left(\alpha\left([nt_2] - [nt_1], 0\right)\right) \leq Cn^{3/2}|t_2 - t_1|^{3/2}\end{aligned}$$

using (3.4), the Markov property for the random walk S and (ii)(a) of Proposition 4.1.

Acknowledgments

The authors are very grateful to the referees for numerous improvements for exposition as well as for the generalization of Theorem 3.2 to the convergence in the space of continuous functions. We are also indebted to the referee who mentioned the possibility to prove Theorem 3.2 by a more functional method used by A. N. Borodin in [5] (see reference [5] of the paper) in the case of an i.i.d. random scenery.

References

[1] A. D. Barbour, R. M. Gerrard and G. Reinert. Iterates of expanding maps. *Probab. Theory Related Fields* **116** (2000) 151–180. MR1743768
[2] H. C. P. Berbee. *Random Walks with Stationary Increments and Renewal Theory*. *Math. Cent. Tracts.* Amsterdam, 1979. MR0547109
[3] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York–London–Sydney, 1968. MR0233396
[4] E. Bolthausen. A central limit theorem for two-dimensional random walks in random sceneries. *Ann. Probab.* **17** (1989) 108–115. MR0972774
[5] A. N. Borodin. Limit theorems for sums of independent random variables defined on a transient random walk. Investigations in the theory of probability distributions, IV. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **85** (1979) 17–29, 237, 244. MR0535455
[6] A. N. Borodin. A limit theorem for sums of independent random variables defined on a recurrent random walk. *Dokl. Akad. Nauk. SSSR* **246** (1979) 786–787. MR0543530
[7] B. Cadre. Etude de convergence en loi de fonctionnelles de processus: Formes quadratiques ou multilinéaires aléatoires, Temps locaux d'intersection de marches aléatoires, Théorème central limite presque sûr. Ph.D. dissertation, Université Rennes 1, 1995.
[8] X. Chen and D. Khoshnevisan. From charged polymers to random walk in random scenery. Preprint, 2008. Available at http://www.math.utk.edu/~xchen/publications.html.
[9] P. Collet, S. Martinez and B. Schmitt. Exponential inequalities for dynamical measures of expanding maps of the interval. *Probab. Theory Related Fields* **123** (2002) 301–322. MR1918536
[10] J. Dedecker and C. Prieur. New dependence coefficients. Examples and applications to statistics. *Probab. Theory Related Fields* **132** (2005) 203–236. MR2199291
[11] J. Dedecker, P. Doukhan, G. Lang, J. R. Leon, S. Louhichi and C. Prieur. *Weak Dependence: With Examples and Applications*. *Lect. Notes in Stat.* **190**. Springer, New York, 2007. MR2338725
[12] R. L. Dobrushin and P. Major. Non-central limit theorems for non-linear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete* **50** (1979) 27–52. MR0550122
[13] C. Dombry and N. Guillotin-Plantard. Discrete approximation of a stable self-similar stationary increments process. *Bernoulli* **15** (2009) 195–222. MR2546804
[14] P. Doukhan and S. Louhichi. A new weak dependence condition and applications to moment inequalities. *Stochastic Process. Appl.* **84** (1999) 313–342. MR1719345
[15] N. Guillotin-Plantard and A. Le Ny. Transient random walks on 2d-oriented lattices. *Theory Probab. Appl.* **52** (2007) 815–826.
[16] N. Guillotin-Plantard and A. Le Ny. A functional limit theorem for a 2d-random walk with dependent marginals. *Electron. Comm. Probab.* **13** (2008) 337–351. MR2415142
[17] N. Guillotin-Plantard and C. Prieur. Central limit theorem for sampled sums of dependent random variables. *ESAIM P&S*. To appear.
[18] F. Hofbauer and G. Keller. Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.* **180** (1982) 119–140. MR0656227
[19] H. Kesten and F. Spitzer. A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete* **50** (1979) 5–25. MR0550121
[20] T. G. Kurtz and P. Protter. Wong-Zakai corrections, random evolutions, and numerical schemes for SEDs. In *Stochastic Analysis* 331–346. Academic Press, Boston, MA, 1991. MR1119837
[21] R. Lang and N. X. Xanh. Strongly correlated random fields as observed by a random walker. *Probab. Theory Related Fields* **64** (1983) 327–340. MR0716490
[22] A. Lasota and J. A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.* **186** (1974) 481–488. MR0335758
[23] M. Maejima. Limit theorems related to a class of operator-self-similar processes. *Nagoya Math. J.* **142** (1996) 161–181. MR1399472
[24] F. Merlevède and M. Peligrad. On the coupling of dependent random variables and applications. In *Empirical Process Techniques for Dependent Data* 171–193. Birkhäuser, Boston, MA, 2002. MR1958781
[25] T. Morita. Local limit theorem and distribution of periodic orbits of Lasota–Yorke transformations with infinite Markov partition. *J. Math. Soc. Japan* **46** (1994) 309–343. MR1264944
[26] Y. A. Rozanov and V. A. Volkonskii. Some limit theorems for random functions I. *Theory Probab. Appl.* **4** (1959) 178–197. MR0121856
[27] F. L. Spitzer. *Principles of Random Walks*, 2nd edition, Springer, New York, 1976. MR0388547
[28] S. A. Utev. Central limit theorem for dependent random variables. *Probab. Theory Math. Statist.* **2** (1990) 519–528. MR1153906