Higher Categories and Rewriting

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From rewriting systems to omega-Cat

Homotopy of omega-Cat

Monoids

Presentations

A *presentation* of a monoid M consists in a pair $(\Sigma, \mathcal{R})$

- an alphabet Σ;
- a set $\mathcal{R} \subset \Sigma^* \times \Sigma^*$ of rewriting rules $r : w \to w'$, where w, w' are words on the alphabet Σ,

such that M is the quotient of the free monoid Σ^* by the congruence generated by $\mathcal{R}$.

Example

$\mathbb{Z}_2$ is presented by $(\{a\}, \{r : aa \to 1\})$

Complete system

Definition

A rewriting system is *complete* if it is noetherian and confluent.

Example

$$\begin{array}{ccc} aaaa & \xrightarrow{raa} & aa \\ {\scriptstyle ara}\downarrow & & \downarrow{\scriptstyle r} \\ aa & \xrightarrow[r]{} & 1 \end{array}$$

Homology

Theorem (Squier 1987)

If a monoid M admits a finite, complete presentation, then $\mathrm{H}_3(M)$ *is of finite type.*

Higher-categorical approach

- A monoid M is a category with a single object.
- The "space of computations" attached to a presentation of M supports a 2-dimensional categorical structure.
- More generally, the notion of resolution of M leads to categories of dimension 2, 3,...,n,...
- This leads to interpret Squier's result in an appropriate homotopical structure on $\omega\mathbf{Cat}$.

Globular sets

The category $\mathbf{O}$

$$0 \underset{\mathrm{t}_0}{\overset{\mathrm{s}_0}{\rightrightarrows}} 1 \underset{\mathrm{t}_1}{\overset{\mathrm{s}_1}{\rightrightarrows}} 2 \underset{\mathrm{t}_2}{\overset{\mathrm{s}_2}{\rightrightarrows}} \cdots$$

- objects are integers $0, 1, 2, \ldots$
- morphisms are generated by $\mathrm{s}_n, \mathrm{t}_n : n \to n{+}1$, with

$$\begin{aligned} \mathrm{s}_{n+1}\mathrm{s}_n &= \mathrm{t}_{n+1}\mathrm{s}_n \\ \mathrm{t}_{n+1}\mathrm{t}_n &= \mathrm{s}_{n+1}\mathrm{t}_n \end{aligned}$$

Globular sets

Definition

A *globular set* is a presheaf on $\mathbf{O}$:

$$X : \mathbf{O}^{op} \to \mathbf{Sets}$$

- globular sets are obtained by glueing together globe-shaped cells.
- looks like simplicial sets with $\mathbf{O}$ replacing Δ, but topologically much more restricted.

Higher categories

Definition

A (strict) ω-*category* C is given by:

- a globular set $\quad C_0 \leftleftarrows C_1 \leftleftarrows C_2 \leftleftarrows \cdots$
- compositions and units satisfying: associativity, exchange...

$$\omega\mathbf{Cat} = \omega\text{-categories} + \omega\text{-functors}$$

Higher categories

Examples

1. set

$$S \leftleftarrows () \leftleftarrows \cdots$$

2. monoid

$$1 \leftleftarrows M \leftleftarrows () \leftleftarrows \cdots$$

3. presentation

$$1 \leftleftarrows \Sigma^* \leftleftarrows \mathcal{R}^*_{/\sim} \leftleftarrows () \leftleftarrows \cdots$$

Polygraphs

Free cell adjunction

Let C be an n-category. Any graph

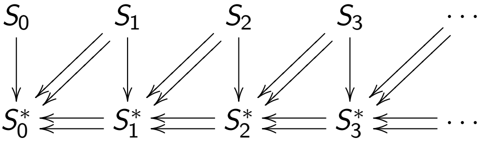
$$C_n \underset{\tau_n}{\overset{\sigma_n}{\leftleftarrows}} S_{n+1}$$

such that $g \in S_{n+1}$, $\sigma_n g \parallel \tau_n g$ for each generator g defines an $(n+1)$-category, the free extension of C by a set S_{n+1} of $(n+1)$-cells.

Polygraphs

Definition

A computad (Street 76) or polygraph (Burroni 91) S is a sequence of sets S_n of n-dimensional cells defining a freely generated n-category in each dimension n.



Examples from rewriting systems

$\Sigma = \{a\}$	graph	$1 \leftleftarrows \Sigma$
	free category	$1 \leftleftarrows \Sigma^*$
$\mathcal{R} = \{r\}$	2-graph	$1 \leftleftarrows \Sigma^* \leftleftarrows \mathcal{R}$
	free 2-category	$1 \leftleftarrows \Sigma^* \leftleftarrows \mathcal{R}^*$
	2-category	$1 \leftleftarrows \Sigma^* \leftleftarrows \mathcal{R}^*/\sim$

- what about higher dimensions ?

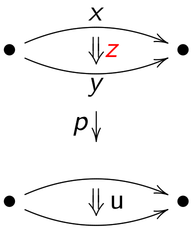
Resolutions

Definition

A *polygraphic resolution* of an ω-category C is a morphism $p : S^* \longrightarrow C$, where S is a polygraph and:

- p_0 is surjective;
- for each pair (x, y) of parallel n-cells in S^*_n and each $u : px \longrightarrow py$, there exists $z : x \longrightarrow y$ such that $pz = u$.

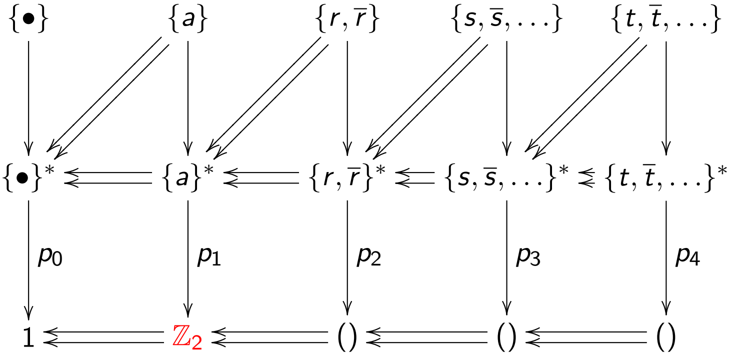
Resolutions



Theorem

Each ω-category admits a polygraphic resolution, which is unique up to "homotopy".

A partial resolution of $\mathbb{Z}_2$



Weak equivalences

Definition

- Two parallel n-cells x, y are *ω-equivalent* if there is a reversible $(n+1)$-cell $u : x \to y$;
- An $(n+1)$-cell $u : x \to y$ is *reversible* if there is a cell $v : y \to x$ such that $u * v$ and $v * u$ are ω-equivalent to 1_x and 1_y respectively.

Definition

A morphism $f : C \to D$ is a *weak equivalence* if:

- for all $d \in D_0$, there is $c \in C_0$ such that $fc \sim d$;
- for each pair (c, c') of parallel n-cells in C and each $d : fc \to fc'$, there exists $u : c \to c'$ such that $fu \sim d$.

We denote by $\mathcal{W}$ the class of weak equivalences.

Globes

n-Globes

- For each n, the *n-globe* $\mathbf{O}^n$ is the free ω-category generated by the globular set with two cells in dimensions $< n$, one cell in dimension n, and none in dimensions $> n$, that is

$$\mathbf{O}^n = \mathbf{O}(-, n)^*$$

- Likewise, $\partial\mathbf{O}^n$ denotes the boundary of the n-globe, obtained from $\mathbf{O}^n$ by removing the unique n-dimensional generator.

Generating cofibrations

Canonical inclusions

We denote by $\mathbf{i}_n$ the inclusion of $\partial\mathbf{O}^n$ in $\mathbf{O}^n$:

$$I = \{\mathbf{i}_n \mid n \geq 0\}$$

I is the set of *generating cofibrations*.

Model structure

Theorem (Lafont, Worytkiewicz & FM)

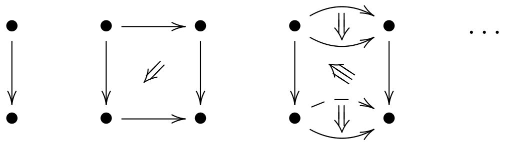
*The class $\mathcal{W}$ of weak equivalences and the set $\mathcal{I}$ of generating cofibrations determine a Quillen model structure on ω**Cat**.*

Fibrations & Cofibrations

The *trivial fibrations* are the morphisms having the right-lifting property with respect to $\mathcal{I}$ and the class $\mathcal{C}$ of *cofibrations* is the class of morphisms having the left-lifting property with respect to all trivial fibrations.
The class $\mathcal{F}$ of fibrations is the class of morphisms having the right-lifting property with respect to all morphisms in $\mathcal{C} \cap \mathcal{W}$.

Cylinders

- $(C^I)_n = \mathrm{Hom}(\mathrm{cyl}[n], C)$;
- C^I is an ω-category;
- there are natural transformations $\pi_1, \pi_2 : C^I \to C$



Properties

- *reversible* cylinders $\Gamma(C) \subset C^I$ define a path object on C ;
- all objects are fibrant;
- cofibrant objects are exactly polygraphs.

$$(\omega\mathbf{Cat})_{cf} = \mathbf{Pol}^*$$

Abelian group objects

Denormalization theorem (Bourn)

There is an equivalence of categories between:

$$\begin{array}{llcl} & \omega\mathbf{Cat}^{ab} & = & \text{abelian group objects in } \omega\mathbf{Cat} \\ \text{and} & \mathbf{Ch} & = & \text{chain complexes} \end{array}$$

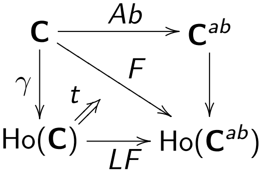
Abelianization functor

$$Ab : \omega\mathbf{Cat} \to \mathbf{Ch}, \qquad C \mapsto (A, \partial)$$

$A_i = \mathbb{Z}C_i/_{\approx}$, where $\mathrm{id}(x) \approx 0$ and $x *_j y \approx x + y$

Homology as a derived functor

Derived functor



$t : LF \circ \gamma \to F$

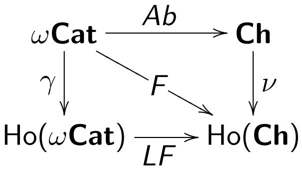
Model structure on **Ch**

- Weak equivalences induce isomorphisms in homology
- $\nu : \mathbf{Ch} \to \mathrm{Ho}(\mathbf{Ch})$

Deriving the abelianization functor

Theorem

Let $F = \nu \circ Ab$. There is a left derived functor LF and for any polygraph S, $(LF \circ \gamma)(S^*) \simeq F(S^*)$.



Proof.

- on cofibrant objects $Ab(S^*) = [S^*] = \mathbb{Z}S$;
- If $f : S^* \to T^*$ is a weak equivalence, then $Ab(f)$ is a quasi-isomorphism.

$\square$