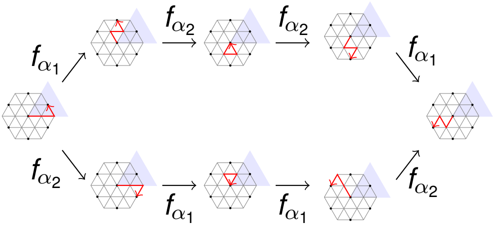
An Introduction to the Littelmann Path Model

Leonard Hardiman



Motivation

Let:

- $\mathfrak{g}$ be a finite dimentional semisimple complex Lie algebra
- $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$
- $R \subseteq \mathfrak{h}^*$ be the set of positive simple roots of $\mathfrak{g}$ (with respect to some choice of Weyl chamber)
- $\lambda \in \mathfrak{h}^*$ be a dominant weight
- V^λ be the unique irreducible representation of $\mathfrak{g}$ with highest weight λ
- $\mathfrak{W}$ is the Weyl group of $\mathfrak{g}$.

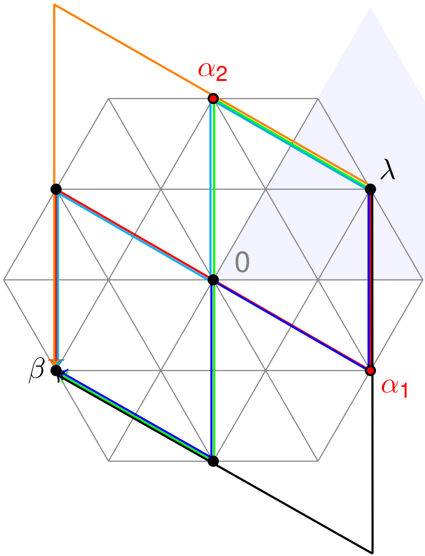
For $\alpha \in R$ let X_α be a non-zero element of $\mathfrak{g}_{-\alpha}$ and let v be a highest weight vector. Then

$$V^\lambda = \text{span}(\{X_{\alpha_1} \circ \ldots \circ X_{\alpha_r}(v) \mid r \in \mathbb{N}, \alpha_i \in R\})$$

and in particular, for any weight space V_β, we have

$$V^\lambda_\beta = \text{span}\left(\left\{X_{\alpha_1} \circ \ldots \circ X_{\alpha_r}(v) \middle| r \in \mathbb{N}, \alpha_i \in R, \lambda - \sum_i \alpha_i = \beta\right\}\right)$$

Illustration: Ajoint Representation of $\mathfrak{sl}_3$



$\rightarrow$: $X_{\alpha_2} \circ X_{\alpha_1} \circ X_{\alpha_1} \circ X_{\alpha_2}$

$\rightarrow$: $X_{\alpha_1} \circ X_{\alpha_2} \circ X_{\alpha_1} \circ X_{\alpha_2}$

$\rightarrow$: $X_{\alpha_1} \circ X_{\alpha_2} \circ X_{\alpha_2} \circ X_{\alpha_1}$

$\rightarrow$: $X_{\alpha_2} \circ X_{\alpha_1} \circ X_{\alpha_2} \circ X_{\alpha_1}$

$\rightarrow$: $X_{\alpha_2} \circ X_{\alpha_2} \circ X_{\alpha_1} \circ X_{\alpha_1}$

$\rightarrow$: $X_{\alpha_1} \circ X_{\alpha_1} \circ X_{\alpha_2} \circ X_{\alpha_2}$

Which of the images (of v) are 0 and can we find a set of images that form a basis of V^{λ}_{β}?

Motivation

The Weyl character formula does not solve this problem. Weyl's formula can be seen as a reformulation of Kostant's multiplicity formula which states that

$$\dim V_\beta^\lambda = \sum_{\mathfrak{w} \in \mathfrak{W}} (-1)^{\mathfrak{w}} P(\mathfrak{w}(\lambda + \rho) - (\beta + \rho))$$

where $P(\beta)$ is the number of ways to write β as a sum of positive roots.

The Littelmann path model sets out to find a character formula that *doesn't overcount* i.e. is not an alternating sum.

Path Operators

Let $\mathbb{E}$ be a Euclidean space, let α be a vector in $\mathbb{E}$ and let Π be the set of paths, $\pi : [0, 1] \to \mathbb{E}$ in $\mathbb{E}$ such that $\pi(0) = 0$.

We shall define a path operator, denoted f_α, that reflects intervals of π in the hyperplane orthogonal to α, so as to move the endpoint of π by $-\alpha$. However the operator should only reflect intervals, $\pi([t_1, t_2])$, that satisfy

$$(\pi(t), \alpha) = \min_{s \in [t,1]} (\pi(s), \alpha) \quad \forall t \in [t_1, t_2]$$

If this is not possible the operator returns the special element θ which is not a path and satisfies the abstract property $f_\alpha(\theta) = \theta \quad \forall \alpha$.

Path Operators

Checking whether or not the endpoint of π can be moved by $-\alpha$ through reflecting sections that satisfy the previously mentioned condition comes down to checking that

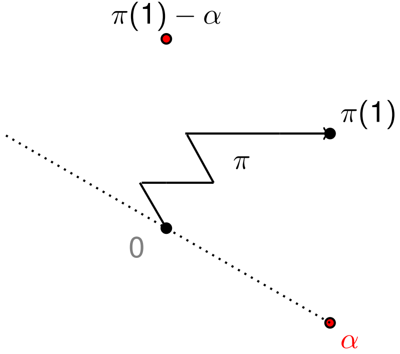
$$(\pi(1), \alpha) - \min_{t \in [0,1]} (\pi(t), \alpha) \geq \frac{\|\alpha\|^2}{2}$$

If this inequality holds then f_α reflects any interval, $\pi([t_1, t_2])$, that satisfies

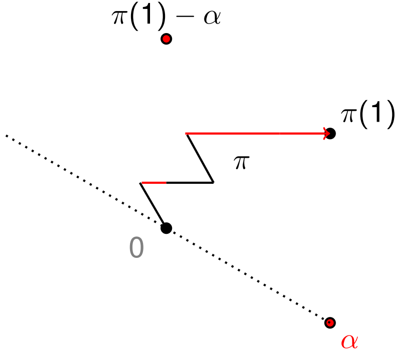
$$(\pi(t), \alpha) = \min_{s \in [t,1]} (\pi(s), \alpha) \quad \forall t \in [t_1, t_2]$$

As soon as the path has been shifted by $-\alpha$ the operator stops reflecting sections.

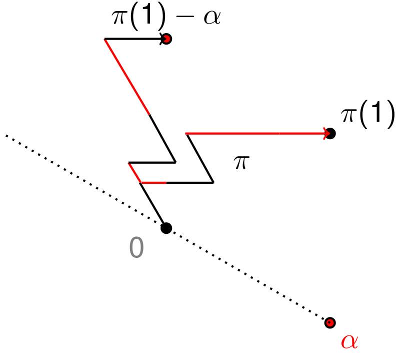
An Example of a Path Operator



An Example of a Path Operator



An Example of a Path Operator



Path Operators

In an similar way one can define another path operator, denoted e_α that moves the endpoint by $+\alpha$ and can only reflect intervals, $\pi([t_1, t_2])$, that satisfy

$$(\pi(t), \alpha) = \min_{s \in [0,t]} (\pi(s), \alpha) \quad \forall t \in [t_1, t_2]$$

This operator also satisfies

$$e_\alpha \circ f_\alpha(\pi) = \pi \quad \text{whenever } f_\alpha \pi \neq \theta$$

and

$$f_\alpha \circ e_\alpha(\pi) = \pi \quad \text{whenever } e_\alpha \pi \neq \theta$$

The Littelmann Path Model

The Littelmann path model is constructed by setting

$$\mathbb{E} = \text{The root system of } \mathfrak{g} \text{ equipped with the Killing form}$$

and considering the, so called, root operators

$$\{f_\alpha, e_\alpha \mid \alpha \in R\}$$

We also restrict our interest to the set, denoted Π, of piecewise linear paths that start in 0 and end in an integral weight. Also, for η an integral weight we denote by $\underline{\eta}$ the straight path that starts at 0 and ends at η.

The Littelmann Path Model

Let $\mathbb{B}$ be a subset of Π. The character of such a subset is given by

$$\mathrm{Char}\,\mathbb{B} := \sum_{\nu \in \mathbb{B}} e^{\nu(1)} \in \mathbb{Z}[\Lambda_W]$$

We note that if $\mathbb{B} \cup \{\theta\}$ is stable under the root operators then $\mathrm{Char}\,\mathbb{B}$ is stable under the Weyl group as

$$s_\alpha \pi := \begin{cases} f_\alpha^k \pi & \text{if } k > 0 \\ e_\alpha^{-k} \pi & \text{otherwise.} \end{cases}$$

(where $k = 2(\pi(1), \alpha)/\|\alpha\|^2$) satisfies

$$s_\alpha^2 = \mathrm{id}$$

and

$$s_\alpha \pi(1) = \mathfrak{w}(\pi(1))$$

where $\mathfrak{w} \in \mathfrak{W}$ is the element of the Weyl group associated with α.

The Path Character Formula

Let $\mathbb{B} \subset \Pi$ be such that $\mathbb{B} \cup \{\theta\}$ is stable under the root operators. Then, from the Weyl character formula, one can deduce the following:

Proposition

$$\mathrm{Char}\, \mathbb{B} = \sum_{\substack{\pi \in \mathbb{B} \\ \underline{\rho} * \pi \in \Pi_0^+}} \mathrm{Char}\, V^{\pi(1)}$$

where Π_0^+ is set of elements of Π who's images are contained in the interior of the dominant Weyl chamber, $V^{\pi(1)}$ denotes the irreducible representation of $\mathfrak{g}$ with highest weight $\pi(1)$ and ρ is the half sum of the positive roots.

Let $\pi \in \Pi$ be such that $\underline{\rho} * \pi \in \Pi_0^+$. Bearing in mind this proposition we are interested in finding $\mathbb{B} \subset \Pi$ such that $\mathbb{B} \cup \{\theta\}$ is stable under the root operators and

$$\{\eta \in \mathbb{B} \mid \underline{\rho} * \eta \in \Pi_0^+\} = \{\pi\}$$

Fundamental Theorem of the Littelmann Path Model

For a given path $\pi \in \Pi$ we consider the smallest set $\mathbb{B} \subset \Pi$ such that $\pi \in \mathbb{B}$ and $\mathbb{B} \cup \{\theta\}$ is stable under the root operators. We denote this set by $\mathbb{B}_\pi$.

Theorem

Let $\pi \in \Pi$ be such that $\underline{\rho} * \pi \in \Pi_0^+$. Then

$$\{\eta \in \mathbb{B}_\pi \mid \underline{\rho} * \eta \in \Pi_0^+\} = \{\pi\}$$

Combining this with the path character formula gives:

Corollary

Let $\pi \in \Pi$ be such that $\underline{\rho} * \pi \in \Pi_0^+$. Then

$$\mathrm{Char}\, V^{\pi(1)} = \mathrm{Char}\, \mathbb{B}_\pi$$

Remarks on the Fundamental Theorem

- When we evalute our character at a weight $\beta \in \mathfrak{h}^*$,

$$\mathrm{Char}\,\mathbb{B}_\pi(\beta) = \sum_{\substack{\eta \in \mathbb{B}_\pi \\ \eta(1)=\beta}} 1$$

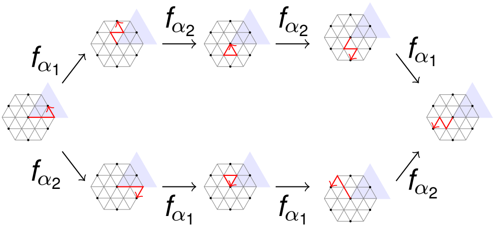
  we get a *non-alternating* sum.
- For a given dominant weight λ, for every $\pi \in \Pi$ such that $\underline{\rho} * \pi \in \Pi_0^+$ and $\pi(1) = \lambda$, we have an (*a priori*) new combinatorial model for V.

What do these models have in common?

The Littelmann Graphs

Let $\pi \in \Pi$ be such that $\underline{\rho} * \pi \in \Pi_0^+$. We construct the *Littelmann graph* of π, denoted $\mathcal{G}_\pi$, in the following way:

- The vertices of $\mathcal{G}_\pi$ are the elements of $\mathbb{B}_\pi$.
- We draw an arrow with colour α between $\eta, \eta' \in \mathbb{B}_\pi$ if $f_\alpha(\eta) = \eta'$.



Theorem

Let $\pi_1, \pi_2 \in \Pi$ be such that $\underline{\rho} * \pi_1, \underline{\rho} * \pi_2 \in \Pi_0^+$ and $\pi_1(1) = \pi_2(1)$. Then

$$\mathcal{G}_{\pi_1} \cong \mathcal{G}_{\pi_2}$$

Generalized Littelwood-Richardson Rule

Let λ and μ be dominant weights and let π_1 and π_2 be such that $\underline{\rho} * \pi_1, \underline{\rho} * \pi_2 \in \Pi_0^+$ and $\pi_1(1) = \lambda$ and $\pi_2(1) = \mu$. We have

$$\begin{aligned}
&\mathrm{Char}\ V^{\lambda} \otimes V^{\mu} \\
&= \mathrm{Char}\ V^{\pi_1(1)} \otimes V^{\pi_2(1)} \\
&= \mathrm{Char}\ V^{\pi_1(1)}\ \mathrm{Char}\ V^{\pi_2(1)} \\
&= \mathrm{Char}\, \mathbb{B}_{\pi_1}\ \mathrm{Char}\, \mathbb{B}_{\pi_2} \\
&= \mathrm{Char}\, \mathbb{B}_{\pi_1} * \mathbb{B}_{\pi_2} \\
&= \sum_{\substack{\eta \in \mathbb{B}_{\pi_2} \\ \underline{\rho} * \pi_1 * \eta \in \Pi_0^+}} \mathrm{Char}\ V^{\lambda + \eta(1)}
\end{aligned}$$

Therefore

$$V^{\lambda} \otimes V^{\mu} \cong \bigoplus_{\substack{\eta \in \mathbb{B}_{\pi_2} \\ \underline{\rho} * \pi_1 * \eta \in \Pi_0^+}} V^{\lambda + \eta(1)}$$

Young Tableaux Theory

One can hope that any reasonable indexing set for a basis of V is in natural bijection with $\mathbb{B}_\pi$ for some choice of π.

To illustrate this we now demonstrate how the Littelmann path model recovers Young tableaux theory. In this context $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{h} = \{x \in \mathfrak{g} \mid X \text{ diagonal}\}$.

We start by considering an identification between certain paths in $\mathfrak{h}^*$ and Young tableaux. Let $L_i \in \mathfrak{h}^*$ be the projection of a diagonal matrix onto it's ith entry. For a given tableau $\mathbb{T}$ let $(i_1, ..., i_N)$ be the entries of the boxes, where we read the entries columnise (from top to bottom, right to left). We associate to $\mathbb{T}$ the path $\pi_{\mathbb{T}} := \underline{L_{i_1}} * ... * \underline{L_{i_n}}$.

1	3	2	2	1
1	1	3	3	
2	1			

$\sim \underline{L_1} * \underline{L_2} * \underline{L_3} * \underline{L_2} * \underline{L_3} * \underline{L_3} * \underline{L_1} * \underline{L_1} * \underline{L_1} * \underline{L_1} * \underline{L_2}$

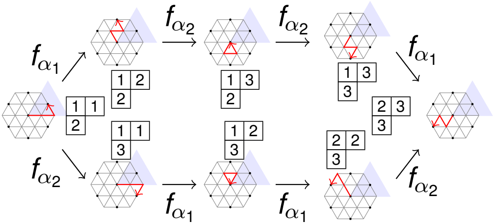
Young Tableaux Theory

Let $p = (a_1, ..., a_n)$ be a partition and let $\mathbb{T}_0$ be the Young tableau of shape p having only 1's as entry in the first row, 2's in the second row etc. Then $\pi_{\mathbb{T}_0}$ satisfies $\underline{\rho} * \pi_{\mathbb{T}_0} \in \Pi_0^+$ and therefore

$$\text{Char}\, \mathbb{B}_{\pi_{\mathbb{T}_0}} = \text{Char}\, V^{\pi_{\mathbb{T}_0}(1)} = \text{Char}\, V^p$$

One can then check that

$$\mathbb{B}_{\pi_{\mathbb{T}_0}} = \{\pi_{\mathbb{T}} \mid \mathbb{T} \text{ semistandard Young tableau of shape } p\}$$



END

Defining Properties of the Root Operators

The root operators satisfy the following properties (under the assumption that their image is not θ):

(A) $\forall \pi \in \Pi, \quad e_\alpha \pi, f_\alpha \pi \in \Pi$

(B) $\forall \pi \in \Pi, \quad e_\alpha \pi(1) = \pi(1) + \alpha$ and $f_\alpha \pi(1) = \pi(1) - \alpha$.

(C) e_α and f_α preserve length

(D) $\forall \pi \in \Pi, \forall k \in \mathbb{Z}, \quad k \cdot e_\alpha(\pi) = e_\alpha^k(k \cdot \pi)$ and $k \cdot f_\alpha(\pi) = f_\alpha^k(k \cdot \pi)$

(E) Let π be in Π, let n be maximal such that $f_\alpha^n(\pi) \neq \theta$ and let m be maximal such that $e_\alpha^m(\pi) \neq \theta$. Then $n - m = 2\frac{(\pi(1),\alpha)}{(\alpha,\alpha)}$

(F) e_α and f_α are continuous

Fundamental Property of Root Operators

The following can be considered the fundamental property of the root operators:

(G) Let $\pi := \pi_1 * ... * \pi_r$ be such that $\pi_i \in \Pi$ and set

$$a_0 = 0 \text{ and } a_i := (\pi_1 * ... * \pi_i(1), \alpha) \quad \forall i \geq 1$$

Let i_0 be minimal such that a_{i_0} is minimal and let i_1 be maximal such that a_{i_1} is minimal. Then

$$e_\alpha \pi = \begin{cases} \theta & \text{if } i_0 = 0 \\ \pi_1 * ... * e_\alpha \pi_{i_0} * ... * \pi_r & \text{else} \end{cases}$$

and

$$f_\alpha \pi = \begin{cases} \theta & \text{if } i_1 = r \\ \pi_1 * ... * f_\alpha \pi_{i_1+1} * ... * \pi_r & \text{else} \end{cases}$$

Proposition

If $\{f'_\alpha, e'_\alpha \mid \alpha \in R\}$ is a set of maps $\Pi \to \Pi \cup \theta$ that satisfies properties (A) to (G) then $f_\alpha = f'_\alpha$ and $e_\alpha = e'_\alpha \ \forall \alpha \in R$.