Lie analogues of Loday algebras and successors of operads

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Outline

1. A brief introduction to Loday algebras and pre-Lie algebras
2. L-dendriform algebras
3. Lie analogues of Loday algebras
4. Successors of operads and Manin black products

- **Dendriform algebras and pre-Lie algebras**

Definition

A *dendriform algebra* $(A, \prec, \succ)$ is a vector space A with two binary operations denoted by $\prec$ and $\succ$ satisfying (for any $x, y, z \in A$)

$$(x \prec y) \prec z = x \prec (y * z), \tag{1}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{2}$$

$$x \succ (y \succ z) = (x * y) \succ z, \tag{3}$$

where $x * y = x \prec y + x \succ y$.

Definition

A *pre-Lie algebra* A is a vector space with a binary operation $(x, y) \to xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \quad \forall x, y, z \in A. \tag{4}$$

- Other names:

1. left-symmetric algebra
2. Koszul-Vinberg algebra
3. quasi-associative algebra
4. right-symmetric algebra
5. $\cdots$

Proposition

Let A be a pre-Lie algebra.

1. *The commutator*

$$[x,y]=xy-yx,\ \ \forall x,y\in A \tag{5}$$

   defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of A.

2. *For any $x,y\in A$, let $L(x)$ denote the left multiplication operator. Then Eq. (4) is just*

$$[L(x),L(y)]=L([x,y]),\ \ \forall x,y\in A, \tag{6}$$

   which means that $L:\mathfrak{g}(A)\to gl(\mathfrak{g}(A))$ with $x\to L(x)$ gives a representation of the Lie algebra $\mathfrak{g}(A)$.

Proposition

Let $(A, \prec, \succ)$ *be a dendriform algebra.*

1. *The binary operation* $\bullet : A \otimes A \to A$ *given by*
$$x * y = x \prec y + x \succ y, \forall x, y \in A, \tag{7}$$
defines an associative algebra.

2. *The binary operation* $\circ : A \otimes A \to A$ *given by*
$$x \circ y = x \succ y - y \prec x, \forall x, y \in A, \tag{8}$$
defines a pre-Lie algebra.

3. *Both* $(A, *)$ *and* $(A, \circ)$ *have the same sub-adjacent Lie algebra* $\mathfrak{g}(A)$ *defined by*
$$[x, y] = x \succ y + x \prec y - y \succ x - y \prec x, \forall x, y \in A. \tag{9}$$

Relationship among Lie algebras, associative algebras, pre-Lie algebras and dendriform algebras in the sense of commutative diagram of categories:

$$\begin{array}{ccc} \text{Lie algebra} & \leftarrow & \text{Pre-Lie algebra} \\ \uparrow & & \uparrow \\ \text{Associative algebra} & \leftarrow & \text{Dendriform algebra} \end{array} \tag{10}$$

◦ **Loday algebras**

Similar algebra structures have a common property of "splitting associativity", that is, expressing the multiplication of an associative algebra as the sum of a string of binary operations. Explicitly, let $(X, *)$ be an associative algebra over a field $\mathbb{F}$ of characteristic zero and $(*_i)_{1\leq i\leq N} : X \otimes X \to X$ be a family of binary operations on X. Then the operation $*$ splits into the N operations $*_1, \cdots, *_N$ if

$$x * y = \sum_{i=1}^{N} x *_i y, \quad \forall x, y \in X. \tag{11}$$

1. $N = 2$: dendriform (di)algebra;
2. $N = 3$: dendriform trialgebra;
3. $N = 4$: quadri-algebra;
4. $N = 8$: octo-algebra;
5. $N = 9$: ennea-algebra;
6. $\cdots$

We pay our main attention to the case that $N = 2^n$, $n = 0, 1, 2, \cdots$.

The "rule" of constructing Loday algebras

1. Operation axioms can be summarized to be a set of "associativity" relations;
2. By induction, for the algebra $(A, *_i)_{1\leq i\leq 2^n}$, besides the natural (regular) module of A on the underlying vector space of A itself given by the left and right multiplication operators, one can introduce the 2^{n+1} operations $\{*_{i_1}, *_{i_2}\}_{1\leq i\leq 2^n}$ such that

$$x *_i y = x *_{i_1} y + x *_{i_2} y, \ \forall x, y \in A, \ 1 \leq i \leq 2^n, \tag{12}$$

and their left and right multiplication operators can give a module of $(A, *_i)_{1\leq i\leq 2^n}$ by acting on the underlying vector space of A itself.

- **Example: quadri-algebras**

Definition

Let A be a vector space with four binary operations denoted by $\searrow, \nearrow, \nwarrow$ and $\swarrow$: $A \otimes A \to A$. $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is called a *quadri-algebra* if for any $x, y, z \in A$,

$$(x \nwarrow y) \nwarrow z = x \nwarrow (y * z), (x \nearrow y) \nwarrow z = x \nearrow (y \prec z), \quad (13)$$

$$(x \wedge y) \nearrow z = x \nearrow (y \succ z), (x \swarrow y) \nwarrow z = x \swarrow (y \wedge z), \quad (14)$$

$$(x \searrow y) \nwarrow z = x \searrow (y \nwarrow z), (x \vee y) \nearrow z = x \searrow (y \nearrow z), \quad (15)$$

$$(x \prec y) \swarrow z = x \swarrow (y \vee z), \quad (x \succ y) \swarrow z = x \searrow (y \swarrow z), \quad (16)$$

$$(x * y) \searrow z = x \searrow (y \searrow z), \quad (17)$$

Definition

(Continued) where

$$x \succ y = x \nearrow y + x \searrow y, x \prec y = x \nwarrow y + x \swarrow y, \tag{18}$$

$$x \vee y = x \searrow y + x \swarrow y, x \wedge y = x \nearrow y + x \nwarrow y, \tag{19}$$

$$x * y = x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y = x \succ y + x \prec y = x \vee y + x \wedge y. \tag{20}$$

◦ **Motivations to introduce "L-dendriform" algebra**

Motivation (I) Extension of the diagram (10)

$$\begin{tikzcd}
\text{Lie} & \text{Pre-Lie} \arrow[l] & \text{L-dendriform} \arrow[l] & \\
& \text{Associative} \arrow[ul] & \text{Dendriform} \arrow[l] \arrow[ul] & \text{Quadri-} \arrow[l] \arrow[ul]
\end{tikzcd} \tag{21}$$

Application: An interpretation of the relationship between pre-Lie algebras and dendriform algebras in terms of the relationship between pre-Lie algebras and L-dendriform algebras.

Motivation (II) The underlying algebraic structure of a pseudo-Hessian structure on a Lie group

Motivation (III) Algebraic structures behind the $\mathcal{O}$-operators of pre-Lie algebras and the related S-equation

◦ **L-dendriform algebras**

Definition

Let A be a vector space with two binary operations denoted by $\triangleright$ and $\triangleleft : A \otimes A \to A$. $(A, \triangleright, \triangleleft)$ is called an *L-dendriform algebra* if for any $x, y, z \in A$,

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright z + (x \triangleleft y) \triangleright z + y \triangleright (x \triangleright z) - (y \triangleleft x) \triangleright z - (y \triangleright x) \triangleright z, \quad (22)$$

$$x \triangleright (y \triangleleft z) = (x \triangleright y) \triangleleft z + y \triangleleft (x \triangleright z) + y \triangleleft (x \triangleleft z) - (y \triangleleft x) \triangleleft z. \quad (23)$$

L-dendriform algebras

Proposition

Let $(A, \triangleright, \triangleleft)$ *be an L-dendriform algebra.*

1. *The binary operation* $\bullet : A \otimes A \to A$ *given by*

$$x \bullet y = x \triangleright y + x \triangleleft y, \forall x, y \in A, \tag{24}$$

   defines a (horizontal) pre-Lie algebra.

2. *The binary operation* $\circ : A \otimes A \to A$ *given by*

$$x \circ y = x \triangleright y - y \triangleleft x, \forall x, y \in A, \tag{25}$$

   defines a (vertical) pre-Lie algebra.

3. *Both* $(A, \bullet)$ *and* $(A, \circ)$ *have the same sub-adjacent Lie algebra* $\mathfrak{g}(A)$ *defined by*

$$[x, y] = x \triangleright y + x \triangleleft y - y \triangleright x - y \triangleleft x, \forall x, y \in A. \tag{26}$$

Remark

Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra. Then Eqs. (22) and (23) can be rewritten as (for any $x, y, z \in A$)

$$x \triangleright (y \triangleright z) - (x \bullet y) \triangleright z = y \triangleright (x \triangleright z) - (y \bullet x) \triangleright z, \tag{27}$$

$$x \triangleright (y \triangleleft z) - (x \triangleright y) \triangleleft z = y \triangleleft (x \bullet z) - (y \triangleleft x) \triangleleft z. \tag{28}$$

The both sides of the above two equations can be regarded as a kind of "generalized associators". In this sense, Eqs. (27) and (28) express certain "generalized left-symmetry" of the "generalized associators".

Definition

Let $(A, \circ)$ be a pre-Lie algebra and V be a vector space. Let $l, r : A \to gl(V)$ be two linear maps. (l, r, V) is called a *module of* $(A, \circ)$ if

$$l(x)l(y) - l(x \circ y) = l(y)l(x) - l(y \circ x), \tag{29}$$

$$l(x)r(y) - r(y)l(x) = r(x \circ y) - r(y)r(x), \forall x, y \in A. \tag{30}$$

The "rule" of introducing the notion of L-dendriform algebra

Proposition

Let A be a vector space with two binary operations denoted by $\triangleright, \triangleleft : A \otimes A \to A$.

1. $(A, \triangleright, \triangleleft)$ *is an L-dendriform algebra if and only if* $(A, \bullet)$ *defined by Eq. (24) is a pre-Lie algebra and* $(L_\triangleright, R_\triangleleft, A)$ *is a module.*
2. $(A, \triangleright, \triangleleft)$ *is an L-dendriform algebra if and only if* $(A, \circ)$ *defined by Eq. (25) is a pre-Lie algebra and* $(L_\triangleright, -L_\triangleleft, A)$ *is a module.*

Relationship between dendriform algebras and L-dendriform algebras

Proposition

Any dendriform algebra $(A, \succ, \prec)$ *is an L-dendriform algebra by letting* $x \triangleright y = x \succ y, x \triangleleft y = x \prec y$.

Proof.

In fact, for a dendriform algebra, both sides of Eqs. (27) and (28) which are the equivalent identities of an L-dendriform algebra, are zero. □

Remark

In the above sense, associative algebras are the special pre-Lie algebras whose associators are zero, whereas dendriform algebras are the special L-dendriform algebras whose "generalized associators" are zero.

Proposition

Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ *be a quadri-algebra.*

1. *The binary operations given by Eq. (18) define a dendriform algebra* $(A, \succ, \prec)$.
2. *The binary operations given by Eq. (19) define a dendriform algebra* $(A, \vee, \wedge)$.
3. *The binary operations given by*

$$x \triangleright y = x \searrow y - y \nwarrow x, x \triangleleft y = x \nearrow y - y \swarrow x, \quad \forall x, y \in A, \tag{31}$$

define an L-dendriform algebra $(A, \triangleright, \triangleleft)$.

Corollary

Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ *be a quadri-algebra.*

1. *The binary operation given by*

$$x \circ y = x \searrow y + x \swarrow y - y \nwarrow x - y \nearrow x = x \rhd y - y \lhd x = x \vee y - y \wedge x, \tag{32}$$

defines a pre-Lie algebra $(A, \circ)$.

2. *Eq. (20) defines an associative algebra* $(A, *)$.

3. *The binary operation given by*

$$x \bullet y = x \searrow y + x \nearrow y - y \nwarrow x - y \swarrow x = x \rhd y + x \lhd y = x \succ y - y \prec x, \tag{33}$$

defines a pre-Lie algebra $(A, \bullet)$.

4. *The binary operation given by*

$$[x, y] = x \searrow y + x \swarrow y + x \nearrow y + x \nwarrow y - (y \searrow x + y \swarrow x + y \nearrow x + y \nwarrow x), \tag{34}$$

defines a Lie algebra $(\mathfrak{g}(A), [,])$.

Commutative diagram:

$$\begin{array}{ccccccccc}
\text{Lie} & \overset{-}{\leftarrow} & \text{Pre-Lie} & \overset{-,+}{\leftarrow} & \text{L-dendriform} & & \\
 & \overset{-}{\nwarrow} & \Uparrow\in & \overset{-}{\nwarrow} & \Uparrow\in & \overset{-}{\nwarrow} & \\
 & & \text{Associative} & \overset{+}{\leftarrow} & \text{Dendriform} & \overset{+}{\leftarrow} & \text{Quadri}
\end{array} \qquad (35)$$

where " $\Uparrow\in$" means the inclusion. "$+$" means the binary operation $x \circ_1 y + x \circ_2 y$ and "$-$" means the binary operation $x \circ_1 y - y \circ_2 x$.

Lie analogues of Loday algebras

Let $(X,[,])$ be a Lie algebra and $(*_i)_{1\leq i\leq N}: X\otimes X\to X$ be a family of binary operations on X. Then the Lie bracket $[,]$ splits into the commutator of N binary operations $*_1,\cdots,*_N$ if

$$[x,y]=\sum_{i=1}^{N}(x*_i y-y*_i x),\ \ \forall x,y\in X. \tag{36}$$

We pay our main attention to the case that $N=2^n$, $n=0,1,2,\cdots$.

"Rule" of construction

A "rule" of constructing the binary operations $*_i$ as follows: the 2^{n+1} binary operations give a natural module structure of an algebra with the 2^n binary operations on the underlying vector space of the algebra itself, which is the beauty of such algebra structures. That is, by induction, for the algebra $(A, *_i)_{1\leq i\leq 2^n}$, besides the natural module of A on the underlying vector space of A itself given by the left and right multiplication operators, one can introduce the 2^{n+1} binary operations $\{*_{i_1}, *_{i_2}\}_{1\leq i\leq 2^n}$ such that

$$x *_i y = x *_{i_1} y - y *_{i_2} x,\ \ \forall x, y \in A,\ 1 \leq i \leq 2^n, \tag{37}$$

and their left or right multiplication operators give a module of $(A, *_i)_{1\leq i\leq 2^n}$ by acting on the underlying vector space of A itself.

1. When $N = 1$, the corresponding algebra $(A, *_i)_{1\le i\le N}$ is a pre-Lie algebra;
2. When $N = 2$, the corresponding algebra $(A, *_i)_{1\le i\le N}$ is an L-dendriform algebra.

Remark

Note that for $n \ge 1$ $(N \ge 2)$, in order to make Eq. (36) be satisfied, there is an alternative (sum) form of Eq. (37)

$$x *_i y = x *_{i_1} y + x *'_{i_2} y, \;\; \forall x, y \in A, \; 1 \le i \le 2^n, \tag{38}$$

by letting $x *'_{i_2} y = -y *_{i_2} x$ for any $x, y \in A$. In particular, in such a situation, it can be regarded as a binary operation $*$ of a pre-Lie algebra that splits into the $N = 2^n$ $(n = 1, 2\cdots)$ binary operations $*_1, ..., *_N$.

$N = 3$: **L-quadri-algebra**

Definition

Let A be a vector space with four bilinear products $\searrow, \nearrow, \nwarrow, \swarrow : A \otimes A \to A$. $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is called an *L-quadri-algebra* if for any $x, y, z \in A$,

$$x \searrow (y \searrow z) - (x * y) \searrow z = y \searrow (x \searrow z) - (y * x) \searrow z, \quad (39)$$

$$x \searrow (y \nearrow z) - (x \vee y) \nearrow z = y \nearrow (x \succ z) - (y \wedge x) \nearrow z, \quad (40)$$

$$x \searrow (y \nwarrow z) - (x \searrow y) \nwarrow z = y \nwarrow (x * z) - (y \nwarrow x) \nwarrow z, \quad (41)$$

$$x \nearrow (y \prec z) - (x \nearrow y) \nwarrow z = y \swarrow (x \wedge z) - (y \swarrow x) \nwarrow z, \quad (42)$$

$$x \searrow (y \swarrow z) - (x \succ y) \swarrow z = y \swarrow (x \vee z) - (y \prec x) \swarrow z, \quad (43)$$

Definition

(Continued) where

$$x \succ y = x \searrow y + x \nearrow y, x \prec y = x \nwarrow y + x \swarrow y, \tag{44}$$

$$x \vee y = x \searrow y + x \swarrow y, x \wedge y = x \nearrow y + x \nwarrow y, \tag{45}$$

$$x * y = x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y = x \succ y + x \prec y = x \vee y + x \wedge y. \tag{46}$$

Commutative diagram

$$
\begin{tikzcd}
\text{Lie} & \text{Pre-Lie} \arrow[l] & \text{L-dendriform} \arrow[l] & \text{L-quadri} \arrow[l] & \\
& \text{Associative} \arrow[ul] & \text{Dendriform} \arrow[l] \arrow[ul] & \text{Quadri} \arrow[l] \arrow[ul] & \text{Octo} \arrow[l] \arrow[ul]
\end{tikzcd}
\tag{47}
$$

Chengming Bai, Ligong Liu, Xiang Ni, Some results on L-dendriform algebras, Journal of Geometry and Physics (2010) 60:940-950.

Ligong Liu, Xiang Ni, Chengming Bai, L-quadri-algebras (in Chinese), Sci. Sin. Math. (2011) 42: 105-124.

Successors of operads

Definition

Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary algebraic operad on the $\mathbb{S}_2$-module $V = V(2)$, concentrated in arity 2 with a $\mathbf{k}[\mathbb{S}_2]$-basis $\mathcal{V}$, such that R is spanned, as an $\mathbb{S}_2$-module, by locally homogeneous elements of the form

$$R := \left\{ r_s := \sum_i c_{s,i}\tau_{s,i} \middle| c_{s,i} \in \mathbf{k}, \tau_{s,i} \in \{t(\mathcal{V}), t \in \mathfrak{R}\}, 1 \leq s \leq k, k \geq 1 \right\} \tag{48}$$

where $\mathfrak{R}$ is a set of representatives of $\mathcal{J}/\sim$. The **bisuccessor** of $\mathcal{P}$ is defined to be the binary algebraic operad $\mathrm{BSu}(\mathcal{P}) = \mathcal{T}(\ \widetilde{V}\ )/(\mathrm{BSu}(R))$ where the $\mathbb{S}_2$-action on $\widetilde{V}$ is given by

Definition

(continued)

$$\binom{\omega}{\prec}^{(12)} := \binom{\omega^{(12)}}{\succ}, \quad \binom{\omega}{\succ}^{(12)} := \binom{\omega^{(12)}}{\prec}, \; \omega \in V, \tag{49}$$

and the space of relations is generated, as an $\mathbb{S}_2$-module, by

$$\mathrm{BSu}(R) := \left\{ \mathrm{Su}_x(r_s) := \sum_i c_{s,i} \mathrm{Su}_x(t_{s,i}) \;\middle|\; x \in Lin(t_{s,i}),\ 1 \leq s \leq k \right\} \tag{50}$$

Successors of operads and Manin black products

The notion of the **bisuccessor** $\mathrm{BSu}(\mathcal{P})$ of a binary operad $\mathcal{P}$ generalizing the relationships among Loday algebras (and their Lie analogues) in the operadic sense.

Example

Let $\mathcal{A}s, Dend, Quad$ be the operad of associative, dendriform and quadri-algebras respectively. Then

$$\mathrm{BSu}(\mathcal{A}s) = \mathcal{D}end, \ \ \mathrm{BSu}(\mathcal{D}end) = \mathcal{Q}uad.$$

$$\mathrm{BSu}(\mathcal{L}ie) = \mathcal{P}reLie, \ \ \mathrm{BSu}(\mathcal{P}reLie) = \mathcal{L}Dend,$$

$$\mathrm{BSu}(\mathcal{L}Dend) = \mathcal{L}Quad.$$

Chengming Bai, Olivia Bellier, Li Guo and Xiang Ni, Splitting of operations, Manin products and Rota-Baxter operators, arXiv:1106.6080 .

Manin black products

Definition

Let $\mathcal{P} = \mathcal{F}(V)/(R)$ and $\mathcal{Q} = \mathcal{F}(W)/(S)$ be two binary quadratic operads with finite-dimensional generating spaces. Define their **Manin black product** by the formula

$$\mathcal{P} \bullet \mathcal{Q} := \mathcal{F}(V \otimes W \otimes \mathbf{k}.\mathrm{sgn}_{\mathbb{S}_2})/(\Psi(R \otimes S)). \tag{51}$$

V. Ginzburg and M. M. Kapranov, Koszul duality for operads, Duke Math. J. 76 (1995), 203-272.

B. Vallette, Manin products, Koszul duality, Loday algebras and Deligne conjecture, J. Reine Angew. Math. 620 (2008) 105-164.

Notation

1. Let V be a (left) $\mathbb{S}_2$-module. The free operad $\mathcal{F}(V)$ on V is given by the $\mathbf{k}$-vector space spanned by binary trees with vertices indexed by elements of V, together with an action of the symmetric groups.
2. Ψ is a $\mathbb{S}_3$-module homomorphism from $\mathcal{F}(V)(3) \otimes \mathcal{F}(W)(3) \otimes \mathbf{k}.\mathrm{sgn}_{\mathbb{S}_3}$ to $\mathcal{F}(V \otimes W \otimes \mathbf{k}.\mathrm{sgn}_{\mathbb{S}_2})(3)$ satisfying certain conditions.

Roughly speaking, Manin black product provides a method of constructing a new operad from two operads.

Example

The operad $\mathcal{L}ie$, of Lie algebras is the neutral element for $\bullet$. That is, for any binary quadratic operad $\mathcal{P}$, we know that

$$\mathcal{P} = \mathcal{L}ie \bullet \mathcal{P} = \mathcal{P} \bullet \mathcal{L}ie. \tag{52}$$

Proposition

(Vallete, Bellier) Let $\mathcal{P}reLie$, $\mathcal{L}Dend$ and $\mathcal{L}Quad$ be the operad of pre-Lie, L-dendriform and L-quadri-algebras respectively. Then

$$\mathcal{L}Dend = \mathcal{P}reLie \bullet \mathcal{P}reLie. \tag{53}$$

$$\mathcal{L}Quad = \mathcal{P}reLie \bullet \mathcal{L}Dend = \mathcal{P}reLie \bullet \mathcal{P}reLie \bullet \mathcal{P}reLie. \tag{54}$$

Remark

The above result is listed partly as one of the conjectures given by Loday in "Some problems in operad theory", arXiv: 1109.3290 (Problem 6).

Theorem

Let $\mathcal{P}$ be a binary quadratic operad. Then we have the isomorphism of operads

$$\mathrm{BSu}(\mathcal{P}) = \mathcal{P}re\mathcal{L}ie \bullet \mathcal{P}. \tag{55}$$

Remark

In the above sense, we know that

1. $\mathcal{P}re\mathcal{L}ie$ plays a role as a "splitting factor";
2. $\mathcal{P}re\mathcal{L}ie$ plays a role of "partition of unit" if $\mathcal{L}ie$ is regarded as a unit for the Manin black product $\bullet$.

Successors of operads and Manin black products

Corollary

1. *Loday algebras*
   - $\mathcal{D}end = \mathcal{P}re\mathcal{L}ie \bullet \mathcal{A}s$;
   - $\mathcal{Q}uad = \mathcal{P}re\mathcal{L}ie \bullet \mathcal{D}end$;
2. *Lie analogues of Loday algebras*
   - $\mathcal{P}re\mathcal{L}ie = \mathcal{P}re\mathcal{L}ie \bullet \mathcal{L}ie$;
   - $\mathcal{LD}end = \mathcal{P}re\mathcal{L}ie \bullet \mathcal{P}re\mathcal{L}ie$;
   - $\mathcal{LQ}uad = \mathcal{P}re\mathcal{L}ie \bullet \mathcal{LD}end$.

The trisuccessor $\mathrm{TSu}(\mathcal{P})$ of a binary quadratic operad $\mathcal{P}$ is the operad of algebras "splitting" the operations into three ones. The algebra playing a similar role of pre-Lie algebras is PostLie algebra.

Definition

A **(left) PostLie algebra** *is a vector space A with two bilinear operations $\circ$ and $[,]$ which satisfy the relations: (for any $x, y, z \in A$)*

$$[x, y] = -[y, x], \tag{56}$$

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0, \tag{57}$$

$$z \circ (y \circ x) - y \circ (z \circ x) + (y \circ z) \circ x - (z \circ y) \circ x + [y, z] \circ x = 0, \tag{58}$$

$$z \circ [x, y] - [z \circ x, y] - [x, z \circ y] = 0. \tag{59}$$

Successors of operads and Manin black products

Theorem

Let $\mathcal{P}$ be a binary quadratic operad. Then we have the isomorphism of operads

$$\mathrm{TSu}(\mathcal{P}) = \mathcal{P}ostLie \bullet \mathcal{P}. \tag{60}$$

Thank You!