Exercise 1. *Introduction*

Binomial coefficients

1. Let k,n be non negative integers. Give three definitions of $\begin{pmatrix} n \\ k \end{pmatrix}$: an algebraic one, a combinatorial one, and its explicit value.
2. Prove that $\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n \\ n-k \end{pmatrix}$.
3. Show that
$$\begin{pmatrix} n \\ k \end{pmatrix} + \begin{pmatrix} n \\ k+1 \end{pmatrix} = \begin{pmatrix} n+1 \\ k+1 \end{pmatrix}.$$
4. What is the value of $\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}$?
5. Prove that
$$\sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} \begin{pmatrix} n_1 \\ k_1 \end{pmatrix} \begin{pmatrix} n_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} n_1+n_2 \\ k \end{pmatrix}.$$

Stirling approximation

1. Recall the Stirling approximation.
2. Show that
$$\frac{1}{2^{2n}} \begin{pmatrix} 2n \\ n \end{pmatrix} \sim \frac{1}{\sqrt{\pi n}},$$
as $n \to \infty$, where $f(n) \sim g(n)$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

Probabilities

1. Let $A, B \subset (\Omega, \mathcal{A}, \mathbb{P})$, be two events. What does it means that they are independent ?
2. What is the definition of the conditional probability $\mathbb{P}(A|B)$? What is the value of $\mathbb{P}(A|B)$ if A and B are independent ?
3. Let X be a non negative random variable. State and prove the Markov inequality.
4. Give the definition of a (discrete time) Markov process.
5. Let G be a general graph, explain what a simple random walk on G is.

Recall that a simple random walk on a graph is called *recurrent* if it returns to its starting point with probability 1, and *transient* otherwise. Recall that a simple random walk $(S_n)_{n\geq 0}$ on a connected graph G, starting from $v \in G$, is recurrent if and only if

$$\sum_{n=0}^{\infty} \mathbb{P}\left(S_n = v\right) = \infty.$$

Exercise 2. *Recurrence/transience theorem for simple random walks on the square lattice* $\mathbb{Z}^d$, $d \geq 1$.

Let $\left(S_n^{(d)}\right)_{n\geq 0}$ be the simple random walk on $\mathbb{Z}^d$ such that $S_0^{(d)} = 0$.

1. $d = 1$ Use Stirling's formula[1] to show that, in one dimension,

$$\mathbb{P}\left(S_{2n}^{(1)} = 0\right) \sim \frac{1}{\sqrt{\pi n}}.$$

Deduce that $(S_n^{(1)})_{n\geq 0}$ is recurrent.

2. $d = 2$ The goal is to prove that the simple random walk on $\mathbb{Z}^2$ is recurrent.

1. By enumerating the different cases, show that

$$\mathbb{P}\left(S_{2n}^{(2)} = 0\right) = \left(\frac{1}{2^{2n}}\binom{2n}{n}\right)^2. \tag{0.1}$$

2. Observe that $\mathbb{P}\left(S_{2n}^{(2)} = 0\right)$ is equal to $\mathbb{P}\left(S_{2n}^{(1)} = 0\right)^2$. Find a probabilistic proof of Equation (0.1).
3. Deduce from Equation (0.1) that $(S_n^{(2)})_{n\geq 0}$ is recurrent.

3. $d = 3$ By a simple enumeration argument, show that

$$\mathbb{P}\left(S_{2n}^{(3)} = 0\right) = \frac{1}{2^{2n}}\binom{2n}{n}\sum_{\substack{j,k\geq 0\\ j+k\leq n}}\left(\frac{n!}{3^n k!j!\,(n-k-j)!}\right)^2$$

and deduce that a simple random walk on $\mathbb{Z}^3$ is transient.

4. $d \geq 3$ Prove that the previous results implies that $\left(S_n^{(d)}\right)_{n\geq 0}$ is transient for $d > 3$.

[1] Stirling's formula is $n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}\left(1 + O\left(n^{-1}\right)\right)$.