

On the Local Time of Random Walks Associated with Gegenbauer Polynomials

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Abstract The local time of random walks associated with Gegenbauer polynomials $P_n^{(\alpha)}(x)$, $x \in [-1, 1]$, is studied in the recurrent case: $\alpha \in [-\frac{1}{2}, 0]$. When α is nonzero, the limit distribution is given in terms of a Mittag-Leffler distribution. The proof is based on a local limit theorem for the random walk associated with Gegenbauer polynomials. As a by-product, we derive the limit distribution of the local time of some particular birth-and-death Markov chains on \mathbb{N} .

Keywords Random walk · Local time · Local limit theorem · Gegenbauer polynomials · Markov chain · Transition kernel · Recurrence · Transience · Birth and death process

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1 Introduction

Random walks on hypergroups have been extensively studied over the last decades. A history of these processes as well as the motivations for studying them are provided in [7]. We here restrict ourselves to discrete polynomial hypergroups. Let $(\alpha_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$ and $(\gamma_n)_{n \in \mathbb{N}}$ be real sequences with the following properties: $\gamma_n > 0$, $\beta_n \geq 0$, $\alpha_{n+1} > 0$ for all $n \in \mathbb{N}$; moreover, $\alpha_0 = 0$, and $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. We

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define the sequence of polynomials $(P_n)_{n \in \mathbb{N}}$ by $P_0(x) = 1$, $P_1(x) = x$, and by the recursive formula

$$xP_n(x) = \alpha_n P_{n-1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x)$$

for all $n \geq 1$ and $x \in \mathbb{R}$. In this case, there exist constants $c(n, m, k)$, $n, m, k \in \mathbb{N}$, such that the following *linearization formula*

$$P_n P_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) P_k$$

holds for all $n, m \in \mathbb{N}$. Since $P_n(1) = 1$ for all $n \in \mathbb{N}$, we have for all $n, m \in \mathbb{N}$,

$$\sum_{k=|n-m|}^{n+m} c(n, m, k) = 1.$$

If the coefficients $c(n, m, k)$ are nonnegative for all $n, m, k \in \mathbb{N}$, then a hypergroup structure on \mathbb{N} is obtained from the generalized convolution \star defined as follows: for all $n, m \in \mathbb{N}$,

$$\delta_n \star \delta_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) \delta_k.$$

The resulting hypergroup $K = (\mathbb{N}, \star)$ is called *the discrete polynomial hypergroup associated with the sequence $(P_n)_{n \in \mathbb{N}}$* . Many classical families of orthogonal polynomials with respect to some positive measure on $[-1, 1]$ satisfy a linearization formula with nonnegative coefficients. A random walk with distribution $\mu \in \mathcal{M}_1(\mathbb{N})$ on the hypergroup (\mathbb{N}, \star) is then defined as a homogeneous Markov chain on \mathbb{N} with Markov kernel given by

$$p(x, y) = \delta_x \star \mu(y), \quad x, y \in \mathbb{N}.$$

This Markov chain is called *random walk associated with the sequence of polynomials $(P_n)_{n \in \mathbb{N}}$* .

Limit theorems (law of large numbers, central limit theorem, local limit theorems, large deviation principle, iterated logarithm law, etc.) for these processes were earlier investigated by M. Ehring [2, 3], M. Voit [13–16]. L. Gallardo et al. [5, 6], Y. Guivarc'h et al. [9], M. Mabrouki [10] have more specifically studied limit theorems for random walks associated with Gegenbauer polynomials. In this paper, we present some extensions of the theory developed by these authors by deriving a limit theorem for the local time of the random walks associated with Gegenbauer polynomials $(P_n^{(\alpha)})_{n \in \mathbb{N}}$ for every $\alpha \in [-1/2, 0]$.

The organization of the paper is as follows: We recall in Sect. 2 some generalities on the Bessel process and its local time. In Sect. 3, Gegenbauer polynomials as well as the definition of random walks associated with these polynomials are given. In Sect. 4, classical limit theorems for these processes are presented. Section 5 is devoted to the study of the local time of these Markov chains and some particular cases are considered.

2 Preliminaries on the Bessel Process and Its Local Time

For every $\alpha \in [-1, +\infty[$, we will denote by $B^{(\alpha)} := (B_t^{(\alpha)})_{t \in \mathbb{R}_+}$ the unique solution of the stochastic differential equation

$$X_t^2 = X_0^2 + 2 \int_0^t \sqrt{X_s^2} dB_s + 2(\alpha + 1)t$$

where $(B_t)_{t \in \mathbb{R}_+}$ is the real Brownian motion. The parameter α is usually called the *index* of the Bessel process $B^{(\alpha)}$. The process $(B_t^{(\alpha)})_{t \in \mathbb{R}_+}$ can also be defined as the \mathbb{R}_+ -valued Feller diffusion whose infinitesimal generator \mathcal{L} is defined as:

$$\mathcal{L}f = \frac{1}{2} \frac{d^2 f}{dx^2} + \frac{2\alpha + 1}{2x} \frac{df}{dx}$$

on the domain

$$\mathcal{D}(\mathcal{L}) = \left\{ f : \mathbb{R}_+ \rightarrow \mathbb{R}; \mathcal{L}f \in C_b(\mathbb{R}_+), \lim_{x \downarrow 0} x^{2\alpha+1} f'(x) = 0 \right\}.$$

The Bessel process has the Brownian scaling property: for every $c > 0$, the processes $(B_{ct}^{(\alpha)})_{t \in \mathbb{R}_+}$ and $(\sqrt{c}B_t^{(\alpha)})_{t \in \mathbb{R}_+}$ have the same law, when $B_0^{(\alpha)} \equiv 0$.

Let us fix $\alpha \in]-1, 0[$. It is well known (see [11]) that there exists a jointly continuous family $(L_t^{(\alpha)}(x))_{x \in \mathbb{R}_+, t \in \mathbb{R}_+}$ of local times such that the occupation formula

$$\int_0^t h(B_s^{(\alpha)}) ds = 2 \int_0^\infty h(x) L_t^{(\alpha)}(x) x^{2\alpha+1} dx$$

holds for every Borel function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. We could also take as a definition of $L_t^{(\alpha)}(0)$ the unique continuous increasing process such that

$$(B_t^{(\alpha)})^{2|\alpha|} - 2|\alpha|L_t^{(\alpha)}(0), \quad t \geq 0,$$

is a martingale.

For every $\alpha > 0$, the unique distribution with Laplace transform given by the Mittag-Leffler function (see [4] p. 453)

$$E_\alpha(x) = \sum_{p=0}^\infty \frac{(-x)^p}{\Gamma(p\alpha + 1)}, \quad x \in \mathbb{R}_+$$

is called the *Mittag-Leffler distribution* and denoted by $\mathcal{M}(\alpha)$. Here, $\Gamma(\cdot)$ denotes the usual Gamma function. The p th moment of $\mathcal{M}(\alpha)$ is equal to

$$\frac{p!}{\Gamma(\alpha p + 1)}.$$

The probability density of $\mathcal{M}(\alpha)$ (see [1]) is equal to

$$f(x) = \frac{1}{\pi} \sum_{k=1}^\infty \frac{(-1)^{k-1}}{(k-1)!} \sin(\pi k\alpha) \Gamma(k\alpha) x^{k-1}, \quad x \in]0, +\infty[.$$

In [8, p. 567, Formula (R8)], the distribution of the random variable $L_1^{(\alpha)}(0)$ is given in terms of the Mittag-Leffler distribution, namely

$$\frac{2^{\alpha+1}\Gamma(\alpha + 1)}{\Gamma(|\alpha|)}L_1^{(\alpha)}(0) \stackrel{\text{law}}{=} \mathcal{M}(|\alpha|). \tag{1}$$

3 Random Walk Associated with Gegenbauer Polynomials

3.1 Generalities on Gegenbauer Polynomials

Gegenbauer polynomials, also called *ultraspherical polynomials*, are defined on $[-1, 1]$ for any $\alpha > -1$ by

$$P_n^{(\alpha)}(x) = \frac{(-1)^n}{2^n(\alpha + 1) \cdots (\alpha + n)}(1 - x^2)^{-\alpha} \frac{d^n}{dx^n} (1 - x^2)^{n+\alpha}.$$

They satisfy the following orthogonality relations:

$$\int_{-1}^1 P_n^{(\alpha)}(x)P_m^{(\alpha)}(x) d\pi_\alpha(x) = \begin{cases} 0 & \text{if } n \neq m, \\ (w_n^{(\alpha)})^{-1} & \text{if } n = m, \end{cases} \tag{2}$$

where $d\pi_\alpha(x) = (1 - x^2)^\alpha \mathbf{1}_{[-1,1]}(x) dx$ and $-n \neq 0$:

$$w_n^{(\alpha)} = \frac{(2n + 2\alpha + 1)\Gamma(n + 2\alpha + 1)}{2^{2\alpha+1}\Gamma(n + 1)\Gamma(\alpha + 1)^2}, \tag{3}$$

$-n = 0$:

$$w_0^{(\alpha)} = \begin{cases} \frac{(2\alpha+1)\Gamma(2\alpha+1)}{2^{2\alpha+1}\Gamma(\alpha+1)^2} & \text{if } \alpha \neq -1/2, \\ 1/\pi & \text{otherwise.} \end{cases} \tag{4}$$

These polynomials satisfy the following properties:

$$\begin{aligned} P_0^{(\alpha)}(x) &\equiv 1, & P_1^{(\alpha)}(x) &\equiv x, \\ P_n^{(\alpha)}(-x) &= (-1)^n P_n^{(\alpha)}(x), \\ P_n^{(\alpha)}(1) &= 1, \end{aligned} \tag{5}$$

and the multiplication formula

$$P_1^{(\alpha)}(x)P_n^{(\alpha)}(x) = \frac{n}{2n + 2\alpha + 1} P_{n-1}^{(\alpha)}(x) + \frac{n + 2\alpha + 1}{2n + 2\alpha + 1} P_{n+1}^{(\alpha)}(x) \tag{6}$$

for every $\alpha > -1$, every $n \in \mathbb{N}^*$ and every $x \in [-1, 1]$.

More generally, when $\alpha \geq -\frac{1}{2}$, we have, for $m \leq n$,

$$P_m^{(\alpha)}(x)P_n^{(\alpha)}(x) = \sum_{r=0}^m C^{(\alpha)}(m, n, r)P_{n-m+2r}^{(\alpha)}(x), \tag{7}$$

where the nonnegative coefficients $C^{(\alpha)}(m, n, r)$ are explicit (see [5]).

Finally, when $\alpha > -\frac{1}{2}$, for any $n \geq 1$,

$$|P_n^{(\alpha)}(x)| < 1, \quad \forall x \in]-1, +1[.$$

3.2 Random Walk Associated with Gegenbauer Polynomials

Let $\mathcal{M}_1(\mathbb{N})$ be the set of probability measures $\mu = \sum_{n \in \mathbb{N}} \mu(n)\delta_n$ on \mathbb{N} . Thanks to formula (7), for every $\alpha \in [-1/2, +\infty[$, we can define a generalized convolution denoted \star as follows:

$$\delta_m \star \delta_n = \sum_r C^{(\alpha)}(m, n, r)\delta_{n-m+2r},$$

and more generally, if μ, ν are in $\mathcal{M}_1(\mathbb{N})$,

$$\mu \star \nu = \sum_{m, n \in \mathbb{N}} \mu(n)\nu(m)\delta_n \star \delta_m.$$

For each $x \in \mathbb{N}$ and for each subset A of \mathbb{N} , we can define the transition kernel from \mathbb{N} to \mathbb{N} as

$$P(x, A) = (\delta_x \star \mu)(A).$$

The random walk associated with Gegenbauer polynomials is defined as the Markov chain with state space \mathbb{N} and transition kernel given by P , and will be denoted by $(S_n)_{n \geq 0}$. In the sequel, for the sake of clarity, we will omit in the notation of the Markov chain the index α and the measure μ from which the process is defined.

The probability to be in a subset A of \mathbb{N} at time n when departing from state x at time 0 is then given by

$$P^{(n)}(x, A) = (\delta_x \star \mu^{(n)})(A),$$

with the notation $\mu^{(n)} = \mu \star \dots \star \mu$ (n times).

It is worth noticing that if the distribution μ is the Dirac mass at point 1, then the random walk associated with Gegenbauer polynomials with index α is the birth-and-death Markov chain on \mathbb{N} with transition probabilities given by

$$p(0, 1) = 1$$

and

$$p(i, i + 1) = 1 - p(i, i - 1) = \frac{1}{2} \left(1 + \frac{\lambda}{i + \lambda} \right)$$

where $\lambda = \alpha + \frac{1}{2} \in [0, +\infty[$.

A natural question is to know if a given Markov chain with state space \mathbb{N} corresponds to a random walk associated with Gegenbauer polynomials. This is true if and only if the transition probabilities of the Markov chain satisfy the following relation (see [5])

$$\begin{aligned} & \frac{i}{2(i + \lambda)} p(i - 1, j) + \frac{i + 2\lambda}{2(i + \lambda)} p(i + 1, j) \\ &= \frac{j + 2\lambda - 1}{2(j + \lambda - 1)} p(i, j - 1) + \frac{j + 1}{2(j + \lambda + 1)} p(i, j + 1) \end{aligned}$$

for some $\lambda \in [0, \frac{1}{2}]$.

The distribution μ is then given by $\mu(n) = p(0, n)$.

Finally, we call (*generalized*) *Fourier transform* of $\mu \in \mathcal{M}_1(\mathbb{N})$ the function $\hat{\mu}$ defined on $[0, \pi]$ by

$$\hat{\mu}(\theta) = \sum_{n \in \mathbb{N}} \mu(n) P_n^{(\alpha)}(\cos(\theta)).$$

From orthogonality relations (2), the coefficient $\mu(n)$ of the measure μ can be obtained from $\hat{\mu}$ by the following formula

$$\mu(n) = w_n^{(\alpha)} \int_0^\pi \hat{\mu}(\theta) P_n^{(\alpha)}(\cos(\theta)) \sin^{2\alpha+1}(\theta) d\theta. \tag{8}$$

In particular, $\hat{\delta}_n(\theta) = P_n^{(\alpha)}(\cos(\theta))$ and, thanks to formula (7),

$$\widehat{(\delta_n \star \delta_m)} = \hat{\delta}_n \hat{\delta}_m.$$

More generally, for every $\mu, \nu \in \mathcal{M}_1(\mathbb{N})$,

$$\widehat{(\mu \star \nu)} = \hat{\mu} \hat{\nu}.$$

4 Limit Theorems

In this section, we denote by $(S_n)_{n \geq 0}$ the random walk associated with Gegenbauer polynomials (as defined in Sect. 3.2) with transition kernel given by $\delta_x \star \mu$ for some $\mu \in \mathcal{M}_1(\mathbb{N})$.

4.1 A Functional Central Limit Theorem

Let $\mathcal{D} = \mathcal{D}([0, +\infty[)$ be the space of càdlàg functions on \mathbb{R}_+ endowed with the Skorokhod topology. We denote by $(B_t^{(\alpha)})_{t \in \mathbb{R}_+}$ the Bessel process on \mathbb{R}_+ of index $\alpha \in [-\frac{1}{2}, +\infty[$ defined in Sect. 2.

Theorem 4.1 [10] *Let $\mu \in \mathcal{M}_1(\mathbb{N})$ with a second order moment and*

$$C = \frac{1}{4(\alpha + 1)} \sum_{n=1}^\infty \mu(n) n(n + 2\alpha + 1).$$

The sequence $(\frac{S_{[nt]}}{\sqrt{2Cn}})_{t \in \mathbb{R}_+}$ converges in \mathcal{D} , as $n \rightarrow +\infty$, to the process $(B_t^{(\alpha)})_{t \in \mathbb{R}_+}$.

4.2 A Large Deviation Principle—A Law of Large Numbers

A large deviation principle for polynomial hypergroups was proved by Ehring [2].

Theorem 4.2 [2] *Let $\mu \in \mathcal{M}_1(\mathbb{N})$ with finite support. Then, the sequence of random variables $(\frac{S_n}{n})_{n \geq 1}$ satisfies a large deviation principle of speed n and good rate function*

$$I(x) = \begin{cases} +\infty & \text{if } x \notin [0, x_0], \\ \sup_{\lambda \geq 0} \{\lambda x - \log(\tilde{\mu}(\lambda))\} & \text{if } x \in [0, x_0], \end{cases}$$

where $x_0 = \max\{x \in \mathbb{N} \mid \mu(x) \neq 0\}$ and $\tilde{\mu}$ is the (generalized) Laplace transform of the measure μ (see [2] for the definition).

In [2], $x = 0$ is proved to be the unique infimum point of the function I , then a weak law of large numbers for $(\frac{S_n}{n})_{n \geq 1}$ holds.

4.3 Local Limit Theorems

By using a (generalized) Fourier calculus, a local limit theorem for the random walk associated with Gegenbauer polynomials for any $\alpha \geq -\frac{1}{2}$ was proved in [3].

Theorem 4.3 *Let us assume that μ is aperiodic (i.e. the support of μ is not a subset of $2\mathbb{N}$) with a finite second order moment.*

Then, for every $x, y \in \mathbb{N}$, as $n \rightarrow +\infty$,

$$p^{(n)}(x, y) \sim \frac{w_y^{(\alpha)} \Gamma(\alpha + 1)}{2(Cn)^{\alpha+1}}$$

where

$$C = \frac{1}{4(\alpha + 1)} \sum_{n=1}^{\infty} \mu(n)n(n + 2\alpha + 1).$$

The random walk associated with Gegenbauer polynomials is then

$$\begin{cases} \text{recurrent if } & \alpha \in [-\frac{1}{2}, 0], \\ \text{transient if } & \alpha \in]0, +\infty[. \end{cases}$$

The properties of recurrence/transience of the Markov chain $(S_n)_{n \in \mathbb{N}}$ were established in [9] by computing the potential kernel of the Markov chain. The previous statement can be extended to the case when the measure μ is not aperiodic. We are only interested in the case $\mu = \delta_1$, but generalization to any periodic measure can easily be done.

Proposition 4.1 *Assume that $\mu = \delta_1$, then for any $x, y \in \mathbb{N}$, as $n \rightarrow +\infty$,*

$$p^{(n)}(x, y) \sim \begin{cases} w_y^{(\alpha)} 2^{\alpha+1} \Gamma(\alpha + 1) n^{-(\alpha+1)} & \text{when } n + x + y \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

We now prove a local limit theorem with a new normalization in space following the same lines as the proof of Theorem 4.3 in [3]. Therefore, in the proof, we just stress on points which differ.

Theorem 4.4 *Let us assume that μ is aperiodic with a second order moment. Then, for every $x \in \mathbb{R}_+^*$, as $n \rightarrow +\infty$,*

$$\sqrt{n} p^{(n)}(\lfloor x\sqrt{n} \rfloor, \lfloor x\sqrt{n} \rfloor) \sim \frac{x}{2C} e^{-\frac{x^2}{2C}} I_\alpha \left(\frac{x^2}{2C} \right) \tag{9}$$

and

$$\sqrt{n} p^{(n)}(0, \lfloor x\sqrt{n} \rfloor) \sim \frac{x^{2\alpha+1} e^{-\frac{x^2}{4C}}}{2^{2\alpha+1} C^{\alpha+1} \Gamma(\alpha + 1)} \tag{10}$$

where I_α is the modified Bessel function of index α .

Proof From formula (8),

$$\begin{aligned} & \sqrt{n} p^{(n)}(\lfloor x\sqrt{n} \rfloor, \lfloor x\sqrt{n} \rfloor) \\ &= \sqrt{n} w_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)} \int_0^\pi \hat{\mu}(\theta)^n \left(P_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)}(\cos(\theta)) \sin^{\alpha+\frac{1}{2}}(\theta) \right)^2 d\theta \\ &= w_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)} \int_0^{\pi\sqrt{n}} \hat{\mu}\left(\frac{\theta}{\sqrt{n}}\right)^n \left(P_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)}\left(\cos\left(\frac{\theta}{\sqrt{n}}\right)\right) \sin^{\alpha+\frac{1}{2}}\left(\frac{\theta}{\sqrt{n}}\right) \right)^2 d\theta \end{aligned}$$

using the change of variables $u = \sqrt{n}\theta$.

The right-hand side is then decomposed as the sum of the following integrals:

$$\begin{aligned} I_0(n) &= w_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)} \int_0^\infty \exp(-C\theta^2) \left(P_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)}\left(\cos\left(\frac{\theta}{\sqrt{n}}\right)\right) \sin^{\alpha+\frac{1}{2}}\left(\frac{\theta}{\sqrt{n}}\right) \right)^2 d\theta, \\ I_1(n, A) &= w_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)} \int_0^A \left[\hat{\mu}\left(\frac{\theta}{\sqrt{n}}\right)^n - \exp(-C\theta^2) \right] \\ &\quad \times \left(P_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)}\left(\cos\left(\frac{\theta}{\sqrt{n}}\right)\right) \sin^{\alpha+\frac{1}{2}}\left(\frac{\theta}{\sqrt{n}}\right) \right)^2 d\theta, \\ I_2(n, A) &= -w_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)} \int_A^{+\infty} \exp(-C\theta^2) \\ &\quad \times \left(P_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)}\left(\cos\left(\frac{\theta}{\sqrt{n}}\right)\right) \sin^{\alpha+\frac{1}{2}}\left(\frac{\theta}{\sqrt{n}}\right) \right)^2 d\theta, \\ I_3(n, A, r) &= w_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)} \int_A^{r\sqrt{n}} \hat{\mu}\left(\frac{\theta}{\sqrt{n}}\right)^n \left(P_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)}\left(\cos\left(\frac{\theta}{\sqrt{n}}\right)\right) \sin^{\alpha+\frac{1}{2}}\left(\frac{\theta}{\sqrt{n}}\right) \right)^2 d\theta, \\ I_4(n, r) &= w_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)} \int_{r\sqrt{n}}^{\pi\sqrt{n}} \hat{\mu}\left(\frac{\theta}{\sqrt{n}}\right)^n \left(P_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)}\left(\cos\left(\frac{\theta}{\sqrt{n}}\right)\right) \sin^{\alpha+\frac{1}{2}}\left(\frac{\theta}{\sqrt{n}}\right) \right)^2 d\theta. \end{aligned}$$

We only give the way of estimating $I_0(n)$ for n large. The integrals $I_j, j = 1, \dots, 4$, can be proved to be negligible as in the proof of Theorem 4.3. From the definition of the $w_x^{(\alpha)}$'s, we easily deduce that as $n \rightarrow +\infty$,

$$w_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)} \sim \frac{x^{2\alpha+1}n^{\alpha+\frac{1}{2}}}{2^{2\alpha}\Gamma(\alpha+1)^2}.$$

Moreover, from formula 8.21.12 in [12], we get

$$\lim_{n \rightarrow +\infty} P_{\lfloor x\sqrt{n} \rfloor}^{(\alpha)} \left(\cos \left(\frac{\theta}{\sqrt{n}} \right) \right) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(\theta x)}{\theta^\alpha x^\alpha}$$

where J_α is the Bessel function of index α , which yields (9) from the dominated convergence theorem, by remarking that

$$\int_{\mathbb{R}_+} e^{-C\theta^2} (\theta x) J_\alpha^2(\theta x) d\theta = \frac{x}{2C} e^{-x^2/2C} I_\alpha \left(\frac{x^2}{2C} \right).$$

We obtain (10) by remarking that we have

$$\int_{\mathbb{R}_+} e^{-C\theta^2} (\theta x)^{\alpha+1} J_\alpha(\theta x) d\theta = \frac{x^{2\alpha+1}}{(2C)^{\alpha+1}} e^{-\frac{x^2}{4C}}. \quad \square$$

5 Limit Distribution of the Local Time

The local time $(N_n(x))_{n \geq 0; x \in \mathbb{N}}$ of the Markov chain $(S_n)_{n \in \mathbb{N}}$ defined in Sect. 3.2 is equal to the number of times the chain visits the site x up to time n , namely

$$N_n(x) = \sum_{k=0}^n \mathbf{1}_{\{S_k=x\}}.$$

For every $x \in \mathbb{N}$, we denote by \mathbb{P}_x the distribution of the Markov chain $(S_n)_{n \geq 0}$ starting from x and by \mathbb{E}_x the corresponding expectation. We prove in the case $\alpha \in [-\frac{1}{2}, 0]$ the following limit theorem for the local time $(N_n(x))_{n \geq 0; x \in \mathbb{N}}$.

Theorem 5.1 *Assume that μ is aperiodic with a finite second order moment.*

- When $\alpha \in [-\frac{1}{2}, 0[$, for every $x, y \in \mathbb{N}$, under \mathbb{P}_x ,

$$\frac{N_n(y)}{n^{|\alpha|}} \xrightarrow{\mathcal{L}} \frac{w_y^{(\alpha)} \Gamma(\alpha+1) \Gamma(|\alpha|)}{2C^{\alpha+1}} \mathcal{M}(|\alpha|) \tag{11}$$

where the $w_x^{(\alpha)}$'s are defined in formulae (3) and (4).

- When $\alpha = 0$, for every $x, y \in \mathbb{N}$, under \mathbb{P}_x ,

$$\frac{N_n(y)}{\log n} \xrightarrow{\mathcal{L}} \frac{(2y+1)}{4C} \mathcal{E}(1)$$

where $\mathcal{E}(1)$ denotes the exponential distribution with parameter one.

Remark From (1), the limit distribution in formula (11) is equal to the law of the random variable

$$\frac{(2y + 2\alpha + 1)\Gamma(y + 2\alpha + 1)}{\Gamma(y + 1)(2C)^{\alpha+1}} L_1^{(\alpha)}(0)$$

where $L_1^{(\alpha)}(0)$ denotes the local time at 0 of the Bessel process with index α .

Proof Assume that $\alpha \in [-\frac{1}{2}, 0[$. For every $x, y \in \mathbb{N}$, we denote by $F_{x,y}$ the generating function of the sequence $(p^{(n)}(x, y))_{n \geq 0}$, namely for every $\lambda \in [0, 1[$,

$$F_{x,y}(\lambda) = \sum_{n=0}^{\infty} \lambda^n p^{(n)}(x, y).$$

From Theorem 4.3, for every $\varepsilon > 0$, there exists n_0 such that for every $n \geq n_0$,

$$\frac{(1 - \varepsilon)w_y^{(\alpha)}\Gamma(\alpha + 1)}{2(Cn)^{\alpha+1}} \leq p^{(n)}(x, y) \leq \frac{(1 + \varepsilon)w_y^{(\alpha)}\Gamma(\alpha + 1)}{2(Cn)^{\alpha+1}}.$$

From Tauberian theorem for power series (see Feller [4], p. 447), we deduce that, as $\lambda \rightarrow 1^-$,

$$F_{x,y}(\lambda) \sim \frac{w_y^{(\alpha)}\Gamma(\alpha + 1)\Gamma(|\alpha|)}{2C^{(\alpha+1)}(1 - \lambda)^{|\alpha|}}. \tag{12}$$

Let $p \geq 1$, by combining all permutations of the same indices j_1, \dots, j_p , we have

$$\begin{aligned} \mathbb{E}_x(N_n(y)^p) &= \sum_{0 \leq j_1, \dots, j_p \leq n} \mathbb{P}_x(S_{j_1} = \dots = S_{j_p} = y) \\ &= p! \sum_{0 \leq j_1 \leq \dots \leq j_p \leq n} p^{(j_1)}(x, y) p^{(j_2 - j_1)}(y, y) \dots p^{(j_p - j_{p-1})}(y, y) \\ &\quad + R_n. \end{aligned} \tag{13}$$

The remainder term R_n contains the sums over the q -tuples $(j_1, \dots, j_q) \in \{0, \dots, n\}^q$ with $q < p$. From Theorem 4.3, we deduce that $R_n = \mathcal{O}(n^{|\alpha|q}) = o(n^{|\alpha|p})$, so it will be negligible in the limit.

We denote by $m_n := m_n(x, p)$ the first sum in the right-hand side of (13) and by G the generating function of the sequence $(m_n)_{n \geq 0}$, that is, for every $\lambda \in [0, 1[$,

$$G(\lambda) = \sum_{n=0}^{\infty} \lambda^n m_n$$

which can be rewritten as

$$\begin{aligned}
 G(\lambda) &= p! \sum_{n=0}^{\infty} \sum_{0 \leq j_1 \leq \dots \leq j_p \leq n} (\lambda^{j_1} p^{(j_1)}(x, y)) (\lambda^{j_2 - j_1} p^{(j_2 - j_1)}(y, y)) \dots \\
 &\quad \times (\lambda^{j_p - j_{p-1}} p^{(j_p - j_{p-1})}(y, y)) \lambda^{n - j_p} \\
 &= p! \sum_{n=0}^{\infty} \sum_{m_1 + \dots + m_{p+1} = n; m_i \geq 0} (\lambda^{m_1} p^{(m_1)}(x, y)) (\lambda^{m_2} p^{(m_2)}(y, y)) \dots \\
 &\quad \times (\lambda^{m_p} p^{(m_p)}(y, y)) \lambda^{m_{p+1}} \\
 &= p! \left(\frac{1}{1 - \lambda} \right) F_{x,y}(\lambda) (F_{y,y}(\lambda))^{p-1}.
 \end{aligned}$$

From (12), we deduce that, as $\lambda \rightarrow 1^-$,

$$G(\lambda) \sim p! \frac{(w_y^{(\alpha)} \Gamma(\alpha + 1) \Gamma(|\alpha|))^p}{2^p C^{p(\alpha+1)} (1 - \lambda)^{|\alpha|p+1}}. \tag{14}$$

Then, from Tauberian theorem for power series (see Feller [4], p. 447), we get as $n \rightarrow +\infty$,

$$\mathbb{E}_x \left(\left(\frac{N_n(y)}{n^{|\alpha|}} \right)^p \right) \sim \frac{p!}{\Gamma(|\alpha|p + 1)} \left(\frac{w_y^{(\alpha)} \Gamma(\alpha + 1) \Gamma(|\alpha|)}{2C^{\alpha+1}} \right)^p =: \beta_p.$$

The Carleman condition

$$\sum_{p=1}^{+\infty} \frac{1}{\beta_{2p}^{1/2p}} = +\infty$$

being satisfied, the limit distribution is uniquely determined and the weak convergence is proved. We characterize the limit distribution by recognizing the moments of the Mittag-Leffler distribution $\mathcal{M}(|\alpha|)$ (see Sect. 2).

The proof in the case $\alpha = 0$ is similar and is omitted. □

When $\mu = \delta_1$, the random walks associated with Gegenbauer polynomials are the birth-and-death Markov chains on \mathbb{N} with transition probabilities $(p(i, j))_{i,j \in \mathbb{N}}$ given by $p(0, 1) = 1$ and

$$p(i, i + 1) = \frac{i + 2\alpha + 1}{2i + 2\alpha + 1}; \quad p(i, i - 1) = \frac{i}{2i + 2\alpha + 1}.$$

When $\alpha \in [-1/2, 0]$, the Markov chain is positive recurrent; we still denote by $(N_n(x))_{n \in \mathbb{N}; x \in \mathbb{N}}$ its local time. Thanks to Proposition 4.1, we can adapt the proof of the previous theorem to provide a complete description of the limit behavior of these local times. When $\alpha = -1/2$, the Markov chain corresponds to the simple random walk on \mathbb{N} with reflection at 0. The mean number of times the Markov chain visits 0 is asymptotically equal to \sqrt{n} . The random walk associated with Gegenbauer polynomials with index $\alpha = 0$ is the birth-and-death Markov chain on \mathbb{N} with transition

probabilities given by

$$p(0, 1) = 1$$

and

$$p(i, i + 1) = \frac{i + 1}{2i + 1}, \quad p(i, i - 1) = \frac{i}{2i + 1}.$$

In that case, the mean number of times the Markov chain visits 0 is asymptotically equal to $\frac{1}{2} \log(n)$. More precisely, we have

Proposition 5.1

- When $\alpha \in [-\frac{1}{2}, 0[$, for every $x, y \in \mathbb{N}$, under the measure \mathbb{P}_x ,

$$\frac{N_n(y)}{n^{|\alpha|}} \xrightarrow{\mathcal{L}} \frac{(2y + 2\alpha + 1)\Gamma(y + 2\alpha + 1)\Gamma(|\alpha|)}{2^{\alpha+1}\Gamma(y + 1)\Gamma(\alpha + 1)} \mathcal{M}(|\alpha|) \quad (15)$$

(with the convention $0 \times \Gamma(0) = 1$).

- When $\alpha = 0$, for every $x, y \in \mathbb{N}$, under the measure \mathbb{P}_x ,

$$\frac{N_n(y)}{\log(n)} \xrightarrow{\mathcal{L}} \mathcal{E}\left(\frac{2}{2y + 1}\right). \quad (16)$$

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References

1. Blumenfeld, R., Mandelbrot, B.B.: Lévy dusts, Mittag-Leffler statistics, mass fractal lacunarity, and perceived dimension. *Phys. Rev. E* **56**(1), 112–118 (1997)
2. Ehring, M.: A large deviation principle for polynomial hypergroups. *J. Lond. Math. Soc. (2)* **53**, 197–208 (1996)
3. Ehring, M.: Large deviations and local limit theorems for Markov chains on one-dimensional hypergroups. (Grosse Abweichungen und lokale Grenzwertsätze für Markov-Ketten auf eindimensionalen Hypergruppen.) PhD Thesis (German), München: Tech. Univ. München (1994)
4. Feller, W.: *An Introduction to Probability Theory and Its Applications*, vol. II, 2nd edn. (1970)
5. Gallardo, L.: Comportement asymptotique des marches aléatoires associées aux polynômes de Gegenbauer et applications. *Adv. Appl. Probab.* **16**, 293–323 (1984)
6. Gallardo, L., Ries, V.: La loi des grands nombres pour les marches aléatoires sur le dual de $SU(2)$. *Studia Math.* **66**, 93–105 (1979)
7. Gallardo, L.: Chaînes de Markov à dérive stable et loi des grands nombres sur les hypergroupes. *Ann. Inst. Henri Poincaré Probab. Stat.* **32**(6), 701–723 (1996)
8. Gradinaru, M., Roynette, B., Vallois, P., Yor, M.: Abel transform and integrals of Bessel local times. *Ann. Inst. Henri Poincaré (B) Probab. Stat.* **35**(4), 531–572 (1999)
9. Guivarc’h, Y., Keane, M., Roynette, B.: *Marches aléatoires sur les groupes de Lie*. *Lect. Notes Math.*, vol. 624. Springer, Berlin (1977)
10. Mabrouki, M.: Principe d’invariance pour les marches aléatoires associées aux polynômes de Gegenbauer et applications. *C.R.A.S.*, t. 299, Série I, no. 19 (1984)
11. Revuz, D., Yor, M.: *Continuous Martingales and Brownian Motion*, 3rd edn. Springer, Berlin (1999)
12. Szego, G.: *Orthogonal polynomials*. *Am. Math. Soc. Colloq. Publ.* (1939)
13. Voit, M.: Central limit theorems for a class of polynomial hypergroups. *Adv. Appl. Probab.* **22**, 68–87 (1990)

14. Voit, M.: Central limit theorems for Markov processes associated with Laguerre polynomials. *J. Math. Anal. Appl.* **182**(3), 731–741 (1994)
15. Voit, M.: A law of the iterated logarithm for Markov chains on \mathbb{N}_0 associated with orthogonal polynomials. *J. Theor. Probab.* **6**(4), 653–669 (1993)
16. Voit, M.: Laws of large numbers for polynomial hypergroups and some applications. *J. Theor. Probab.* **3**(2), 245–266 (1990)