We continue the study of the face percolation on hexagonal graph for $p = \frac{1}{2}$ (it is the last exercise sheet on this topic).

Exercise 1. *Hurwitz's theorem.*

(1) Recall the *argument principle* seen in the lesson and a sketch of its proof.

(2) Let us consider $(f_n)_{n\in\mathbb{N}}$ a sequence of holomorphic functions on a connected open set Ω converging uniformly to a limit f on all compact subsets of Ω.

   (a) Prove that f is holomorphic.

   (b) Let us suppose that f has a zero of order m at z_0 (i.e. $f^{(0)}(z_0) = f^{(1)}(z_0) = \ldots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$, where $f^{(i)}$ is the i^{th}-derivative of f), show that for any $\rho > 0$ small enough, for sufficiently large $k \in \mathbb{N}$, f_k has precisely m zeros in the disk $D(z_0, \rho)$ including multiplicity.

   Remark. This is Hurwitz's theorem.

(3) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on a connected open set Ω converging uniformly to a limit f on all compact subsets of Ω.

   (a) Show that if each f_n is non-zero everywhere, then f is either identically zero or is also nowhere zero.

   (b) Show that if each f_n is injective, then f is either constant or is also injective.

Exercise 2. *Hitting distribution*

Consider the equilateral triangle $T \subset \mathbb{C}$ whose vertices are $0, e^{\pm\pi i/6}$. Fix a hexagonal discretisation T_δ of T as usual, and consider the critical face percolation on T_δ.

Let us color the complement of the triangle in the right half plane $\{\Re z > 0\} \setminus T_\delta$ black on the top side (positive imaginary part) and white on the bottom (negative imaginary part). Each site percolation configuration gives us an interface: the (well-defined) path defining the interface between the black colouring on the upper half plane and the white on the lower half plane (there can also be some sign clusters in T_δ above and below this path). We would like to study the hitting distribution of this path on the right side $\left[e^{\pi i/6}, e^{-\pi i/6}\right]$: where does the interface end? In terms of the percolation on T_δ, this corresponds to the distribution of the highest white face $W \in \left[e^{\pi i/6}, e^{-\pi i/6}\right]$ connected to the bottom $\left[0, e^{-\pi i/6}\right]$ (if there is no such face, we set its location as $e^{-\pi i/6}$).

(1) Recall Cardy's theorem for the limit of the crossing probability in a general bounded simply connected domain Ω.

(2) Using Cardy's formula, show that $\lim_{\delta\to 0} \mathbb{P}_{T_\delta}[\Im(W) > h] = \frac{1}{2} - h$ for $h \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Conclude that W converges in distribution to the uniform variable on $\left[e^{-i\pi/6}, e^{i\pi/6}\right]$ as $\delta \to 0$.

(3) Now consider the general conformal triangle with two straight line segments from 0 to $e^{\pm\pi i/3}$. The remaining segment connecting $e^{\pm\pi i/3}$ is the parabola $y^2 = \frac{9}{4} - 3x$ between $y = \pm\frac{\sqrt{3}}{2}$. What is the hitting distribution in this case?

Exercise 3. *Crossing probability*

Using the notations from Exercise 2 of last exercise sheet, we have seen that there exists $C, \alpha > 0$ such that

$$p_{r,R} \leq C\left(\frac{r}{R}\right)^{\alpha}.$$

Prove that there exists $c, \beta > 0$ such that

$$c\left(\frac{r}{R}\right)^{\beta} \leq p_{r,R}$$