Modular Invariants as Representations of the Tube Category

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Plan

1. The Mathematics of CFT
2. Motivating example: $\hat{\mathfrak{su}}(2)$ WZW model
3. The tube category
4. Extending the trace
5. A return to the motivating example

Full 2D Conformal Field Theory

$$\swarrow \qquad\qquad \searrow$$

Open CFT | Closed CFT

Let $\mathcal{C}$ be the category of modules over the associated vertex operator algebra. In particular $\mathcal{C}$ is a *Modular Tensor Category* i.e. a linear category which is:

- semisimple (with finitely many isomorphism classes of simple objects)
- monoidal
- rigid
- equipped with a spherical pivotal structure
- equipped with a non-degenerate braiding

A module category over $\mathcal{C}$ | A modular invariant of $\mathcal{C}$

In an open CFT for any two boundary conditions $a, b \in A$ we have a Hilbert space ${}_aH_b$. Decomposing into simple modules over the VOA gives

$$ {}_aH_b = \bigoplus_{I \in \mathrm{Irr}(\mathcal{C})} {}_aH_b^I. $$

$$ \begin{aligned} \mathcal{M} \colon \mathcal{C} &\to A_{\mathbb{C}}, A_{\mathbb{C}}\text{-Bimod} = \mathrm{End}(\mathrm{Mod}\, A_{\mathbb{C}}) \\ I &\mapsto H^I. \end{aligned} $$

A module category over $\mathcal{C}$ is a linear category $\mathcal{B}$ together with a monoidal functor

$$ \mathcal{M} \colon \mathcal{C} \to \mathrm{End}(\mathcal{B}). $$

Within the context of this talk we assume that all module categories are semisimple with finitely many isomorphism classes of simple objects.

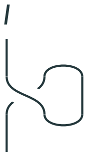
The Mathematics of CFT

Modular Invariants

The value of the closed CFT on a torus is given by the *partition function*:

$$Z(\tau) = \sum_{I,J \in \mathrm{Irr}(\mathcal{C})} Z_{IJ}\, \chi_I(\tau)\chi_J(\tau)^*.$$

where $\tau \in \mathbb{H}$ parametrizes the conformal structure on the torus and χ_I is the character of I. MTCs come with a special representation of $\mathrm{PSL}_2(\mathbb{Z})$ given by their *modular data*:

$\mathcal{T}_{IJ} := \delta_{I,J}$ 

$\mathcal{S}_{IJ} :=$ 

.

Conformal invariance is equivalence to the $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$-matrix of multiplicities (Z_{IJ}) commuting with the modular data. We define a *modular invariant* to be any such non-negative integer matrix Z such that $Z_{\mathbf{1}\mathbf{1}} = 1$.

Motivating Example

The $\hat{\mathfrak{su}}(2)$ WZW model

In the $\hat{\mathfrak{su}}(2)$ WZW model the modular category of modules over the vertex operator algebra is given by

$$\mathcal{C} = \mathrm{Rep}_k(\hat{\mathfrak{su}}(2)) \quad \text{for some } k \in \mathbb{N}_{\geq 1}.$$

This is a semisimple category with $k+1$ simple objects which we denote

$$X_i \quad \text{for } i \in \{1, \ldots, k+1\}$$

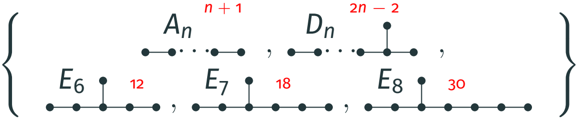
Furthermore the category is tensor generated by the single self dual object X_2 (the fundamental representation).

Motivating Example

Module Categories

Let $\mathcal{Q}$ a double Dynkin quiver of type A, D or E, i.e.

$\mathcal{Q} \in$



Assigning such a quiver with Coxeter number $k+2$ to a module category over $\mathcal{C}$ given by

$$\begin{aligned} \mathcal{M}_{\mathcal{Q}} \colon \mathcal{C} &\to \mathcal{Q}_0, \mathcal{Q}_0\text{-Bimod} = \mathrm{End}(\mathrm{Mod}\text{-}\mathcal{Q}_0) \\ X_2 &\mapsto \mathcal{Q}_1 \end{aligned}$$

gives a complete list of the irreducible finite semisimple module categories over $\mathcal{C}$. [EO04]

Motivating Example

Modular Invariants

The modular data of $\mathcal{C}$ is given by

$$\mathcal{S}_{ab} = (-1)^{a+b}\sqrt{\frac{2}{k+2}}\sin\left(\pi\frac{ab}{k+2}\right), \quad \mathcal{T}_{ab} = (-1)^{a-1}\exp\left(\pi i\frac{a^2}{2k+4}\right)\delta_{a,b}$$

i.e. the *Kac-Peterson matrices.* The corresponding modular invariants are given as partition functions below [CIZ87],

$$\mathcal{A}_{k+1} = \sum_{l=1}^{k+1} |\chi_l|^2 , \qquad \forall k \geq 1$$

$$\mathcal{D}_{\frac{k}{2}+2} = \sum_{l=1}^{k+1} \chi_l \, \chi^*_{\sigma^{l-1}l} , \qquad \text{whenever } \frac{k}{2} \text{ is even}$$

$$\mathcal{D}_{\frac{k}{2}+2} = |\chi_1 + \chi_{\sigma 1}|^2 + |\chi_3 + \chi_{\sigma 3}|^2 + \cdots + 2|\chi_{\frac{k}{2}}|^2 , \qquad \text{whenever } \frac{k}{2} \text{ is odd}$$

$$\mathcal{E}_6 = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2 , \qquad \text{for } k = 10$$

$$\mathcal{E}_7 = |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + \chi_9 (\chi_3 + \chi_{15})^* + (\chi_3 + \chi_{15}) \chi_9^* + |\chi_9|^2 , \qquad \text{for } k = 16$$

$$\mathcal{E}_8 = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2 , \qquad \text{for } k = 28.$$

Motivating Example

First Clue: the ADE Pattern

As the notation suggests, $\widehat{\mathfrak{su}}(2)$ modular invariants satisfy an ADE pattern.

$$|\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2$$

$\mathcal{E}_6$ double quiver

$\mathcal{E}_6$ modular invariant

Motivating Example

First Clue: the ADE Pattern

As the notation suggests, $\hat{\mathfrak{su}}(2)$ modular invariants satisfy an ADE pattern.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2$$

$\mathcal{E}_6$ adjacency matrix

$\mathcal{E}_6$ modular invariant

Motivating Example

First Clue: the ADE Pattern

As the notation suggests, $\hat{\mathfrak{su}}(2)$ modular invariants satisfy an ADE pattern.

$$\left\{-2\cos\left(\frac{\pi * l}{h}\right)\right\}$$
for $l = 1,4,5,7,8,11$

$$|\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2$$

$\mathcal{E}_6$ adjacency matrix eigenvalues

$\mathcal{E}_6$ modular invariant

Motivating Example

Second Clue: The Suggestion of a Trace

To illustrate we consider a different double Dynkin quiver:

$$\begin{aligned} \mathcal{M}_{\mathcal{Q}} \colon \mathcal{C} &\to \mathcal{Q}_0, \mathcal{Q}_0\text{-Bimod} \\ X_2 &\mapsto \mathcal{Q}_1 \end{aligned}$$

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{D}_4$ module category

$\mathcal{D}_4$ modular invariant

Motivating Example

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$$\begin{aligned} \mathcal{M}_{\mathcal{Q}} : \mathcal{C} &\to \mathcal{Q}_0, \mathcal{Q}_0\text{-Bimod} \\ X_2 &\mapsto \mathcal{Q}_1 \end{aligned}$$

$$T = \sum_{ij} Z_{ij}\, X_i \otimes X_j$$

$\downarrow$ trace

$$\begin{aligned} \operatorname{Tr} \mathcal{M}_{\mathcal{Q}} : \mathcal{C} &\to \underline{\text{Vect}} \\ X_2^n &\mapsto \langle n\text{-cycles in } \mathcal{Q} \rangle \end{aligned}$$

$$\begin{aligned} T^\sharp : \mathcal{C} &\to \underline{\text{Vect}} \\ X &\mapsto \operatorname{Hom}_{\mathcal{C}}(X, T) \end{aligned}$$

$\mathcal{D}_4$ module category

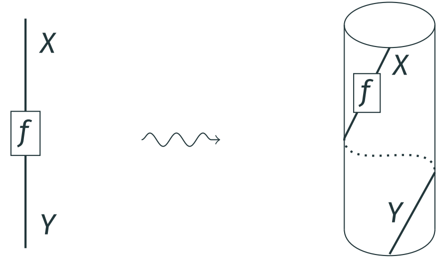
$\mathcal{D}_4$ modular invariant

$$\operatorname{Tr} \mathcal{M}_{\mathcal{Q}}(X_2^n) \cong T^\sharp(X_2^n)$$

The Tube Category

An Introduction

Let $\mathcal{C}$ be a spherical fusion category and let f be in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.



$\mathcal{TC}$ shares the same objects as $\mathcal{C}$ but has more morphisms. Morphisms in $\mathcal{TC}$ are described by diagrams in $\mathcal{C}$ drawn on a cylinder.

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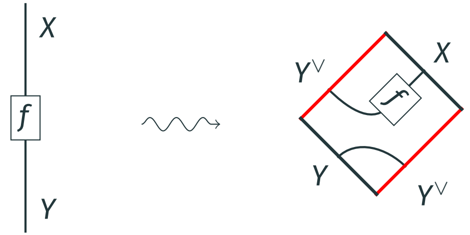

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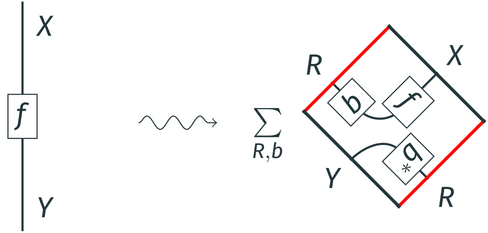


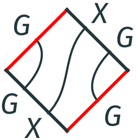
$$\mathrm{Hom}_{\mathcal{TC}}(X, Y) = \bigoplus_{R} \mathrm{Hom}_{\mathcal{C}}(RX, YR)$$

$\mathcal{TC}$ shares the same objects as $\mathcal{C}$ but has more morphisms. Morphisms in $\mathcal{TC}$ are described by diagrams in $\mathcal{C}$ drawn on a cylinder.

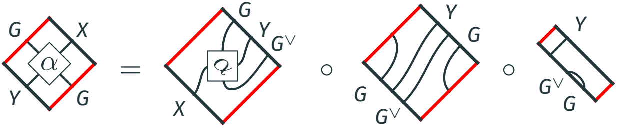
The Tube Category

Representations of $\mathcal{TC}$

We consider the following morphism in $\mathcal{TC}$

$c_{G,X} =$ 

Any morphism in $\mathcal{TC}$ may be re-written in the following form:



Therefore a representation $\mathcal{F} \colon \mathcal{TC} \to \underline{\text{Vect}}$ is simply a representation of $\mathcal{C}$ together with an action of the $c_{G,X}$ morphisms i.e. a collection of isomorphisms $\kappa_{G,X} \colon \mathcal{F}(GX) \to \mathcal{F}(XG)$ which are natural in G and X and satisfy $\kappa_{GH,X} = \kappa_{H,XG} \circ \kappa_{G,HX}$.

The Tube Category

Equivalence of Representations with $\mathcal{Z}(\mathcal{C})$

$\mathcal{Z}(\mathcal{C})$ is a category with objects (F,τ) where $F \in \mathcal{C}$ and $\tau\colon (-\otimes F) \to (F \otimes -)$ is a half-braiding i.e. satisfies:

$$\tau_{GH} = (\tau_G \otimes \mathrm{id}_H) \circ (\mathrm{id}_G \otimes \tau_H).$$

As $\mathcal{C}$ is fusion the Yoneda embedding $F \mapsto F^\sharp$ is an equivalence. This induces an equivalence $(F,\tau) \mapsto (F^\sharp, \tau^\sharp)$ between $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{RC})$ where

$$\begin{array}{ccc} \tau^\sharp\colon (-\otimes F)^\sharp & \to & (F \otimes -)^\sharp. \\ \| & & \| \\ F^\sharp \circ (G \otimes -) & & F^\sharp \circ (-\otimes G) \end{array}$$

The Tube Category

Equivalence of Representations with $\mathcal{Z}(\mathcal{C})$

$\mathcal{Z}(\mathcal{C})$ is a category with objects (F, τ) where $F \in \mathcal{C}$ and $\tau\colon (- \otimes F) \to (F \otimes -)$ is a half-braiding i.e. satisfies:

$$\tau^\sharp_{GH} = (\tau_G \otimes \mathrm{id}_H)^\sharp \circ (\mathrm{id}_G \otimes \tau_H)^\sharp.$$

$$\kappa_{GH,X} = \kappa_{H,XG} \circ \kappa_{G,HX}$$

As $\mathcal{C}$ is fusion the Yoneda embedding $F \mapsto F^\sharp$ is an equivalence. This induces an equivalence $(F, \tau) \mapsto (F^\sharp, \tau^\sharp)$ between $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{RC})$ where

$$\begin{array}{cccc} \tau^\sharp_{G^\vee}\colon & (G^\vee \otimes F)^\sharp & \to & (F \otimes G^\vee)^\sharp. \\ & \| & & \| \\ & F^\sharp \circ (G \otimes -) & & F^\sharp \circ (- \otimes G) \end{array}$$

$$\kappa_{G,X}\colon F^\sharp(GX) \to F^\sharp(XG)$$

An object in $\mathcal{Z}(\mathcal{C}) = \mathcal{Z}(\mathcal{RC})$ is an object in $\mathcal{RC}$ together with together with isomorphisms $\kappa_{G,X}$ as above, i.e. an object in $\mathcal{RTC}$.

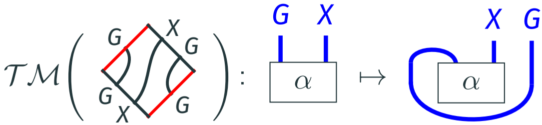
Extending the Trace

Construction

Let $\mathcal{M} \colon \mathcal{C} \to \mathcal{D}$ be a monoidal, pivotal functor. We consider the (contravariant) trace of $\mathcal{M}$ i.e. the object in $\mathcal{RC}$ given by

$$\begin{aligned} \operatorname{Tr}\mathcal{M} \colon \mathcal{C} &\to \underline{\mathrm{Vect}} \\ X &\mapsto \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}(X), \mathbf{1}_{\mathcal{D}}) \end{aligned}$$

To extend $\operatorname{Tr}\mathcal{M}$ to a representation of $\mathcal{TC}$, which we denote $\mathcal{TM}$, we have to specify the value of $\mathcal{TM}$ on the $c_{G,X}$ morphisms. Setting



defines a unique object in $\mathcal{RTC}$ (where the blue colour denotes evaluation under $\mathcal{M}$).

Extending the Trace

The Associated Modular Invariant

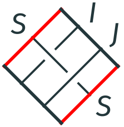
Let $\mathcal{M}\colon \mathcal{C} \to \mathrm{End}(\mathcal{B})$ be a module category which *induces a pivotal structure on its full image*. We may extend $\mathrm{Tr}\,\mathcal{M}$ to a representation of the tube category $\mathcal{TM}$.

$$\mathcal{TM} \in \mathcal{RTC} = \mathcal{Z}(\mathcal{C}) = \mathcal{C} \boxtimes \bar{\mathcal{C}}$$

when $\mathcal{C}$ is modular. $\mathcal{TM}$'s irreducible multiplicities therefore give an integer $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$-matrix $Z(\mathcal{TM})$. As $\mathcal{TC}$ admits a *complete set of primitive idempotents* these multiplicities may also be obtained by evaluating $\mathcal{TM}$ on these idempotents.

$$Z(\mathcal{TM})_{IJ} = \dim \mathrm{Im}\,\mathcal{TM}\left(\epsilon_I^J\right)$$

where

$$\epsilon_I^J = \frac{1}{d(\mathcal{C})} \bigoplus_S d(S)$$  $$\in \mathrm{End}_{\mathcal{TC}}(XY).$$

Extending the Trace

Prior Approches to Relating Module Categories and Modular Invariants

- **α-induction** (1998) — *Böckenhauer, Evans, Kawahigashizk*
  - ⋆ Stated and proved within an operator algebraic framework [BE98].

- **Full Centre Construction** (2006) — *Runkel, Fjelstad, Fuchs, Schweigert*
  - ⋆ Any module category may be realised (non uniquely) as

$$\mathcal{M} : \mathcal{C} \to \mathrm{End}(\mathrm{Mod}\text{-}A)$$
$$X \mapsto - \otimes X$$

    where A is an algebra object in $\mathcal{C}$.
  - ⋆ The *full centre* of A is an object in $\mathcal{Z}(\mathcal{C})$. Under certain assumptions on A its full centre will be a modular invariant [RFFS07].

Motivating Example

Decategorification of the Module Category

We start by recalling the A-D-E pattern appearing in the classification of $\hat{\mathfrak{su}}(2)$ modular invariants.

$$\mathcal{M}_{\mathcal{Q}} \colon \mathcal{C} \to \mathrm{End}(\mathcal{B}) \qquad \text{where } \mathcal{B} = \text{Mod-}\mathcal{Q}_0$$

$$\rightsquigarrow$$

$$R_1 \colon \mathcal{K}_{\mathbb{C}}(\mathcal{C}) \to \mathrm{End}(\mathcal{K}_{\mathbb{C}}(\mathcal{B}))$$

where $\mathcal{K}(\mathcal{C})$ denotes the Grothendieck ring of $\mathcal{C}$ and $\mathcal{K}_{\mathbb{C}}(\mathcal{C})$ denotes $\mathcal{K}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$.

The A-D-E pattern describes how the diagonal entries to the modular invariant encode the weights of R_1.

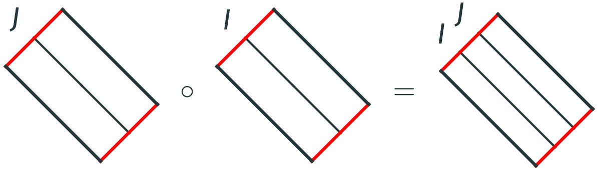
The $\mathcal{TM}$ construction reveals another representation of $\mathcal{K}_{\mathbb{C}}(\mathcal{C})$!

The Tube Category

Recovering the (complexified) Grothendieck Ring

$\mathrm{End}_{\mathcal{TC}}(\mathbf{1})$ and $\mathcal{K}_{\mathbb{C}}(\mathcal{C})$ are canonically isomorphic algebras.

- $\mathrm{End}_{\mathcal{TC}}(\mathbf{1}) = \bigoplus_S \mathrm{End}(S) = \bigoplus_S \mathbb{C}$
- composition in $\mathrm{End}_{\mathcal{TC}}(\mathbf{1})$ corresponds to the tensor product in $\mathcal{K}_{\mathbb{C}}(\mathcal{C})$.



We therefore obtain another representation of $\mathcal{K}_{\mathbb{C}}(\mathcal{C})$,

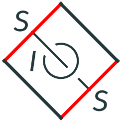
$$R_2 \colon \mathcal{K}_{\mathbb{C}}(\mathcal{C}) \to \mathrm{End}(\mathcal{TM}(\mathbf{1}))$$

Theorem. R_1 and R_2 are isomorphic representations of $\mathcal{K}_{\mathbb{C}}(\mathcal{C})$.

Motivating Example

The Conclusion

As $\mathcal{TC}$ admits a complete set of primitive idempotents, $\mathcal{K}_{\mathbb{C}}(\mathcal{C}) = \mathrm{End}_{\mathcal{TC}}(\mathbf{1})$ is a semisimple commutative algebra generated by the primitive idempotents of $\mathbf{1}$:

$$\mathbf{1}_I := \frac{1}{d(I)d(\mathcal{C})} \sum_S d(S)$$


$$\text{for } I \in \mathrm{Irr}(\mathcal{C}).$$

Therefore the weight spaces of R_2 are given by $\mathcal{TM}(\mathbf{1}_I)$ for $I \in \mathrm{Irr}(\mathcal{C})$. However, $(\mathbf{1}, \mathbf{1}_I)$ and $(II^\vee, \epsilon_I^{I\vee})$ are isomorphic idempotent objects in the Karoubi envelope and so the diagonal entries to the modular invariant are precisely $\mathcal{TM}(\mathbf{1}_I)$ for $I \in \mathrm{Irr}(\mathcal{C})$i.e. the dimension of the corresponding weight space.

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