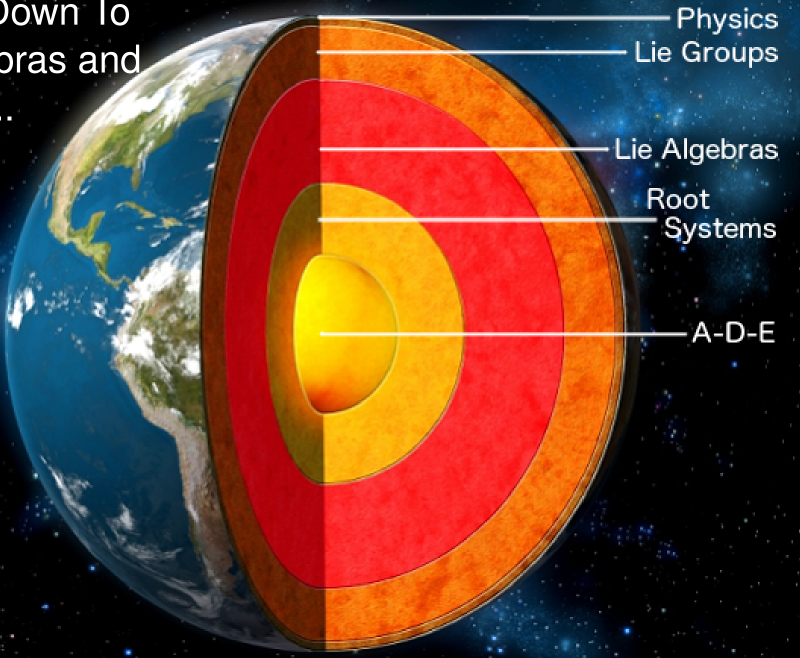
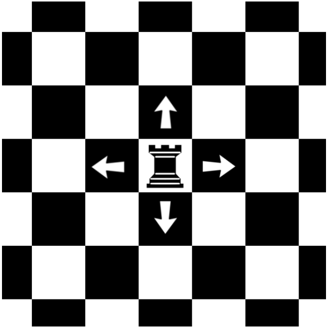
Lie Algebras and ADE-ology

Leonard Hardiman

Drilling Down To Lie Algebras and Beyond...



Discrete and Continuous Symmetries



Discrete Symmetry

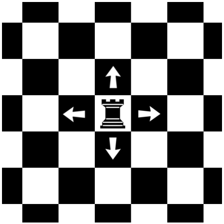
Continuous Symmetry

Topological Groups

In mathematical formalisations of physics,

- Symmetry is modelled by the action of a *group* on a *vector space*
- Continuity is described by a set equipped with a *topology* (a notion of proximity)

It is therefore natural that *continuous symmetry* is modelled by the action of a (connected) *topological group* on a vector space.



$G \cong \mathbb{Z}^2$ (discrete group)

$G \cong \mathbb{R}/\mathbb{Z}$ (continuous group)

Lie Groups

where $\theta \in \mathbb{R}$.

$$f : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$$
$$\theta \mapsto [\theta]_1$$

is a local isomorphism.

We are interested in the case when G is locally isomorphic to $\mathbb{K}^n$ for some $n \in \mathbb{N}$. Such a group is called a *topological group manifold*.

Once equipped with a differential structure the group is called a *Lie group*.

Representations of Lie Groups

An action of a Lie group, G, on a vector space, V, is equivalent to a morphism (i.e. a smooth group homomorphism)

$$\rho : G \to \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ is the *general linear group* of V. Such a ρ is called a *representation* of G.

It turns out that ρ is determined by its differential at the identity

$$\rho : T_e G \to T_e GL(V) \cong \mathfrak{gl}(V)$$

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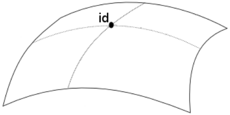

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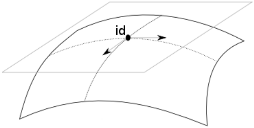
ρ : 

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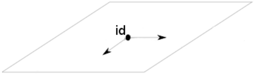
$d\rho_e :$ 

$\longrightarrow \quad T_e\,\mathrm{GL}(V)$

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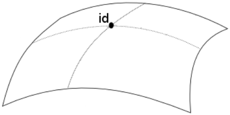
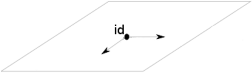
We have seen how to translate a map

$\rho :$ 

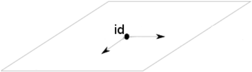
$\longrightarrow \mathfrak{gl}(V)$

into a *linear* map

$d\rho_e :$ 

$\longrightarrow \mathfrak{gl}(V)$

but of *all* the maps

$f :$ 

$\longrightarrow \mathfrak{gl}(V)$

which ones come from this procedure?

Lie Algebras

We consider the map

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(T_e G) \\ g &\mapsto d\Big(h \mapsto ghg^{-1}\Big)_e \end{aligned}$$

we differentiate at the identity to get a map

$$\mathrm{ad} : T_e G \to \mathrm{End}(T_e G)$$

finally we define a bracket on $T_e G$ by setting

$$[X, Y] := \mathrm{ad}(X)(Y)$$

In general any vector space equipped with such a bracket is called a *Lie algebra* (often denoted $\mathfrak{g}$).

Representations of Lie Algebras

One can check that, for any representation of a Lie group, ρ, the differential satisfies

$$d\rho([X, Y]) = [d\rho(X), d\rho(Y)] \tag{1}$$

In general a map from any Lie algebra,

$$\phi : \mathfrak{g} \to \mathrm{GL}(V)$$

is called a representation of $\mathfrak{g}$ if it satisfies (1).

A Lie algebra is called simple if it has no proper ideals (an ideal is a subspace I such that $[\mathfrak{g}, I] \subset I$). Just as finite simple groups are the 'building blocks' of finite groups, so simple Lie algebra are the 'building blocks' of Lie algebras.

From now on we shall consider only representations of *simple* Lie algebras.

Cartan Subalgebra

We recall that two commuting diagonalizable matrices are *simultaneously* diagonalizable.

This motivates the study of subalgebras of $\mathfrak{g}$ that

- are diagonalizable (action is diagonalizable w.r.t. any representation)
- are abelian (action is *simultaneously* diagonalizable w.r.t. any representation)

A subalgebra that is maximal w.r.t. these properties is called a *Cartan subalgebra* (often denoted $\mathfrak{h}$).

Every simple Lie algebra posses a unique (up to conjugation) Cartan subalgebra.

Weights of a Representation

We start with a representation of $\mathfrak{g}$,

$$\phi : \mathfrak{g} \to \mathrm{GL}(V)$$

and we consider it's restriction to $\mathfrak{h}$ (a Cartan subalgebra),

$$\phi : \mathfrak{h} \to \mathrm{GL}(V)$$

By the properties of $\mathfrak{h}$, V admits the decomposition,

$$V = \bigoplus_{\beta \in W_\phi} V_\beta$$

where $V_\beta = \{v \in V \mid \phi(H)(v) = \beta(H) \cdot v \; \forall H \in \mathfrak{h}\}$ and $W_\phi \subset \mathfrak{h}^*$ is called the set of *weights* of ϕ.

Root Systems

One representation of $\mathfrak{g}$ is extremely important: the *adjoint representation*,

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathrm{GL}(\mathfrak{g}) \\ X &\mapsto (Y \mapsto [X, Y]) \end{aligned}$$

The subspace of $\mathfrak{h}^*$ spanned by the weights of $\mathtt{ad}$ (called roots) contains the weights of *all* the representations of $\mathfrak{g}$. This subspace is called the *root system*.

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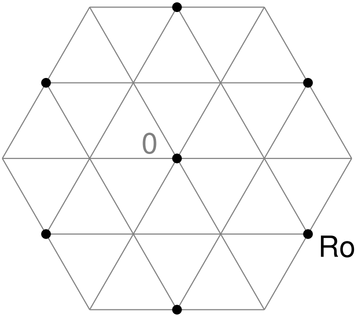

Root System of $\mathfrak{sl}_3$

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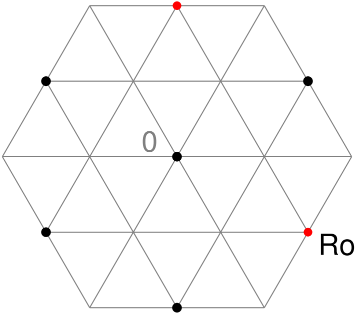
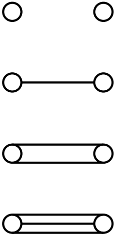


Root System of $\mathfrak{sl}_3$

Dynkin Diagrams

It turns out that a root system is determined by the angles between it's positive simple roots.

To every root system we associate a diagram that tracks these angles. We draw one node for each positive simple root. Then for every pair of nodes we draw a certain number of parallel lines between them, this number is determined by the angle:



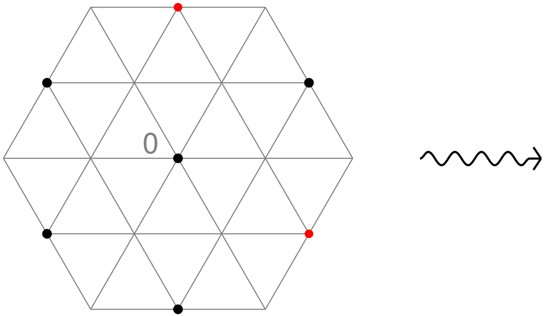
$$\theta = \frac{\pi}{2}$$

$$\theta = \frac{2\pi}{3}$$

$$\theta = \frac{3\pi}{4}$$

$$\theta = \frac{5\pi}{6}$$

Dynkin Diagrams



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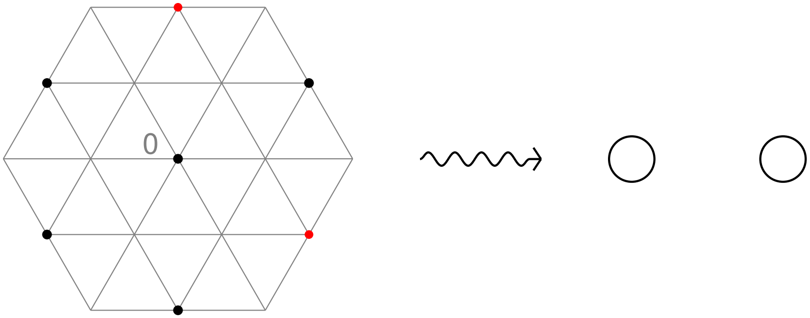

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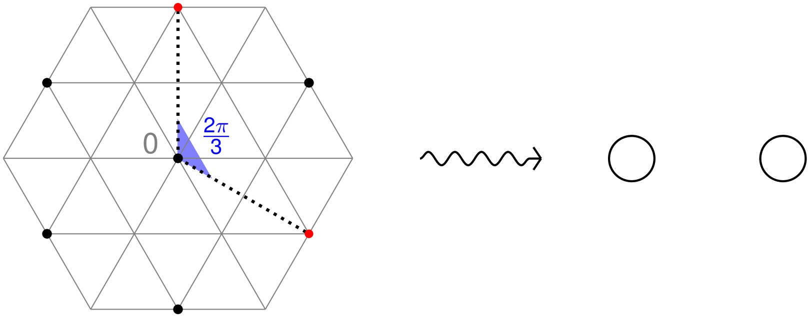

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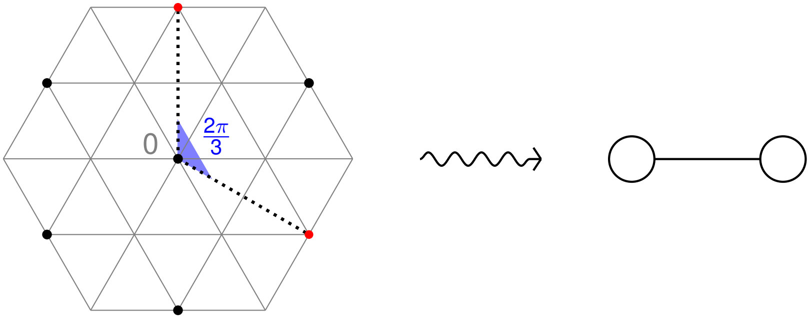
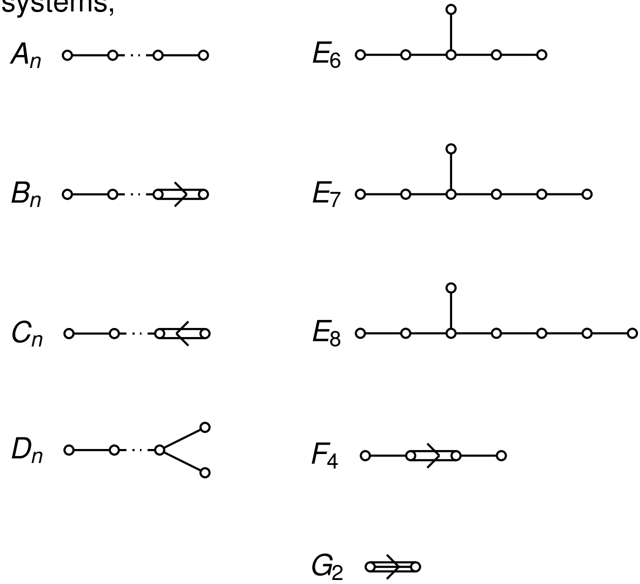


Classification of Complex Simple Lie Algebras

The following diagrams are the only possibile Dynkin Diagrams of root systems,



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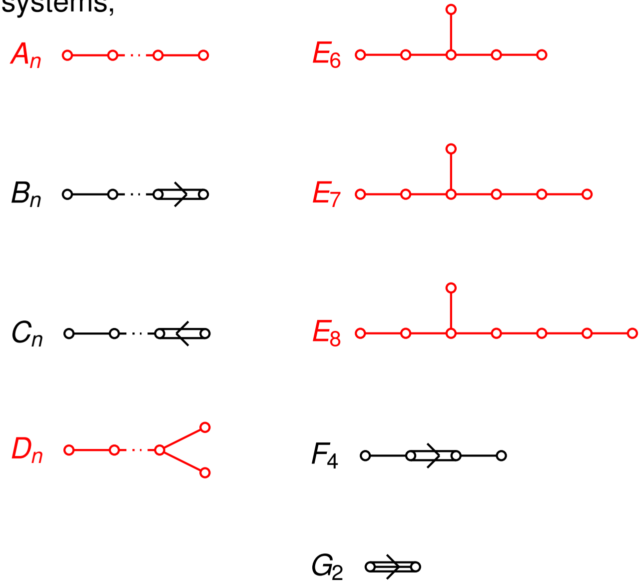

The A-D-E metapattern

Occurrences of the ADE scheme are found in:

- Simply Laced Complex Simple Lie Algebras (Killing and Cartan 1888-94);
- Finite proper subgroups of $\mathrm{SL}_2(\mathbb{C})$, and the associated Platonic solids (Klein 1884);
- Kleinian singularities (Slodowy 1983);
- Subfactors of finite index (Jones 1983);
- Physical invariants of WZW models associated to $\mathfrak{sl}_2$ (Cappelli, Itzykson and Zuber 1987)
- Finite reflection groups of crystallographic and of simply-laced type (Bourbaki 1981, Humphreys 1990);
- Finite type quivers (Gabriel 1972);
- Algebraic solutions to the hypergeometric equation (Hille 1976);

James Green's PSS (19/03/15)
Alex Collins's PSS (30/10/12)

My PhD?