CRITICAL PAIRS IN 2-DIMENSIONAL REWRITING SYSTEMS

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OPERADS AND REWRITING

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- term algebras

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In this talk, I will be interested in extending the procedures of **unification** in higher dimensions.

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`Can we generalize it to higher dimensions?`

In this talk, I will be interested in extending the procedures of **unification** in dimension 2.

REWRITING SYSTEMS

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A **rewriting system** consists of

- a set of *terms* generated by a free construction:
  - free monoid: *string rewriting systems*
  - free term algebra: *term rewriting systems*
- a set of *rewriting rules*: $r : t \to u$

Example

$\Sigma = \{a, b\}$ $\qquad$ $\text{terms} = \Sigma^*$ $\qquad$ $\text{rules} = \{ba \to ab\}$

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A term t **rewrites** to a term t' when there exists

- a rule $r : u \to u'$
- a context C such that $t = C[u]$ and $t' = C[u']$

Example

$\Sigma = \{a, b\}$ $\quad$ $\text{terms} = \Sigma^*$ $\quad$ $\text{rules} = \{ba \to ab\}$

$$aa\color{red}{ba}ab \xrightarrow{aa\color{red}{r}ab} aa\color{red}{ab}ab$$

CONVERGENT REWRITING SYSTEMS

- A rewriting system can be **terminating** when there is no infinite reduction path

$$\begin{array}{c} t \\ \downarrow \\ t_1 \\ \downarrow \\ t_2 \\ \downarrow \\ \vdots \end{array}$$

CONVERGENT REWRITING SYSTEMS

- A rewriting system can be **terminating**
- A rewriting can be **confluent** when

$$\begin{tikzcd}
 & t \arrow[dl, "*"'] \arrow[dr, "*"] & \\
u & & v
\end{tikzcd}$$

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  when both terminating and confluent

In a convergent rewriting system, every term has a **normal form**:
canonical representative of terms modulo rewriting.

Why
are those properties
interesting?

PRESENTATIONS OF MONOIDS

A **presentation**

$$\langle G \mid R \rangle$$

of a monoid M consists of

- a set G of *generators*
- a set $R \subseteq G^* \times G^*$ of *relations*

such that

$$M \qquad \cong \qquad G^* / \equiv_R$$

Example

- $\mathbb{N} \cong \langle a \mid \rangle$
- $\mathbb{N}/2\mathbb{N} \cong \langle a \mid aa = 1 \rangle$
- $\mathbb{N} \times \mathbb{N} \cong \langle a, b \mid ba = ab \rangle$
- $\mathfrak{S}_n \cong \langle \sigma_1, \ldots, \sigma_n \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i^2 = 1,\ \sigma_i \sigma_j = \sigma_j \sigma_i \rangle$
- ...

PRESENTATIONS OF MONOIDS

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2. Show that the rewriting system is terminating.
3. Show that the rewriting system is confluent.
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Normal forms are:

a^n and $a^n b$

They are in bijection with $\mathbb{N} \times (\mathbb{N}/2\mathbb{N})$!

How do we show
that a rewriting system
is confluent?

CRITICAL PAIRS

Given a *terminating* rewriting system the following are equivalent:

1. the rewriting system is **confluent**

$$
\begin{tikzcd}
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& w &
\end{tikzcd}
$$

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2. the rewriting system is **locally confluent**

$$
\begin{tikzcd}
 & t \arrow[dl] \arrow[dr] & \\
u \arrow[dr, dotted, "*"'] & & v \arrow[dl, dotted, "*"] \\
 & w &
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$$

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Critical pairs are:

$$\begin{tikzcd} & bba \arrow[dl] \arrow[dr] & \\ a & & bab \end{tikzcd} \qquad \begin{tikzcd} & bbb \arrow[dl] \arrow[dr] & \\ b & & b \end{tikzcd}$$

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Critical pairs are joinable:

$$\begin{tikzcd}
 & bba \arrow[dl] \arrow[dr] & \\
a \arrow[dr, equal] & & bab \arrow[dl, "*"] \\
 & a &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & bbb \arrow[dl] \arrow[dr] & \\
b \arrow[dr, equal] & & b \arrow[dl, equal] \\
 & b &
\end{tikzcd}$$

```
string rs  =  presentation of a monoid
  term rs  =  presentation of ?
```

TERM REWRITING SYSTEMS

- A **signature** (Σ, α) consists of
  - a set Σ of *generators*
  - an *arity* function $\alpha : \Sigma \to \mathbb{N}$
- Terms are elements of the free algebra Σ^* over this signature

Example

The TRS of commutative monoids: $\Sigma = \{m : 2,\ e : 0\}$

$$R = \left\{ \begin{array}{lrl} \alpha : & m(m(x,y),z) & \to m(x,m(y,z)) \\ \lambda : & m(e,x) & \to x \\ \rho : & m(x,e) & \to x \\ \gamma : & m(x,y) & \to m(y,x) \end{array} \right\}$$

PRESENTATIONS OF LAWVERE THEORIES

String rewriting systems correspond to presentations of monoids.

PRESENTATIONS OF LAWVERE THEORIES

Term rewriting systems correspond to presentations of **Lawvere theories**.

PRESENTATIONS OF LAWVERE THEORIES

A **Lawvere theory** is a category $\mathcal{C}$

- whose objects are integers
- which is cartesian
- whose cartesian product is given on objects by addition

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TRS $((\Sigma, \alpha), R)$ induce a LT whose morphisms $m \to n$ are n-uples of terms with variables $x_1, \ldots, x_m$, considered modulo relations.

Presentation: $\mathcal{C} \quad \cong \quad \Sigma^* / \equiv_R$

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Consider the TRS of commutative monoids: $\Sigma = \{m : 2,\ e : 0\}$

It presents the Lawvere theory whose morphisms $M : m \to n$ are $(m \times n)$-matrices with coefficients in $\mathbb{N}$.

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$$[m(m(x_1, x_1), x_2)\ ;\ e\ ;\ x_2] \quad : 2 \to 3$$

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$$[m(m(x_1, x_1), x_2)\ ;\ e\ ;\ x_2]\ \ : 2 \to 3 \qquad \rightsquigarrow \qquad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

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$$\begin{array}{r} [m(m(x_1, x_1), x_2)\ ;\ e\ ;\ x_2] \\ [m(x_1, m(x_1, x_2))\ ;\ e\ ;\ m(e, x_2)] \end{array} : 2 \to 3 \qquad \rightsquigarrow \qquad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

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Use rewriting theory!

`Can we use the same techniques`
`in order to build`
`presentations of` n`-categories?`

POLYGRAPHS

[Street76,Burroni93,Power90]

PRESENTING n-CATEGORIES

We want to generalize rewriting systems

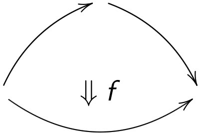
dimension	rewr. syst.	presents
1	string	monoid

$$\xrightarrow{a}\ \xrightarrow{b}\ \xrightarrow{c}$$

PRESENTING n-CATEGORIES

We want to generalize rewriting systems

dimension	rewr. syst.	presents
1	string	monoid
2	term	Lawvere th.



PRESENTING n-CATEGORIES

We want to generalize rewriting systems

dimension	rewr. syst.	presents
0	element	set
1	string	monoid
2	term	Lawvere th.

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PRESENTING n-CATEGORIES

We want to generalize rewriting systems

dimension	rewr. syst.	presents
0	element	0-category
1	string	monoid
2	term	Lawvere th.

$$\text{set} \quad = \quad \text{0-category}$$

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We want to generalize rewriting systems

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PRESENTING n-CATEGORIES

We want to generalize rewriting systems

dimension	rewr. syst.	presents
0	element	0-category
1	string	1-category
2	term	Lawvere th.

monoid $=$ 1-category with only one object

Generalization: $\xrightarrow{a} \xrightarrow{b} \quad \rightsquigarrow \quad x \xrightarrow{a} y \xrightarrow{b} y$

PRESENTING n-CATEGORIES

We want to generalize rewriting systems

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1	string	1-category
2	term	Lawvere th.

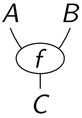
PRESENTING n-CATEGORIES

We want to generalize rewriting systems

dimension	rewr. syst.	presents
0	element	0-category
1	string	1-category
2	term	cartesian category

Lawvere th. $=$ cartesian category with $\mathbb{N}$ as objects

Generalization: 

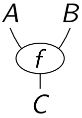
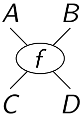
$\rightsquigarrow$ 

PRESENTING n-CATEGORIES

We want to generalize rewriting systems

dimension	rewr. syst.	presents
0	element	0-category
1	string	1-category
2	term	monoidal category

cartesian category $=$ monoidal category in which every object is a comonoid

Generalization:  $\rightsquigarrow$ 

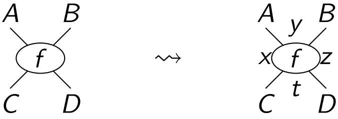
PRESENTING n-CATEGORIES

We want to generalize rewriting systems

dimension	rewr. syst.	presents
0	element	0-category
1	string	1-category
2	term	2-category

monoidal category $\quad = \quad$ 2-category with only one object

Generalization:



A 0-signature

Σ_0

Example

signature

x y

POLYGRAPHS

A 0-rewriting system

$$\begin{tikzcd} & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \\ \Sigma_0 & \end{tikzcd}$$

Example

signature

$x \qquad y$

rules

$$\begin{tikzcd} x \arrow[r, "a"] & y \arrow[loop above, "b"] \end{tikzcd}$$

POLYGRAPHS

A 1-signature = a 0-rewriting system

$$\begin{tikzcd} & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \\ \Sigma_0 & \end{tikzcd}$$

Example

signature

$$\begin{tikzcd} x \arrow[r, "a"] & y \arrow[loop above, "b"] \end{tikzcd}$$

POLYGRAPHS

A 1-signature generates a *category*

$$
\begin{tikzcd}
 & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \arrow[d, "i_1"] \\
\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"]
\end{tikzcd}
$$

Example

signature

$$
\begin{tikzcd}
x \arrow[r, "a"] & y \arrow[loop above, "b"]
\end{tikzcd}
$$

terms

$$
x \xrightarrow{a} y \xrightarrow{b} y \xrightarrow{b} y
$$

A 1-rewriting system

$$\begin{tikzcd}
& \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \arrow[d, "i_1"] & \Sigma_2 \arrow[dl, shift right, "s_1"'] \arrow[dl, shift left, "t_1"] \\
\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] &
\end{tikzcd}$$

such that $s_0^* \circ s_1 = s_0^* \circ t_1$ and $t_0^* \circ s_1 = t_0^* \circ t_1$

Example

signature

$$\begin{tikzcd}
x \arrow[r, "a"] & y \arrow[loop above, "b"]
\end{tikzcd}$$

terms

$$x \xrightarrow{a} y \xrightarrow{b} y \xrightarrow{b} y$$

rules

$$\begin{tikzcd}
& y \arrow[r, "b"] & y \arrow[dr, "b"] & \\
x \arrow[ur, "a"] \arrow[rrr, bend right, "a"'] \arrow[rrr, phantom, "r\Downarrow"] & & & y
\end{tikzcd}$$

A 2-signature = a 1-rewriting system

$$
\begin{tikzcd}
 & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \arrow[d, "i_1"'] & \Sigma_2 \arrow[dl, shift right, "s_1"'] \arrow[dl, shift left, "t_1"] \\
\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] &
\end{tikzcd}
$$

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Example

signature

$$
\begin{tikzcd}
 & y \arrow[r, "b"] \arrow[d, phantom, "r\Downarrow"] & y \arrow[dr, "b"] \arrow[d, phantom, "\Downarrow s"] & \\
x \arrow[ur, "a"] \arrow[rrr, bend right, "a"'] & & & y
\end{tikzcd}
$$

POLYGRAPHS

A 2-signature generates a 2-category

$$
\begin{tikzcd}
 & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \arrow[d, "i_1"] & \Sigma_2 \arrow[dl, shift right, "s_1"'] \arrow[dl, shift left, "t_1"] \arrow[d, "i_2"] \\
\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] & \Sigma_2^* \arrow[l, shift right, "s_1^*"'] \arrow[l, shift left, "t_1^*"]
\end{tikzcd}
$$

such that $s_0^* \circ s_1 = s_0^* \circ t_1$ and $t_0^* \circ s_1 = t_0^* \circ t_1$

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 & y \arrow[r, "b"] \arrow[d, phantom, "r\Downarrow"] & y \arrow[dr, "b"] \arrow[d, phantom, "\Downarrow s"] & \\
x \arrow[ur, "a"] \arrow[rrr, bend right, "a"'] & & & y
\end{tikzcd}
$$

POLYGRAPHS

A 2-rewriting system

$$\begin{tikzcd}
 & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \arrow[d, "i_1"] & \Sigma_2 \arrow[dl, shift right, "s_1"'] \arrow[dl, shift left, "t_1"] \arrow[d, "i_2"] & \Sigma_3 \arrow[dl, shift right, "s_2"'] \arrow[dl, shift left, "t_2"] \\
\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] & \Sigma_2^* \arrow[l, shift right, "s_1^*"'] \arrow[l, shift left, "t_1^*"] &
\end{tikzcd}$$

such that $s_1^* \circ s_2 = s_1^* \circ t_2$ and $t_1^* \circ s_2 = t_1^* \circ t_2$

Example

signature

$$\begin{tikzcd}
 & y \arrow[r, "b"] \arrow[d, phantom, "r\Downarrow"] & y \arrow[dr, "b"] \arrow[d, phantom, "\Downarrow s"] & \\
x \arrow[ur, "a"] \arrow[rrr, bend right, "a"'] & & & y
\end{tikzcd}$$

rules

$$\begin{tikzcd}
 & y \arrow[r, "b"] \arrow[dr, phantom, "r\Downarrow \overset{\alpha}{\Rrightarrow} \Downarrow s"] & y \arrow[dr, "b"] & \\
x \arrow[ur, "a"] \arrow[rrr, bend right, "a"'] & & & y
\end{tikzcd}$$

POLYGRAPHS

A 2-rewriting system

$$
\begin{tikzcd}
 & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \arrow[d, "i_1"'] & \Sigma_2 \arrow[dl, shift right, "s_1"'] \arrow[dl, shift left, "t_1"] \arrow[d, "i_2"'] & \Sigma_3 \arrow[dl, shift right, "s_2"'] \arrow[dl, shift left, "t_2"] \\
\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] & \Sigma_2^* \arrow[l, shift right, "s_1^*"'] \arrow[l, shift left, "t_1^*"] &
\end{tikzcd}
$$

Right notion of *n-rewriting system*: n-**polygraphs**.

`An example: a presentation of` **Bij**

[Lafont03]

A PRESENTATION OF Bij

The category **Bij** has

- objects are integers $[n] = \{0, \ldots, n-1\}$
- morphisms $f : [m] \to [n]$ are bijections

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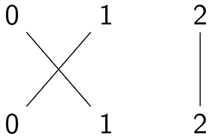
$$\begin{tikzcd} {[3]} \arrow[d, "f"'] \\ {[3]} \end{tikzcd}$$



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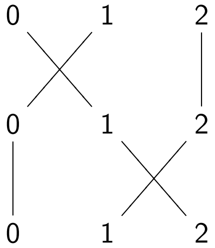
$$
\begin{tikzcd}
{[3]} \arrow[d, "f"'] \\
{[3]} \arrow[d, "g"'] \\
{[3]}
\end{tikzcd}
$$



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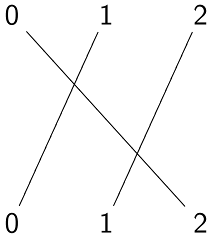
$$\begin{tikzcd}[3] \arrow[d, "g \circ f"'] \\ [3]\end{tikzcd}$$



Vertical composition $\circ$

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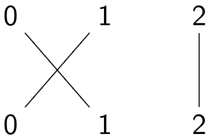
$$\begin{array}{c}[3]\\ \Big\downarrow{\scriptstyle f}\\ [3]\end{array}$$



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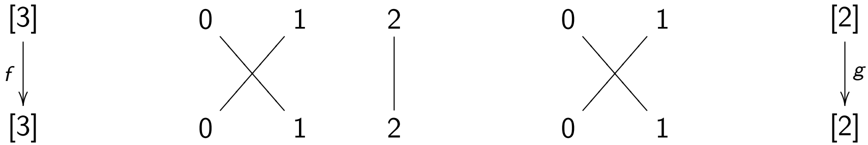
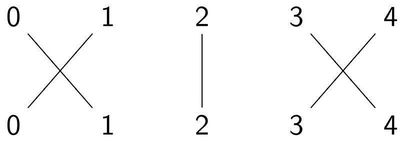


A PRESENTATION OF Bij

The category **Bij** has

- objects are integers $[n] = \{0, \ldots, n-1\}$
- morphisms $f : [m] \to [n]$ are bijections

$$\begin{tikzcd}
{[5]} \arrow[d, "f \otimes g"'] \\
{[5]}
\end{tikzcd}$$



Horizontal composition $\otimes$

A PRESENTATION OF Bij

We want to give a presentation of **Bij**, i.e. describe it as

- a free category on sets of typed generators for 0-, 1- and 2-cells
- quotiented by relations between 2-cells in the generated 2-category

A PRESENTATION OF Bij

Bij is presented by the 3-polygraph such that [Lafont03]

- $\Sigma_0 = \{*\}$

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$$\begin{tikzcd}
 & * \arrow[dr, "1"] & \\
* \arrow[ur, "1"] \arrow[dr, "1"'] \arrow[rr, phantom, "\gamma\Downarrow"] & & * \\
 & * \arrow[ur, "1"'] &
\end{tikzcd}$$

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$$\begin{tikzcd} & * \arrow[r] \arrow[d, phantom, "\gamma\Downarrow"] & * \arrow[dr] & \\ * \arrow[ur] \arrow[r] \arrow[dr] & * \arrow[ur] \arrow[dr] \arrow[d, phantom, "\gamma\Downarrow"] \arrow[rr, phantom, "\gamma\Downarrow"] & & * \\ & * \arrow[r] & * \arrow[ur] & \end{tikzcd} \overset{y}{\Rrightarrow} \begin{tikzcd} & * \arrow[r] \arrow[dr] & * \arrow[dr] \arrow[d, phantom, "\gamma\Downarrow"] & \\ * \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\gamma\Downarrow"] & & * \arrow[r] \arrow[d, phantom, "\gamma\Downarrow"] & * \\ & * \arrow[ur] \arrow[r] & * \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} & * \arrow[dr] \arrow[d, phantom, "\gamma\Downarrow"] & \\ * \arrow[ur] \arrow[r] \arrow[dr] & * \arrow[r] \arrow[d, phantom, "\gamma\Downarrow"] & * \\ & * \arrow[ur] & \end{tikzcd} \overset{s}{\Rrightarrow} * \longrightarrow * \longrightarrow *$$

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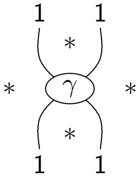

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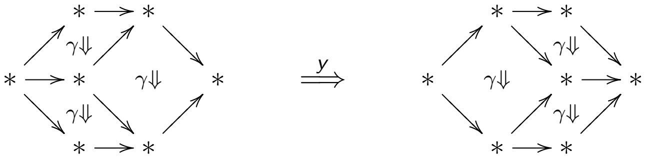

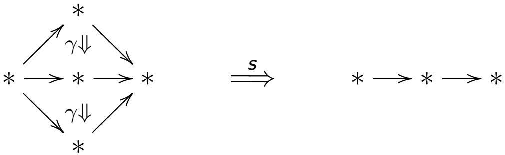

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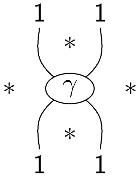

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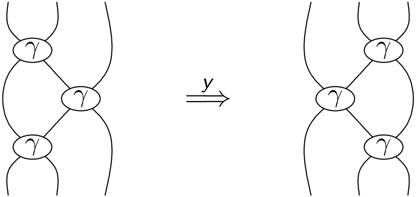
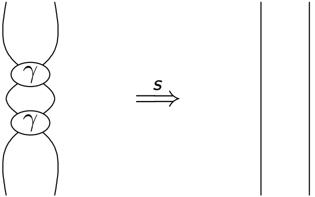
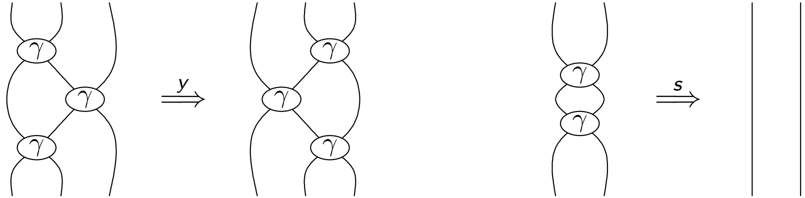


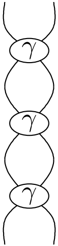


A PRESENTATION OF Bij

The rules



of the rewriting system induce critical pairs



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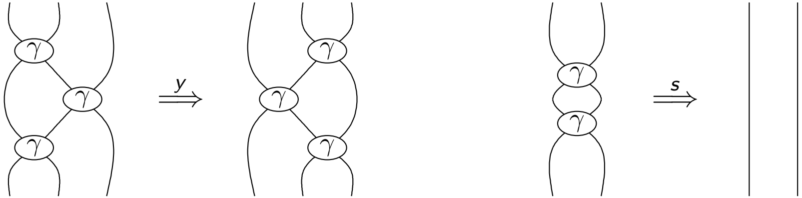

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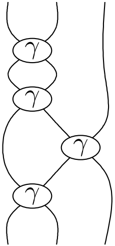

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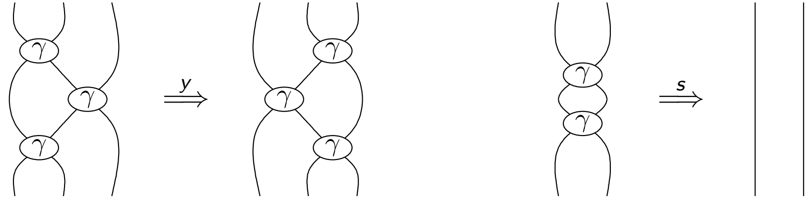

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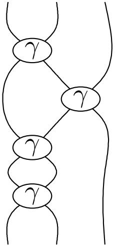

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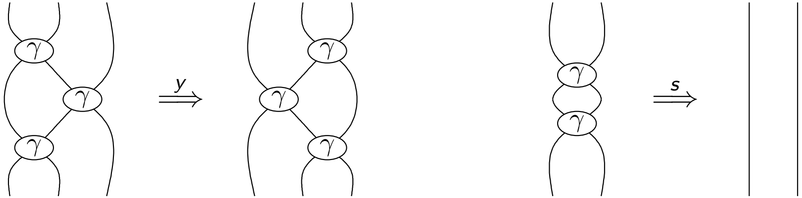

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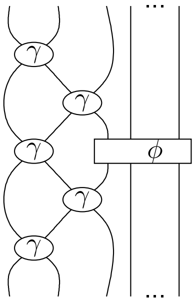


Critical pairs are computed using a **unification** procedure.

We want to extend it to 2-dimensional rewriting systems

Contrarily to term rewriting systems
we can have an infinite number
of critical pairs...

IDEA:

change the definition

of critical pairs

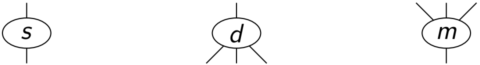
CRITICAL PAIRS
IN THE MULTICATEGORY OF
COMPACT CONTEXTS

TWO PROBLEMS WITH CRITICAL PAIRS

Consider the 2-rewriting system Σ with

$$\Sigma_0 = \{*\} \qquad \Sigma_1 = \{1\} \qquad \Sigma_2 = \{s : 1 \to 1,\ d : 1 \to 3,\ m : 3 \to 1\}$$

the generators for 2-cells are drawn respectively as



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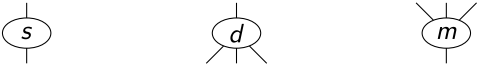
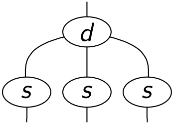
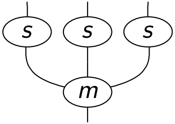


with rules



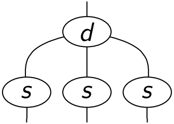
$\Rightarrow \dots$ and



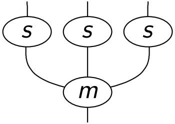
$\Rightarrow \dots$

TWO PROBLEMS WITH CRITICAL PAIRS

The two rules



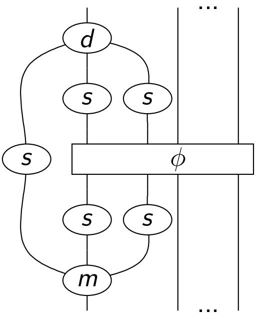
$\Rightarrow \ldots$ and



$\Rightarrow \ldots$

induce an infinite number of critical pairs:

variables on the border:



TWO PROBLEMS WITH CRITICAL PAIRS

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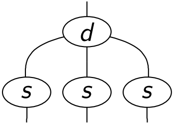
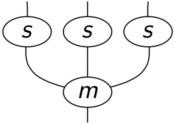


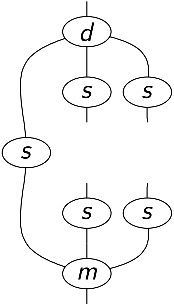
$\Rightarrow \dots$ and



$\Rightarrow \dots$

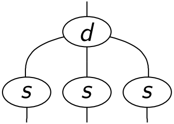
induce an infinite number of critical pairs:

variables on the border:
use compact morphisms!



TWO PROBLEMS WITH CRITICAL PAIRS

The two rules



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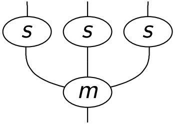

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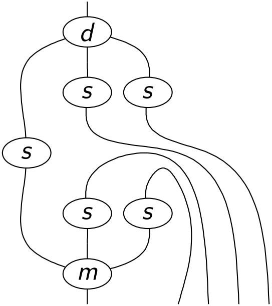
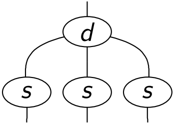
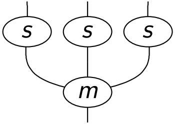


TWO PROBLEMS WITH CRITICAL PAIRS

The two rules



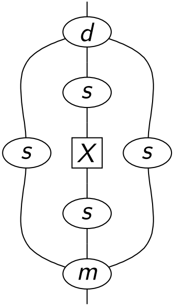
$\Rightarrow \ldots$ and



$\Rightarrow \ldots$

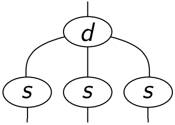
induce an infinite number of critical pairs:

variables inside:

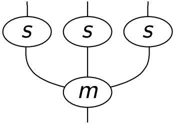


TWO PROBLEMS WITH CRITICAL PAIRS

The two rules



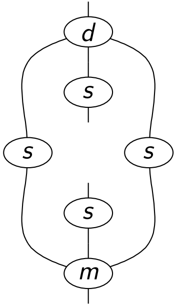
$\Rightarrow \ldots$ and



$\Rightarrow \ldots$

induce an infinite number of critical pairs:

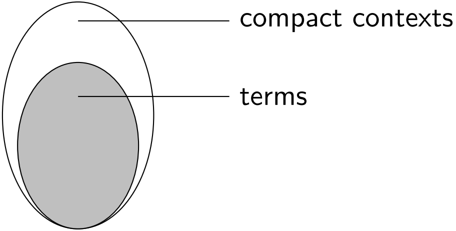
variables inside:
use contexts!
($=$ multicategory of
terms with metavariables)



BACK TO A FINITE NUMBER OF CRITICAL PAIRS

Theorem

The 2-category of "terms" generated by a signature can be embedded into the ***multicategory of compact contexts****.*



In other words, there is a finite number of *generating families* of critical pairs in those rewriting systems.

THE MULTICATEGORY OF CONTEXTS

Consider a 2-polygraph Σ

$$\begin{tikzcd}
 & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \arrow[d, "i_1"] & \Sigma_2 \arrow[dl, shift right, "s_1"'] \arrow[dl, shift left, "t_1"] \\
\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] &
\end{tikzcd}$$

and a family $\mathcal{X} = \{X_1 : f_1 \Rightarrow g_1, \ldots, X_n : f_n \Rightarrow g_n\}$ of 2-globes, with $f_i, g_i \in \Sigma_1^*$ parallel 1-cells considered as *formal variables*

THE MULTICATEGORY OF CONTEXTS

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We define the 2-polygraph $\Sigma[\mathcal{X}]$ as

$$\begin{tikzcd}
 & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \arrow[d, "i_1"] & \Sigma_2 \uplus \mathcal{X} \arrow[dl, shift right, "s_1"'] \arrow[dl, shift left, "t_1"] \\
\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] &
\end{tikzcd}$$

THE MULTICATEGORY OF CONTEXTS

Consider a 2-polygraph Σ

$$
\begin{tikzcd}
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\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] & \Sigma_2^* \arrow[l, shift right, "s_1^*"'] \arrow[l, shift left, "t_1^*"]
\end{tikzcd}
$$

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\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] & (\Sigma_2 \uplus \mathcal{X})^* \arrow[l, shift right, "s_1^*"'] \arrow[l, shift left, "t_1^*"]
\end{tikzcd}
$$

THE MULTICATEGORY OF CONTEXTS

Consider a 2-polygraph Σ

$$\begin{tikzcd}
 & \Sigma_1 \arrow[dl, shift right, "s_0"'] \arrow[dl, shift left, "t_0"] \arrow[d, "i_1"'] & \Sigma_2 \arrow[dl, shift right, "s_1"'] \arrow[dl, shift left, "t_1"] \arrow[d, "i_2"'] \\
\Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0^*"'] \arrow[l, shift left, "t_0^*"] & \Sigma_2^* \arrow[l, shift right, "s_1^*"'] \arrow[l, shift left, "t_1^*"]
\end{tikzcd}$$

Substitution

Given

- a 2-cell $\alpha : f \Rightarrow f'$ in Σ_2^*
- and a 2-cell $\beta : g \Rightarrow g'$ in $\Sigma_2[X : f \Rightarrow f']^*$

we can define

- a 2-cell $\alpha[\beta/X] : f \Rightarrow f'$ in Σ_2^*

which corresponds to the 2-cell α where all occurrences of X have been replaced by β.

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\end{tikzcd}$$

Definition

We can thus define the **multicategory of contexts** $\mathcal{K}_\Sigma$ whose

- objects are globes $f \Rightarrow g$, i.e. parallel 1-cells in Σ_1^*
- operations in $\mathcal{K}_\Sigma(f_1 \Rightarrow g_1, \ldots, f_n \Rightarrow g_n; f \Rightarrow g)$ are 2-cells $\kappa : f \Rightarrow g$ in $\Sigma_2[X_1 : f_1 \Rightarrow g_1, \ldots, X_n : f_n \Rightarrow g_n]^*$
- composition is given by generalizing the previous substitution

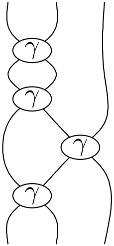
CRITICAL PAIRS

Definition

Suppose that we are given two 2-cells

$$\alpha_1 : f_1 \Rightarrow g_1 \qquad \text{and} \qquad \beta_1 : f_1 \Rightarrow g_1$$

in a 2-polygraph Σ. A **most general unifier** is



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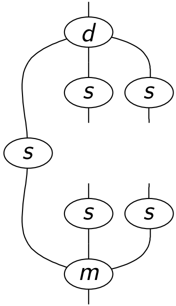
$$\kappa_1 \in \mathcal{K}_\Sigma(f_1 \Rightarrow g_1; f \Rightarrow g) \qquad \text{and} \qquad \kappa_2 \in \mathcal{K}_\Sigma(f_2 \Rightarrow g_2; f \Rightarrow g)$$

of *linear* contexts such that

1. *unifier*: $\kappa_1(\alpha_1) = \kappa_2(\alpha_2)$
2. *minimal*: if $\kappa_1 = \kappa_1'' \circ \kappa_1'$ and $\kappa_2 = \kappa_2'' \circ \kappa_2'$ where (κ_1', κ_2') is a unifier then $\kappa_1'' = \mathrm{id}$ and $\kappa_2'' = \mathrm{id}$
3. *overlapping*: there is no binary context κ such that $\kappa_1 = (\mathrm{id}, \alpha_2)$ and $\kappa_2 = (\alpha_1, \mathrm{id})$

COMPACT 2-CATEGORIES

Now, we want to represent morphisms with "holes" in the border



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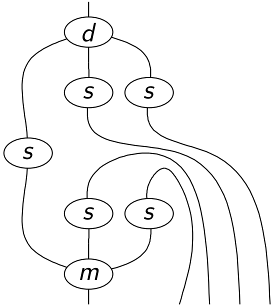


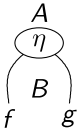
COMPACT 2-CATEGORIES

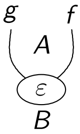
Definition

Given a 2-category $\mathcal{C}$, a 1-cell $f : A \to B$ is **left adjoint** to a 1-cell $g : B \to A$, which we write

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

when there exists two 2-cells $\eta : A \to f \otimes g$ and $\varepsilon : g \otimes f \to B$





such that

$$(f \otimes \varepsilon) \circ (\eta \otimes f) = f \quad \text{and} \quad (\varepsilon \otimes g) \circ (g \otimes \eta) = g$$

COMPACT 2-CATEGORIES

Definition

A 2-category $\mathcal{C}$ is **compact** when every 1-cell admits both a left and a right adjoint.

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Lemma

A 2-category $\mathcal{C}$ generates a free compact 2-category $\mathcal{A}_\mathcal{C}$, which is

- 0-*cells: same as* $\mathcal{C}$,
- 1-*cells:* f^n *with* f *a 1-cell of* $\mathcal{C}$ *and* $n \in \mathbb{Z}$,
- 2-*cells:*
  - $\alpha : f^0 \Rightarrow g^0$ *for* $\alpha : f \Rightarrow g$ *a 2-cell of* $\mathcal{C}$,
  - $\eta_{f^n} : \mathrm{id} \Rightarrow f^{n-1} \otimes f^n$
  - $\varepsilon_{f^n} : f^n \otimes f^{n-1} \Rightarrow \mathrm{id}$
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Theorem

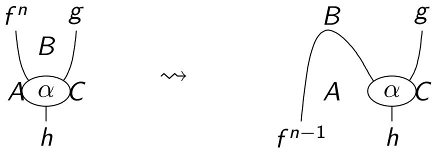
A 2-category $\mathcal{C}$ embeds fully and faithfully in the free compact category it generates.

ROTATIONS

In the free compact 2-category $\mathcal{A}_{\mathcal{C}}$, the following Hom-sets

$$\mathcal{A}_{\mathcal{C}}(f^n \otimes g, h) \qquad \cong \qquad \mathcal{A}_{\mathcal{C}}(g, f^{n-1} \otimes h)$$

are isomorphic:



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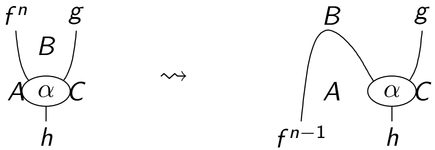

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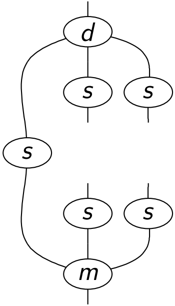
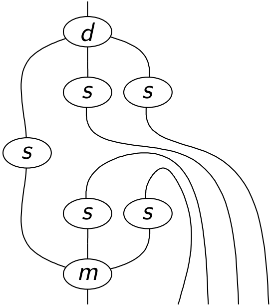
Rotations are unary operations in $\mathcal{K}_{\mathcal{A}_\mathcal{C}}$.

ROTATIONS

The diagram



is a representation of



up to rotation!

THE MULTICATEGORY OF COMPACT CONTEXTS

A 2-polygraph Σ generates a 2-category $\mathcal{C}$,

- which can be embedded in the free compact 2-category $\mathcal{A}_{\mathcal{C}}$
- and the 2-cells $\alpha : f \Rightarrow g$ of $\mathcal{A}_{\mathcal{C}}$ can be seen as nullary contexts in $\mathcal{K}_{\mathcal{A}_{\mathcal{C}}}(; f \Rightarrow g)$

UNIFICATION IN THE MULTICATEGORY OF COMPACT CONTEXTS

Theorem

Given a 3-polygraph R with underlying 2-polygraph Σ generating a 2-category $\mathcal{C}$, there exists a ***finite*** *number of contexts*

$$\kappa^i \quad \in \quad \mathcal{K}_{\mathcal{A}_\mathcal{C}}(f_1^i \Rightarrow g_1^i \ , \ \ldots \ , \ f_{k_i}^i \Rightarrow g_{k_i}^i ; f^i \Rightarrow g^i)$$

such that

- *for any nullary contexts $\kappa_1, \ldots, \kappa_{k_i}$ and unary context κ such that the composite $\kappa \circ \kappa^i \circ (\kappa_1, \ldots, \kappa_{k_i}) \in \mathcal{K}_{\mathcal{A}_\mathcal{C}}(; f' \Rightarrow g')$ is of the form κ_α, for some 2-cell $\alpha : f' \Rightarrow g'$ of $\mathcal{C}$, the 2-cell α is an unifier of left members of two rewriting rules of R,*
- *and moreover any such unifier can be obtained in this way (in particular the critical pairs).*

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Remark

There is unicity up to rotations.

UNIFICATION
IN THE MULTICATEGORY OF
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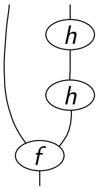
UNIFICATION IN TRS

Suppose that we have a TRS

$f : 2 \qquad h : 1 \qquad\qquad f(x, h(h(y))) \Rightarrow \dots \qquad h(h(f(x, y))) \Rightarrow \dots$

In order to generate critical pairs,
we unify a subterm of $f(x, h(h(y)))$ with $h(h(f(x, y)))$



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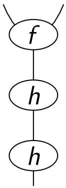

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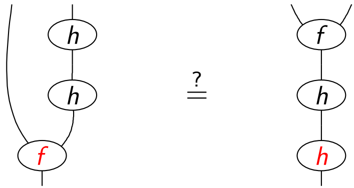

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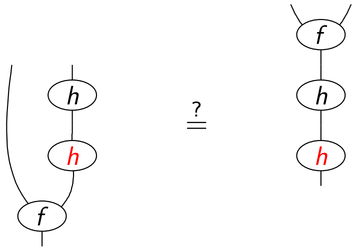

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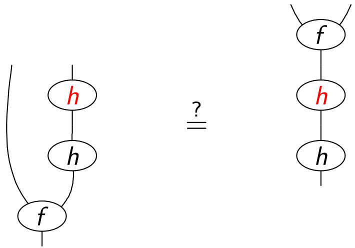

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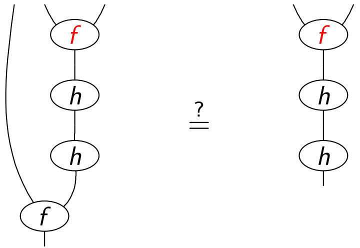
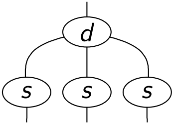


UNIFICATION IN 2-REWRITING SYSTEMS

We want to compute the critical pairs generated by



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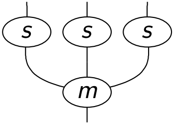

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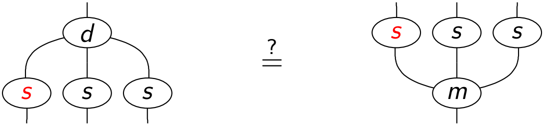

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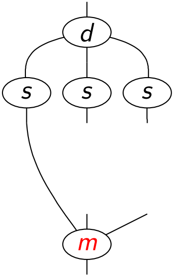

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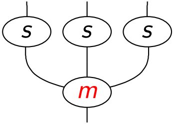

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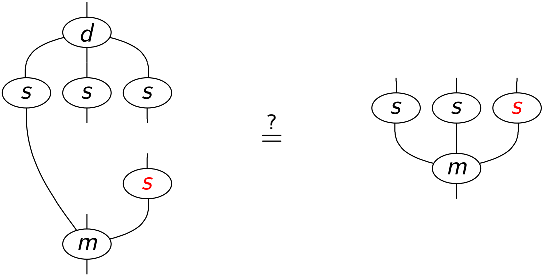

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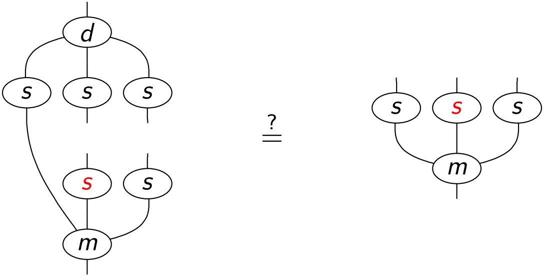

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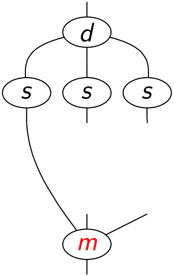

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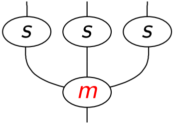

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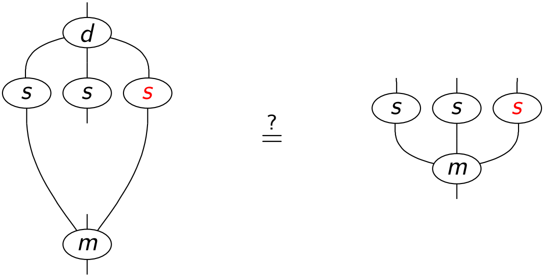

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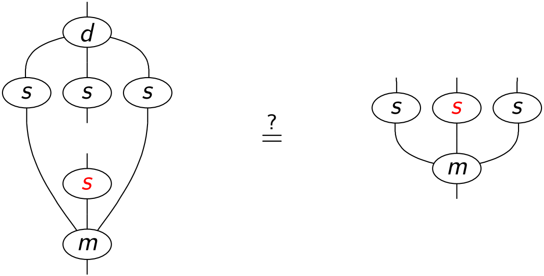

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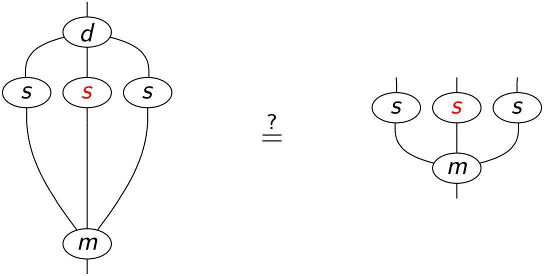

METACONFLUENCE

Remark

When a 2-rewriting system is confluent, its critical pairs (in our generalized sense) are not necessarily confluent. But at least, we get a finite description of the critical pairs!

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- The operations in the multicategory of compact contexts can be represented in an effective way.

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- Precise formulation of the algorithm.

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- Precise formulation of the algorithm.
- An implementation was realized.

FUTURE WORKS

- Generalize techniques developed by Guiraud and Malbos to this setting
- Generalize to higher dimensions
- Towards automated tools for studying higher categories?
- ...

THANKS!

Any questions?