We continue the study of the face percolation on hexagonal graph for $p=\frac{1}{2}$ (it is the last exercise sheet on this topic).

Exercise 1. *Hurwitz's theorem.*

(1) Recall the *argument principle* seen in the lesson and a sketch of its proof.

Solution. If f is a holomorphic function in a simply connected open set Ω and if γ is a closed curve in Ω on which f has no zeros, then

$$\frac{1}{2i\pi}\oint_{\gamma}\frac{f'(z)}{f(z)-w}dz$$

is the number of solutions (with their multiplicity) of $f(z)=w$.

The proof goes as follows, we apply the Residue theorem :

$$\frac{1}{2i\pi}\oint_{\gamma}\frac{f'(z)}{f(z)-w}dz=\sum_{i}Res\left(\frac{f'(z)}{f(z)-w},z_i\right)$$

where z_i are the solutions of $f(z)=w$. We have to compute the residue of $\frac{f'(z)}{f(z)-w}$ at z_i. If z_i is a solution (with multiplicity m) of $f(z)=w$, this means that locally around z_i,

$$f(z)-w=a_m(z-z_i)^m+...$$

with $a_m\neq 0$. Hence

$$\frac{f'(x)}{f(z)-w}=\frac{ma_m(z-z_i)^{m-1}+...}{a_m(z-z_i)^m+...}$$

hence the residue is equal to m, the multiplicity of the solution. This allows us to conclude.

(2) Let us consider $(f_n)_{n\in\mathbb{N}}$ a sequence of holomorphic functions on a connected open set Ω converging uniformly to a limit f on all compact subsets of Ω.

(a) Prove that f is holomorphic.

Solution. The holomorphicity of f can be checked locally, hence we can suppose that Ω is simply connected. In this case, f_n is holomorphic if and only if for any curve γ, $\oint_\gamma f_n=0$. Any continuous loop γ belongs to a compact subset of Ω and since the sequence converges uniformly on any compact set then $\oint_\gamma f_n\to\oint_\gamma f$.

(b) Let us suppose that f has a zero of order m at z_0 (i.e. $f^{(0)}(z_0)=f^{(1)}(z_0)=\ldots=f^{(m-1)}(z_0)=0$ and $f^{(m)}(z_0)\neq 0$, where $f^{(i)}$ is the i^{th}-derivative of f), show that for any $\rho>0$ small enough, for sufficiently large $k\in\mathbb{N}$, f_k has precisely m zeros in the disk $D(z_0,\rho)$ including multiplicity.

Solution. Let us suppose that f has a zero of order m at z_0. Let us remark that this implies that f is not identically equal to 0. Using the fact that zeros of holomorphic functions are isolated, we can find $\rho>0$ small enough such that f has no zero on $D(z_0,\rho)\cup\partial D(z_0,\rho)$ except z_0. We can apply the argument principle :

$$\frac{1}{2i\pi}\oint_{\partial D(z_0,\rho)}\frac{f'(z)}{f(z)}dz$$

is equal to the number of zeros of f in $D(z_0,\rho)$. Using the fact that $f_n\to f$ uniformly, there exists $K\in\mathbb{N}$ such that for $k>K$, f_k does not vanish on $\partial D(z_0,\rho)$. Hence for k big enough, $\frac{1}{2i\pi}\oint_{\partial D(z_0,\rho)}\frac{f_k'(z)}{f_k(z)}dz$ is well defined and is the number of zeros of f_k in the disk $D(z_0,\rho)$ including multiplicity. Using the uniform convergence again of $f_n\to f$, we get:

$$\frac{1}{2i\pi}\oint_{\partial D(z_0,\rho)}\frac{f_k'(z)}{f_k(z)}dz\to\frac{1}{2i\pi}\oint_{\partial D(z_0,\rho)}\frac{f'(z)}{f(z)}dz.$$

Since the r.h.s and the l.h.s. are in $\mathbb{N}$, for k big enough, $\frac{1}{2i\pi}\oint_{\partial D(z_0,\rho)}\frac{f_k'(z)}{f_k(z)}dz=\frac{1}{2i\pi}\oint_{\partial D(z_0,\rho)}\frac{f'(z)}{f(z)}dz$. This implies that for any $\rho>0$ small enough, for sufficiently large $k\in\mathbb{N}$, f_k has precisely m zeros in the disk $D(z_0,\rho)$ including multiplicity.

Remark. This is Hurwitz's theorem.

(3) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on a connected open set Ω converging uniformly to a limit f on all compact subsets of Ω.

(a) Show that if each f_n is non-zero everywhere, then f is either identically zero or is also nowhere zero.

Solution. Let us suppose that f_n is non-zero everywhere, and let us suppose that f is not identically zero. Then, if $f(z_0) = 0$, then there exists $m \in \mathbb{N}$ such that z_0 is a zero of f of order m. Using the Hurwitz's theorem, this implies that f_k should have at least m zeros including multiplicity for k big enough: this is not possible by assumption. Thus f is nowhere zero.

(b) Show that if each f_n is injective, then f is either constant or is also injective.

Solution. Let us suppose that f_n is injective, and let us suppose that f is not constant. Then let us suppose that there exists two zeros of $f(.) - \omega$, for $\omega \in \mathbb{C}$, using Hurwitz's theorem, this implies that $f_n(.) - \omega$ has two zeros for n big enough: this is not possible by assumption. Thus f is injective.

Exercise 2. *Hitting distribution*

Consider the equilateral triangle $T \subset \mathbb{C}$ whose vertices are $0, e^{\pm\pi i/6}$. Get the hexagonal discretisation T_δ of T as usual, and consider the critical face percolation on T_δ.

Let us color the complement of the triangle in the right half plane $\{\Re z > 0\} \setminus T_\delta$ black on the top side (positive imaginary part) and white on the bottom (negative imaginary part). Each site percolation configuration gives us an interface: the well-defined path defining the interface between the black colouring on the upper half plane and the white colouring on the lower half plane (there can also be some islands of each signs above and below this path). We would like to study the hitting distribution of this path on the right side $\left[e^{\pi i/6}, e^{-\pi i/6}\right]$: where does the interface end? In terms of the percolation on T_δ, this corresponds to the distribution of the highest white face $W \in \left[e^{\pi i/6}, e^{-\pi i/6}\right]$ connected to the bottom $\left[0, e^{-\pi i/6}\right]$ (if there is no such face, we set its location as $e^{-\pi i/6}$).

(1) Recall Cardy's theorem for the limit of the crossing probability in a general bounded simply connected domain Ω.

Solution. It is in your lesson.

(2) Using Cardy's formula, show that $\lim_{\delta\to 0} \mathbb{P}_{T_\delta}[\Im(W) > h] = \frac{1}{2} - h$ for $h \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Conclude that W converges in distribution to the uniform variable on $\left[e^{-i\pi/6}, e^{i\pi/6}\right]$ as $\delta \to 0$.

Solution. Let us remark that :

$$\lim_{\delta\to 0} \mathbb{P}_{T_\delta}[\Im(W) > h] = \lim_{\delta\to 0} \mathbb{P}\left[\exists \text{ white crossing from } \left[\frac{\sqrt{3}}{2} + ih, e^{i\pi/6}\right] \text{ to } \left[0, e^{-i\pi/6}\right]\right]$$

Thus by Cardy's formula, we have that :

$$\lim_{\delta\to 0} \mathbb{P}_{T_\delta}[\Im(W) > h] = \frac{\left|e^{i\pi/6} - \left(\frac{\sqrt{3}}{2} + ih\right)\right|}{1} = \frac{1}{2} - h.$$

This implies that $\Im(W)$ is uniform in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and thus, W is uniform on $\left[e^{-i\pi/6}, e^{i\pi/6}\right]$ as $\delta \to 0$.

(3) Now consider the deformed triangle made out of two straight line segments from 0 to $e^{\pm\pi i/3}$ and a third segment connecting $e^{\pm\pi i/3}$ through the parabola $y^2 = \frac{9}{4} - 3x$ between $y = \pm\frac{\sqrt{3}}{2}$. What is the hitting distribution in this case?

Solution. Let us call $\tilde{T}$ the deformed triangle. Let us remark that

$$\begin{aligned} \psi : \mathbb{C} &\to \mathbb{C} \\ z &\to z^2 \end{aligned}$$

sends T onto $\tilde{T}$. Indeed, it is easy to see that the segments $\left[0, e^{i\pi/6}\right]$ and $\left[0, e^{-i\pi/6}\right]$ are sent on $\left[0, e^{i\pi/3}\right]$ and $\left[0, e^{-i\pi/3}\right]$. Also if $z = \frac{\sqrt{3}}{2} + ih$ then $\psi(z) = \frac{3}{4} - h^2 + \sqrt{3}hi$. Thus

$$\Im(\psi(z))^2 = 3h^2 = \frac{9}{4} - 3\Re(\psi(z))$$

which shows that the segment $\left[e^{-i\pi/6}, e^{i\pi/6}\right]$ is sent onto the segment of parabola $y^2 = \frac{9}{4} - 3x$ between $y = \pm\frac{\sqrt{3}}{2}$.

Now, let us remark that Cardy's formula tells us that the crossing probability is a conformal invariant. This implies that the law of the hitting distribution is also conformally invariant, since

$$\lim_{\delta\to 0}\mathbb{P}\left(\Im\left(\tilde{W}\right)>\Im\left(\tilde{z}\right)\right)=\lim_{\delta\to 0}\mathbb{P}\left(\exists\text{ white crossing from }\left[\left(\frac{1}{3}\left(\frac{9}{4}-\tilde{z}^2\right)+i\tilde{z}\right),e^{i\pi/3}\right]_{\tilde{T}}\text{ to }\left[0,e^{-i\pi/3}\right]\right)$$

and we can then apply Cardy's theorem. Let us denote by $\tilde{W}$ the hitting point for the deformed triangle $\tilde{T}$: the conformal invariance tells us that $\tilde{W}$ and $\psi(W)$ have the same law. Thus for $\tilde{z}$ in the segment of $\tilde{T}$ connecting $e^{\pm\pi i/3}$,

$$\lim_{\delta\to 0}\mathbb{P}\left(\Im\left(\tilde{W}\right)>\Im\left(\tilde{z}\right)\right)=\lim_{\delta\to 0}\mathbb{P}\left(\Im\left(\psi(W)\right)>\Im\left(\psi(\sqrt{3}/2+ih)\right)\right)$$

where we defined h by the relation $\tilde{z}=\psi(\sqrt{3}/2+ih)$ and thus :

$$\begin{aligned}\lim_{\delta\to 0}\mathbb{P}\left(\Im\left(\tilde{W}\right)>\Im\left(\tilde{z}\right)\right)&=\lim_{\delta\to 0}\mathbb{P}\left(\Im\left(\psi(W)\right)>\Im\left(\psi(\sqrt{3}/2+ih)\right)\right)\\&=\lim_{\delta\to 0}\mathbb{P}\left(\Im\left(W\right)>\Im\left(\sqrt{3}/2+ih\right)=h\right)\\&=\frac{1}{2}-h\end{aligned}$$

Now let us remark that we have already seen that $\Im(\psi\left(z\right))=\sqrt{3}\Im\left(z\right)$ thus $h=\Im\left(\psi^{-1}\left(\tilde{z}\right)\right)=\frac{1}{\sqrt{3}}\Im\left(z\right)$. This implies that

$$\lim_{\delta\to 0}\mathbb{P}\left(\Im\left(\tilde{W}\right)>\Im\left(\tilde{z}\right)\right)=\frac{1}{2}-\frac{1}{\sqrt{3}}\Im\left(z\right).$$

This implies that $\Im\left(\tilde{W}\right)$ is uniform between $\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$, and thus

$$\tilde{W}\sim\frac{1}{3}\left(\frac{9}{4}-U^2\right)+iU$$

where $U\sim Unif\left(\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]\right)$.

Exercise 3. *Crossing probability*

Using the notations from Exercise 2 of last exercise sheet, we have seen that there exists $C,\alpha>0$ such that

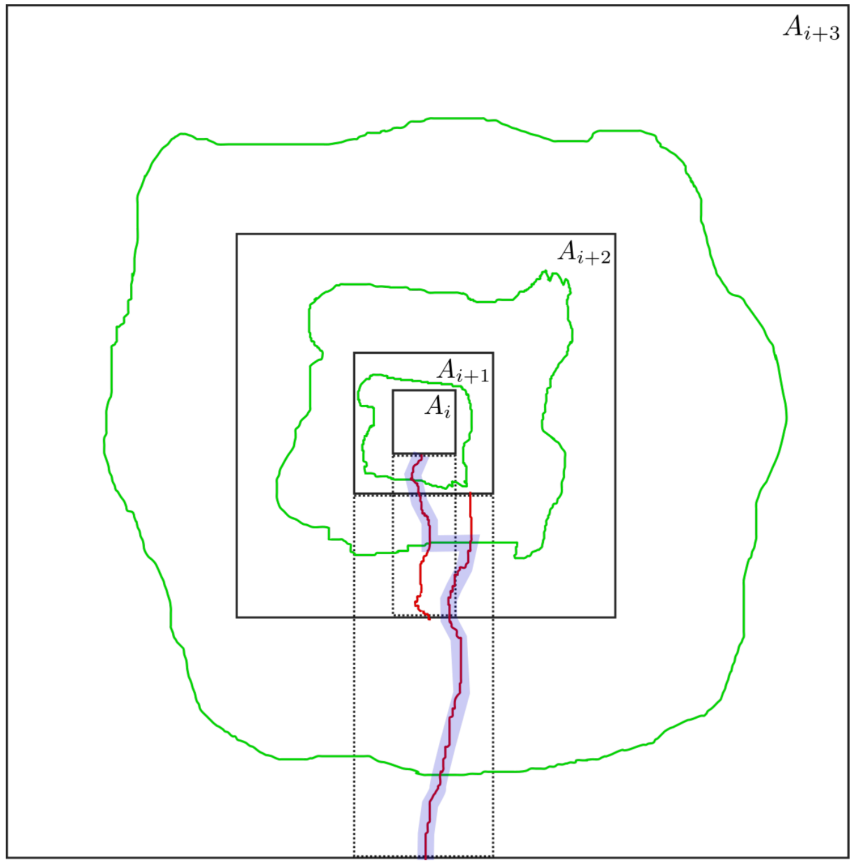
$$p_{r,R}\leq C\left(\frac{r}{R}\right)^{\alpha}.$$

Prove that there exists $c,\beta>0$ such that

$$c\left(\frac{r}{R}\right)^{\beta}\leq p_{r,R}$$

Solution. We need to prove that the probability is not too low : we need to prove that there exists an event $\mathcal{E}$ which happens with a probability which is bounded by below by $c\left(\frac{r}{R}\right)^{\beta}$ such that if this event happens then the event $r\rightsquigarrow R$ happens.

The solution is given by the drawing given below : we impose that there exists $\sim\log\left(\frac{R}{r}\right)$ black circles in some maximal concentric family of (squared-shape) annuli of similar ratio (ratio 3/1), and we impose that there exists a top-bottom crossing in the vertical rectangle which upper side is the bottom of the square A_i, and which bottom side belongs to the bottom of the square A_{i+2} : this gives again $\sim\log\left(\frac{R}{r}\right)$ crossings. Using FKG inequality, the probability of this events is bigger than the product of the probability of each crossing. Since we consider crossing events in similar shapes, we can lower bound each probability by a constant $p>0$ (RSW inequalities) which allows us to have a lower bound $\sim p^{2\log\left(\frac{R}{r}\right)}$ and gives us the lower bound $c\left(\frac{r}{R}\right)^{\beta}\leq p_{r,R}$ with $c,\beta>0$.



A_{i+3}
A_{i+2}
A_{i+1}
A_i