Classifying Conformal Field Theories

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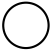
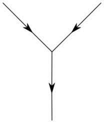
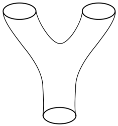
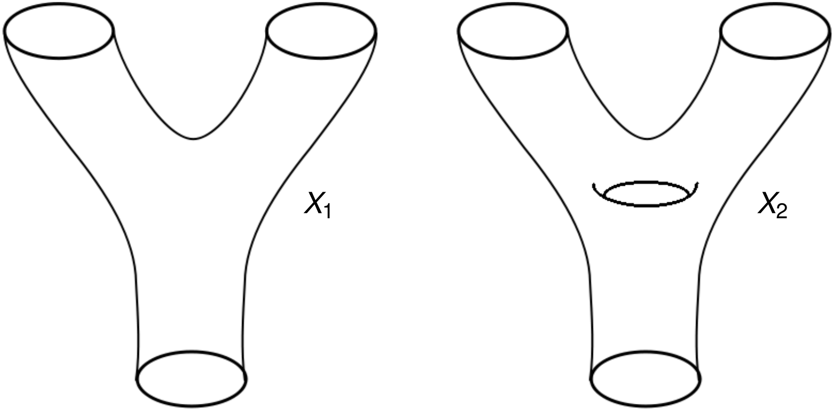
String Theory

String theory replaces the *point-like particles* of particle physics with one-dimensional objects called *strings*.

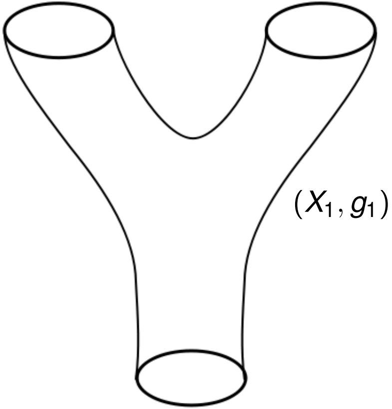
Quantum Field Theory	String Theory
	
	

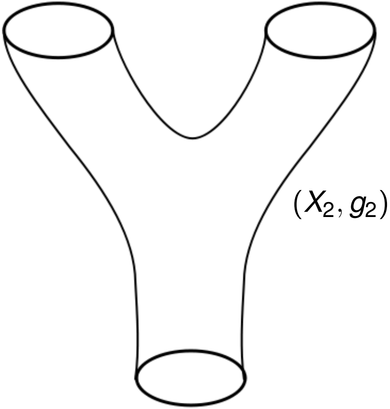
Topological Quantum Field Theories



Setting $X_1 \sim X_2$ if X_1 is topologically equivalent to X_2 would lead us towards a *topological quantum field theory*.

Conformal Quantum Field Theories

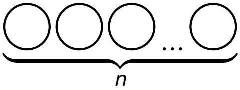




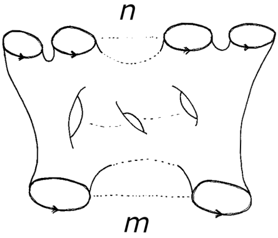
Setting $(X_1, g_1) \sim (X_2, g_2)$ if (X_1, g_1) is conformally equivalent to (X_2, g_2) leads us towards a *conformal field theory*.

Segal's Category

$$\mathrm{Obj}(\mathcal{C}) = \left\{ \underline{n} := \coprod_{n} S_1 \;\middle|\; n \in \mathbb{N} \right\}$$

$\underline{n} =$ 

$\mathrm{Hom}(\underline{n}, \underline{m}) = \{[X] \mid$ X is a Rimannian Manifold with n incoming boundaries and m outgoing boundaries.$\}$



Definition of a Conformal Field Theory

Let $H \in \mathbf{VECT}$ be a Hilbert space. Then a conformal field theory on H is a monoidal functor

$$\mathcal{U} : \mathcal{C} \to \mathbf{VECT}$$

that sends $\underline{1}$ to H (and satisfies certain other axioms).

Remark

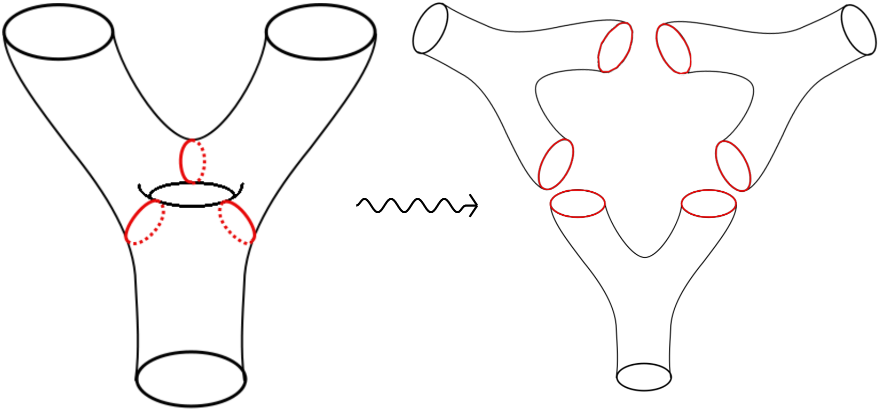
The fact that $\mathcal{U}$ is monoidal implies $\mathcal{U}(\underline{n}) = H^{\otimes n}$. Therefore $X \in \mathrm{Hom}(\underline{n}, \underline{m})$ is sent onto a linear map from $H^{\otimes n}$ to $H^{\otimes m}$.

An important question in the field is the classification of possible conformal field theories. In other words how much data do we need to construct such a functor?

The answer can be summed up via the following mnemonic:

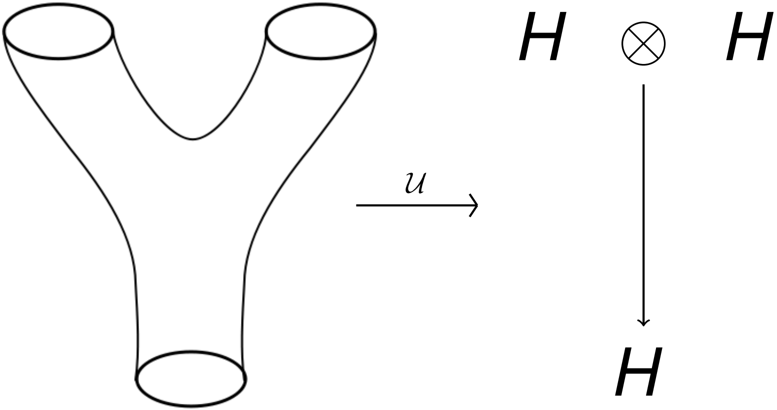
Donut Pants

Pants (i.e. Trousers)



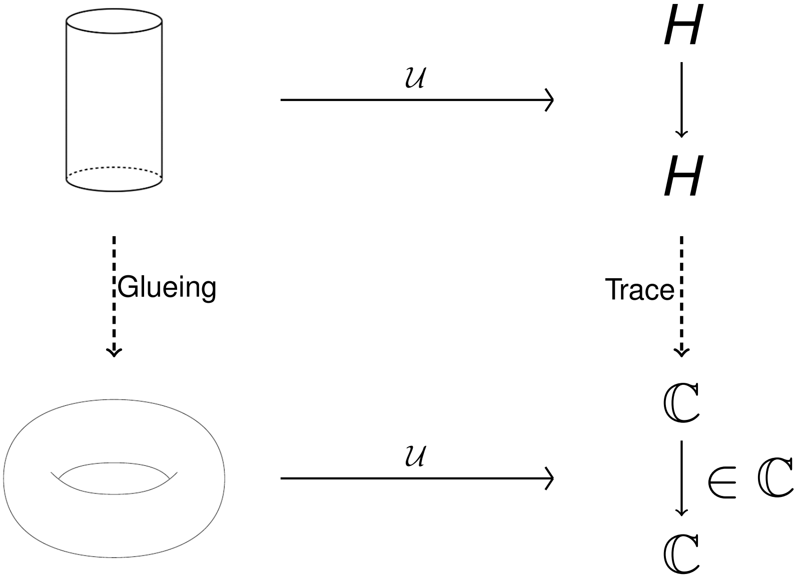
All elements in $\mathrm{Hom}(\underline{n}, \underline{m})$ can be cut up into *pairs of pants*.

Pants (i.e. Trousers)

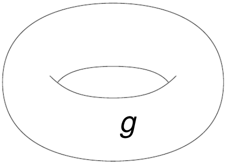


Under our functor $\mathcal{U}$ a pair of pants (equipped with a conformal structure) is sent onto a linear map $H \otimes H \to H$. The Hilbert space H is therefore equipped with an infinite sequence of inner products, this gives rise to a V.O.A.

Donuts (i.e. Glueing)



Modularity of the Partition Function

$\mathcal{Z}$: 

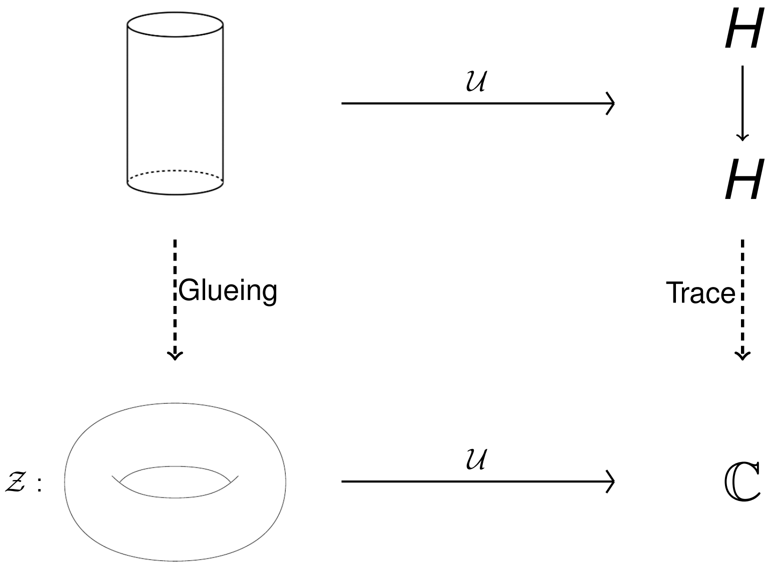
$\xrightarrow{\mathcal{U}}$ $\mathbb{C}$

Conformal structures on the torus are parametrized by $\tau \in \mathbb{C}^+/\mathrm{PSL}_2(\mathbb{Z})$.

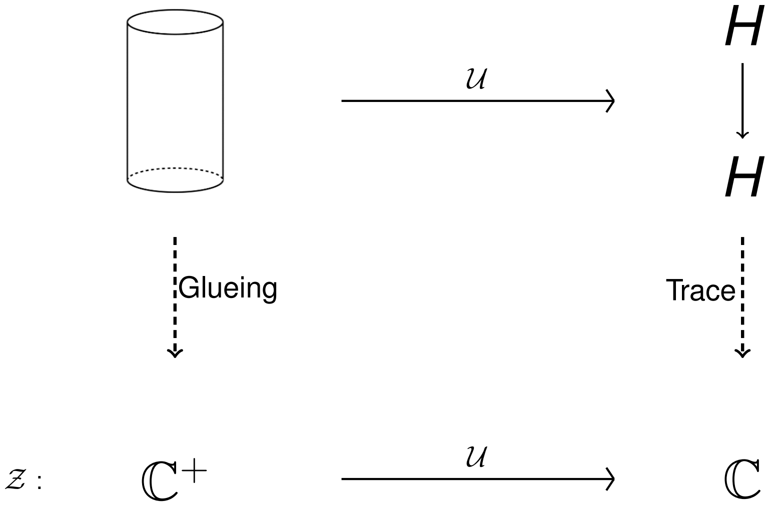
$$\mathcal{Z} : \quad \mathbb{C}^+ \xrightarrow{\mathcal{U}} \mathbb{C}$$

and $\mathcal{Z}$ is a *modular* function (i.e. invariant under the action of $\mathrm{PSL}_2(\mathbb{Z})$).

WZW Models



WZW Models



H
H
Glueing
Trace
$\mathcal{Z}$:
$\mathcal{U}$
$\mathcal{U}$
$\mathbb{C}^+$
$\mathbb{C}$

WZW Models

$$
\begin{array}{ccc}
 & & H \\
\mathfrak{g} \oplus \bar{\mathfrak{g}} & \xrightarrow{\quad \mathcal{U} \quad} & \downarrow \\
 & & H \\
\Big\downarrow \text{Glueing} & & \text{Trace} \Big\downarrow \\
\mathcal{Z} : \quad \mathbb{C}^{+} & \xrightarrow{\quad \mathcal{U} \quad} & \mathbb{C}
\end{array}
$$

Where $\mathfrak{g}$ is an affine algebra. We therefore get a representation of $\mathfrak{g} \oplus \bar{\mathfrak{g}}$. Any such representation is a direct sum of irreducible representations. When we take the trace of this representation we get the character, which will be the sum of the characters of the irreducibles.

$$\mathcal{Z}(\tau) = \sum_{\lambda,\mu} M_{\lambda\mu}\, H_\lambda \otimes \overline{H_\mu}, \qquad M_{\lambda\mu} \in \mathbb{Z}^{\geq 0}$$

Modularity in WZW models

The partition function

$$\mathcal{Z}(\tau) = \sum_{\lambda,\mu} M_{\lambda\mu}\, H_\lambda \otimes \overline{H_\mu}, \qquad M_{\lambda\mu} \in \mathbb{Z}^{\geq 0}$$

is completely determined by the integer matrix $M = (M_{\lambda\mu})$.

For every affine algebra $\mathfrak{g}$ there exists matrices S and T such that

$$\mathcal{Z} \text{ modular} \iff M \text{ commutes with } S \text{ and } T.$$

Therefore the problem of classifying WZW models has been reduced to the purely algebraic problem of finding integer matrices that commute with S and T! Such matrices are called physical invariants.

The CIZ Classicafation

Physical invariants of WZW models for which $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ were classified by Cappelli, Itzykson and Zuber in 1987.

Their classification had an **A-D-E** meta-pattern!

The goal is to understand why this occurs. Current research is focused around the fact that $\mathbf{TL}(-[2]_q)$ is a subcategory in $\mathrm{Rep}(\widehat{\mathfrak{sl}_2})$ and representations of $\mathbf{TL}(-[2]_q)$ that generate rational conformal theories are classified by **A-D-E**.