Graphical Approach to the Drinfeld Centre

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Plan

Part 1. Categorical considerations

- Graphical calculus and Drinfeld centre
- Tube category
- Equivalence with Drinfeld centre

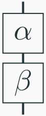
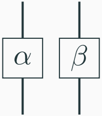
Part 2. Applications in conformal field theory

- Modular invariants as representations of the tube category
- Extending the trace of a module category

Graphical Calculus

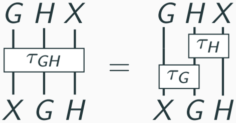
Objects and Morphisms

Let $\mathcal{C}$ be a monoidal category. Within the graphical calculus of $\mathcal{C}$:

objects are depicted as labelled strands (except the tensor identity)		morphisms are depicted as labelled boxes	
composition is depicted by vertical juxtaposition		tensor product is depicted by horizontal juxtaposition	

We adopt the convention that diagrams are read top to bottom. Even when not using graphical calculus we often omit the '$\otimes$' symbol.

A *half-braiding* on X in $\mathcal{C}$ is a collection of natural isomorphisms

$\tau_G \colon GX \to XG$ such that 

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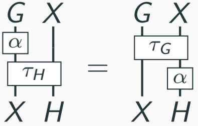
for all G, H in $\mathcal{C}$. From a graphical perspective the condition that τ_G is natural allows us to 'push' morphisms though τ:



.

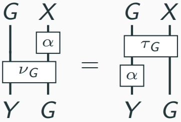
Note that this only works when pushing morphisms from the top-left to the bottom-right, endomorphisms of X cannot be pushed through.

Graphical Calculus

Drinfeld Centre

The *Drinfeld centre* of $\mathcal{C}$ categorifies the notion of the centre of a group or monoid. Whearas the classical centre consists of all elements which commute with everything, in the categorical context we have to specify *how* they commute. This is encoded by a half-braiding.

- Objects in $Z(\mathcal{C})$ are pairs (X, τ) where X is in $\mathcal{C}$ and τ is a half-braiding on X.
- $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((X, \tau), (Y, \nu))$ is given by the subspace $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ defined by the condition that $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ satisfies



$\forall G$ in $\mathcal{C}$.

In other words $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((X, \tau), (Y, \nu))$ is the space of morphisms which *can* be pushed through top-right to bottom-left.

Applications: Mathematical physics and the representation theory of monoidal categories in general.

Graphical Calculus

Dual Objects

We now suppose that $\mathcal{C}$ is rigid, i.e. for every X in $\mathcal{C}$ there is an object $X^\vee$ and ${}^\vee X$ together with morphisms

such that the 'S-relations' holds:

Furthermore we chose a *pivotal structure* in $\mathcal{C}$ i.e. a natural isomorphism ${}^\vee X \to X^\vee$. This allows us to identify left and right duals. We can now consider the left and right dimensions of an object X,

$d_r(X) =$ 

and $d_l(X) =$ 

.

Finally we suppose that our pivotal structure is *spherical*, i.e.

$$d_l(X) = d_r(X) =: d(X).$$

At times we will also suppose $\mathcal{C}$ is equipped with a *braiding* $\sigma_{XY} \colon XY \to YX$ which is depicted by the over-crossing,

.

Naturality of the braiding allows morphisms to pass over and under strands,

.

A choice of braiding on $\mathcal{C}$ determines a functor

$$\Phi \colon \mathcal{C} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{C})$$

$$X \boxtimes Y \mapsto \Big(XY, \quad \Big).$$

The braiding is called *factorisable* if is Φ is an equivalence.

For the rest of this talk we assume that $\mathcal{C}$ is a *linear category* over an algebraically closed field (i.e. enriched over $\underline{\mathsf{Vect}}$). We also assume that $\mathcal{C}$ is *semisimple* and admits a finite complete set of simple objects $\mathcal{I}$.

Such categories are equivalent to $\mathcal{RC} := \mathsf{Fun}^{\mathsf{op}}(\mathcal{C}, \underline{\mathsf{Vect}})$. Indeed, the functor

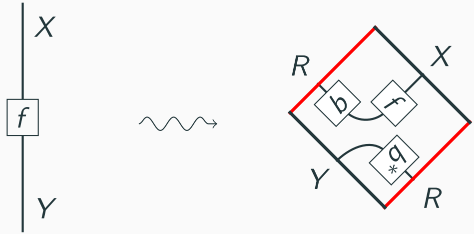
$$\flat\colon \mathcal{RC} \to \mathcal{C}$$
$$F \mapsto \bigoplus_S F(S) \cdot S$$

(where the sum ranges over $\mathcal{I}$) is a inverse to the Yoneda embedding $\sharp$. When such a category is equipped with a rigid monoidal product it is known as a *fusion category*. Its dimension is given by $d(\mathcal{C}) = \sum_S d(S)^2$.

The Tube Category

An Introduction

Let $\mathcal{C}$ be a spherical fusion category and let f be in $\text{Hom}_{\mathcal{C}}(X, Y)$.



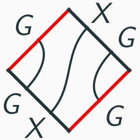
$$\text{Hom}_{\mathcal{TC}}(X, Y) = \bigoplus_R \text{Hom}_{\mathcal{C}}(RX, YR)$$

$\mathcal{TC}$ shares the same objects as $\mathcal{C}$ but has more morphisms. Morphisms in $\mathcal{TC}$ are described by diagrams in $\mathcal{C}$ drawn on a cylinder.

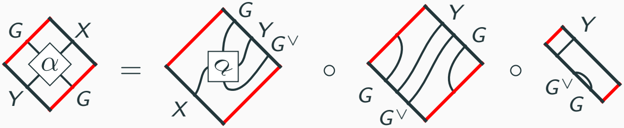
The Tube Category

Representations of $\mathcal{TC}$

We consider the following morphism in $\mathcal{TC}$

$c_{G,X} =$ 

Any morphism in $\mathcal{TC}$ may be re-written in the following form:



Therefore a (contravariant) representation $\mathcal{F}\colon \mathcal{TC} \to \underline{\text{Vect}}$ is simply a representation of $\mathcal{C}$ together with an action of the $c_{G,X}$ morphisms i.e. a collection of isomorphisms $\kappa_{G,X}\colon \mathcal{F}(GX) \to \mathcal{F}(XG)$ which are natural in G and X and satisfy $\kappa_{GH,X} = \kappa_{H,XG} \circ \kappa_{G,HX}$.

The Tube Category

Equivalence of Representations with $\mathcal{Z}(\mathcal{C})$

$\mathcal{Z}(\mathcal{C})$ is a category with objects (F, τ) where $F \in \mathcal{C}$ and $\tau \colon (- \otimes F) \to (F \otimes -)$ is a half-braiding i.e. satisfies:

$$\tau^\sharp_{GH} = (\tau_G \otimes \mathrm{id}_H)^\sharp \circ (\mathrm{id}_G \otimes \tau_H)^\sharp.$$

$$\kappa_{GH,X} = \kappa_{H,XG} \circ \kappa_{G,HX}$$

As $\mathcal{C}$ is fusion the Yoneda embedding $F \mapsto F^\sharp$ is an equivalence. This induces an equivalence $(F, \tau) \mapsto (F^\sharp, \tau^\sharp)$ between $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{RC})$ where

$$\begin{array}{ccc} \tau^\sharp_{G^\vee} \colon (G^\vee \otimes F)^\sharp & \to & (F \otimes G^\vee)^\sharp. \\ \| & & \| \\ F^\sharp \circ (G \otimes -) & & F^\sharp \circ (- \otimes G) \end{array}$$

$$\kappa_{G,X} \colon F^\sharp(GX) \to F^\sharp(XG)$$

An object in $\mathcal{Z}(\mathcal{C}) = \mathcal{Z}(\mathcal{RC})$ is an object in $\mathcal{RC}$ together with together with isomorphisms $\kappa_{G,X}$ as above, i.e. an object in $\mathcal{RTC}$.

The Tube Category

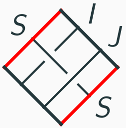
Primitive Idempotents when $\mathcal{C}$ is Modular

We now suppose that $\mathcal{C}$ is equipped with a *factorisable* braiding, in particular, we have a canonical equivalence

$$\Phi\colon \mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{Z}(\mathcal{C}) = \mathcal{RTC}.$$

Therefore elements of the form $\Phi(I \boxtimes J)$ for $I, J \in \mathcal{I}$ form a complete set of simples in $\mathcal{RTC}$. Helpfully, the simple representation $\Phi(I \boxtimes J)$ is represented by the idempotent

$$\epsilon_I^J = \frac{1}{d(\mathcal{C})} \sum_S d(S)$$



$$\in \mathrm{End}_{\mathcal{TC}}(IJ).$$

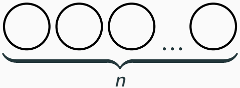
Therefore every representation of $\mathcal{TC}$ is represented by an idempotent. This effectively allows us to translate any computation in $\mathcal{Z}(\mathcal{C})$ into a computation of idempotents in $\mathcal{TC}$.

Conformal Field Theory

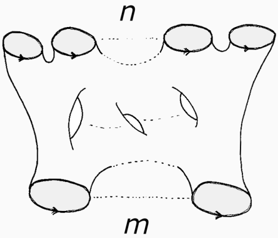
The cobordism category $\underline{\text{Seg}}^{\text{op/cl}}$

$$\text{Obj}(\underline{\text{Seg}}^{\text{cl}}) = \left\{ \underline{n} := \bigsqcup_{n} S_1 \;\middle|\; n \in \mathbb{N} \right\}$$

$\underline{n} =$



$$\text{Hom}_{\underline{\text{Seg}}^{\text{cl}}}(\underline{n}, \underline{m}) = \left\{ [X] \mid \begin{array}{l} X \text{ is a Riemann surface with } n \text{ incoming} \\ \text{boundaries and } m \text{ outgoing boundaries.} \end{array} \right\}$$



There is an analogous category $\underline{\text{Seg}}^{\text{op}}$ where the objects are 1-dimensional manifolds with two boundary points labelled by *boundary conditions*.

Full 2D Conformal Field Theory

$$\mathcal{F}\colon \underline{\mathsf{Seg}}^{\mathsf{op/cl}} \to \underline{\mathsf{Vect}}$$

$\swarrow$ Open CFT $\qquad$ $\searrow$ Closed CFT

The *vertex operator algebra* (VOA) structure axiomatises (one chiral half of) the genus zero part of $\mathcal{F}$. The category of modules $\mathcal{C}$ over a (rational) VOA is a *modular tensor category* i.e. a linear category which is:

- spherical
- fusion
- equipped with a factorisable braiding

Extra data:

Module category over $\mathcal{C}$ $\qquad$ Modular invariant of $\mathcal{C}$

A module category over $\mathcal{C}$ is a linear category $\mathcal{B}$ together with a monoidal functor

$$\mathcal{M}\colon \mathcal{C} \to \mathsf{End}(\mathcal{B}).$$

In other words it is simply a categorification of a module over a ring.

Conformal Field Theory

Modular Invariants

The value of a closed CFT $\mathcal{F}$ on a torus is given by the *partition function* Z.

$$\mathcal{F}(\mathbb{T}_\tau) =: Z(\tau) = \sum_{I,J\in\mathcal{I}} Z_{IJ}\, \chi_I(\tau)\chi_J(\tau)^* \in \mathbb{C}$$

where $\tau \in \mathbb{H}$, $T_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ and χ_I is the character of I. Conformal invariance implies invariance of Z under the action of the mapping class group on $\mathbb{H}$. MTCs come with a special representation of $\mathrm{PSL}_2(\mathbb{Z})$ given by their *modular data*:

$\mathcal{T}_{IJ} := \delta_{I,J}$ 

$\mathcal{S}_{IJ} :=$ 

.

Conformal invariance of $Z(\tau)$ is equivalent to the $\mathcal{I} \times \mathcal{I}$-matrix of multiplicities (Z_{IJ}) commuting with the modular data. We define a *modular invariant* to be any such non-negative integer matrix Z.

As described before, a complete set of simple objects in $\mathcal{RTC}$ is indexed by $\mathcal{I} \times \mathcal{I}$ so the isomorphism classes in $\mathcal{RTC}$ are in bijection with non-negative integer $\mathcal{I} \times \mathcal{I}$-matrices.

$$\begin{aligned} Z\colon \mathcal{RTC} &\to \mathsf{Mat}_{\mathcal{I}\times\mathcal{I}}(\mathbb{N}) \\ F &\mapsto \left(\dim \operatorname{Im} F(\epsilon_I^J)\right) \end{aligned}$$

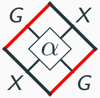
We may therefore assign to any $F \in \mathcal{RTC}$ a *candidate* modular invariant by simply considering its isomorphism class. In fact we can go further and *define* a modular invariant to be a representation of $\mathcal{TC}$ satisfying its own notion of invariance under the modular group.

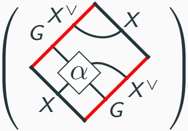
Under what condition should an object $F \in \mathcal{RTC}$ be a modular invariant?

tr F 

Theorem (to appear, H.)

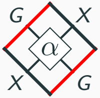
The isomorphism class $Z(F)$ gives a modular invariant if and only if tr F *is invariant under the action of* $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{MCG}(\mathbb{T}^2)$.

tr F 

= tr F 

For $\alpha = \epsilon_I^J$ this is equivalent to

$$(\mathcal{T}^{-1} Z(F) \mathcal{T})_{IJ} = Z(F)_{IJ}$$

tr F 

= tr F 

For $\alpha = \epsilon_I^J$ this is equivalent to

$$(\mathcal{S}^{-1} Z(F) \mathcal{S})_{IJ} = Z(F)_{IJ}$$

Conformal Field Theory

A Unified Approach to Open/Closed CFT

‘closing up’

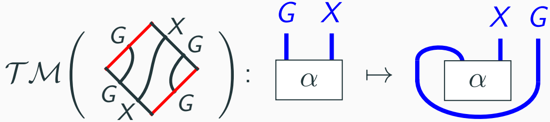
Open CFT	Closed CFT
module category	modular invariant
$\mathcal{M}\colon \mathcal{C} \to \mathsf{End}(\mathcal{B})$	$\mathcal{F}\colon \mathcal{TC} \to \underline{\mathsf{Vect}}$

‘trace’

There is a notion of trace for monoidal functors. However it simply gives the following representation of $\mathcal{C}$:

$$\begin{aligned} \mathsf{Tr}\,\mathcal{M}\colon \mathcal{C} &\to \underline{\mathsf{Vect}} \\ X &\mapsto \mathsf{Hom}_{\mathsf{End}(\mathcal{B})}(\mathbf{1}, \mathcal{M}(X)). \end{aligned}$$

To extend $\mathrm{Tr}\,\mathcal{M}$ to a representation of $\mathcal{TC}$, which we denote $\mathcal{TM}$, we have to specify the value of $\mathcal{TM}$ on the $c_{G,X}$ morphisms. Setting



defines a unique object in $\mathcal{RTC}$ (where the blue colour denotes evaluation under $\mathcal{M}$) when $\mathcal{M}$ ‘induces a pivotal structure’.

Theorem (Theorem 6.7, H.)

$\mathcal{TM}$ is a symmetric, commutative Frobenius algebra in $\mathcal{RTC}$.

This result together with the work of Kong and Runkel [KR, Theorem 3.4] implies that $\mathcal{TM}$ is a modular invariant if and only if $d(\mathcal{TM}) = d(\mathcal{C})$.

Leonard Hardiman.
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