For exercises 1, 2 and 3, we consider the Ising model with + boundary conditions on the square lattice inside the open unit disc $\mathbb{D} \subset \mathbb{R}^2$. We denote by $\mathbb{D}_\delta$ the discretisation $\mathbb{D} \cap \delta\mathbb{Z}^2$.

Exercise 1. *Low-temperature expansion*

(1) Recall the partition function of the Ising model on $\mathbb{D}_\delta$ with + boundary conditions:

$$Z_{\mathbb{D}_\delta,+} = \sum_{\sigma \in \{\pm 1\}^{\mathbb{D}_\delta,+}} e^{\beta \sum_{xy \in \mathcal{E}} \sigma_x \sigma_y}.$$

Express Z using the low-temperature expansion, i.e. expand Z using the relation

$$e^{\beta\sigma_x\sigma_y} = e^{\beta}\left(\delta_{\sigma_x\sigma_y=1} + e^{-2\beta}\delta_{\sigma_x\sigma_y=-1}\right)$$

.

Solution. We get the following expression by developping the product over the edges xy of $\mathbb{D}_\delta$ with the sum $\left(\delta_{\sigma_x\sigma_y=1} + e^{-2\beta}\delta_{\sigma_x\sigma_y=-1}\right)$:

$$\begin{aligned} Z &= \sum_{\sigma \in \{\pm 1\}^{\mathbb{D}_\delta,+}} \prod_{xy \in \mathcal{E}} e^{\beta}\left(\delta_{\sigma_x\sigma_y=1} + e^{-2\beta}\delta_{\sigma_x\sigma_y=-1}\right) \\ &= e^{\beta|\mathcal{E}|} \sum_{C \in \mathcal{C}(\mathcal{E}^*)} e^{-2\beta|C|} \end{aligned}$$

where $\mathcal{C}(\mathcal{E}^*)$ denotes the *set of cluster configurations* i.e. subsets of $\mathcal{E}^*$, the set of edges of the graph dual to $\mathbb{D}_\delta$, corresponding to loops surrounding the sign clusters of a spin configuration.

(2) What is the expectation of a spin at a given site expressed using the low-temperature expansion ?

Solution. The value of a spin at a given site x is the parity of the number of loops N which are surrounding the site x in the low-temperature representation. Thus

$$\mathbb{E}^{\beta}_{\mathbb{D}_\delta,+}(\sigma_x) = \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}[N \text{ is even}] - \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}[N \text{ is odd}].$$

(3) Show that there is $\beta > 0$ such that

$$\liminf_{\delta \to 0} \mathbb{E}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 0.99.$$

Hint : We are looking at large β, hence you can use the previous question to obtain a lower bound on $\mathbb{E}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right)$.
Additional hint if needed: $\mathbb{E}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 1 - 2\mathbb{P}(N > 0)$ is big because $\mathbb{P}(N > 0)$ is small, where N is the number of loops in the low temperature expansion which surrounds 0. It remains to count how many loops of size l surround $(0,0)$.

Solution. We know that

$$\mathbb{E}^{\beta}_{\mathbb{D}_\delta,+}(\sigma_x) \geq \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(N = 0) - \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(N > 0) = 1 - 2\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(N > 0).$$

Let us prove that $\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(N > 0)$ is bounded above by $\sum_{\ell \geq 1} \ell 4^{\ell} e^{-2\beta\ell}$, which goes to zero as $\beta \to \infty$, hence for large β we have the desired lower bound. .

Using the low temperature expansion, the event $N > 0$ corrpesonds to all the possible cluster configurations C of dual edges containing a loop l surrounding $(0,0)$. Any such cluster configuration C can be mapped to a cluster configuration C' "compatible with l" defined by $C' = C \setminus l$, i.e. corresponding to the sign clusters of the spin configuration corresponding to C after flipping all the spins contained inside l. Hence we obtain:

$$\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(N > 0) = \frac{1}{Z} \sum_{C \mid \exists l \text{ surrounding } (0,0)} e^{-2\beta|C \setminus l|} e^{-2\beta|l|} = \sum_{l \text{ surrounding } (0,0)} e^{-2\beta|l|} \left(\frac{\sum_{C' \text{ compatible with } l} e^{-2\beta|C'|}}{Z} \right)$$

where $|C|$ denotes the number of edges in C. Since we then have that

$$\frac{\sum_{C' \text{ compatible with } l} e^{-2\beta|C'|}}{Z} \leq 1,$$

we have that $\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(N>0)$ is bounded above by $\sum_{l \text{ surrounds } (0,0)} e^{-2\beta|l|}$. We also recall (see exercise 3 (2)(d) of sheet 8 for a similar argument) that a loop of length ℓ surrounding $(0,0)$ can only cross the positive x axis at the first $O(\ell)$ points and, once we know where it crosses, we have a maximal of 4^ℓ different loops of length ℓ starting from this point, we indeed get that $\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(N>0)$ is bounded above by $c\sum_{\ell\geq 1}\ell 4^\ell e^{-2\beta\ell}$.

Exercise 2. *Coupling and stochastic domination*

(1) Recall the Markov Chain for the Ising model that you have seen in class (the Glauber dynamics).

Solution. The Markov Chain you have seen consists in the following steps:
(a) Start from an arbitrary configuration,
(b) Make random flips:
  (i) Compute the energy of the current configuration H_σ.
  (ii) Pick a vertex x at random, consider the configuration ρ obtained by flipping the spin x of σ, and compute its energy H_ρ
  (iii) If $H_\rho \leq H_\sigma$, replace σ by ρ. If $H_\rho > H_\sigma$, replace σ by ρ with probability $e^{-\beta H_\rho}/e^{-\beta H_\sigma}$.

(2) Consider the following Heat Bath Dynamics :
(a) Pick a vertex x at random,
(b) Sample the spin σ_x at random by giving probability

$$\mathbb{P}(\sigma_x = 1) = \frac{e^{-\beta\mathcal{H}(\sigma^+)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}$$

where σ^+ and σ^- denote the configuration σ with the spin σ_x forced to be $+1$ and -1 respectively. Prove that the Ising measure is the invariant probability measure of this dynamics. *Hint : check the detailed balance equation.*

Solution. We will prove the detailed balance equation :

$$\pi_{Ising}(\sigma) P_{HeatBath}(\sigma,\rho) = \pi_{Ising}(\rho) P_{HeatBath}(\rho,\sigma).$$

If ρ is not of the form σ^+ or σ^-, the detailed balance equation is trivially true since $P_{HeatBath}(\sigma,\rho) = P_{HeatBath}(\rho,\sigma) = 0$. Now, let us suppose there exists a vertex x such that $\rho = \sigma^+$, then

$$\pi_{Ising}(\sigma) P_{HeatBath}(\sigma,\rho) = \pi_{Ising}(\sigma) P_{HeatBath}\left(\sigma,\sigma^+\right) = e^{-\beta\mathcal{H}(\sigma)}\frac{e^{-\beta\mathcal{H}(\sigma^+)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}$$

and

$$\pi_{Ising}(\rho) P_{HeatBath}(\rho,\sigma) = \pi_{Ising}\left(\sigma^+\right) P_{HeatBath}\left(\sigma^+,\sigma\right) = e^{-\beta\mathcal{H}(\sigma^+)}\frac{e^{-\beta\mathcal{H}(\sigma)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}.$$

This proves that the detailed balance equation is valid and the Ising measure is the invariant probability measure of this dynamics.

(3) There is a partial ordering between spin configurations $\sigma \in \{\pm 1\}^{\mathbb{D}_\delta}$: $\sigma \leq \sigma'$ if $\sigma_a \leq \sigma'_a$ for all $a \in \mathbb{D}_\delta$. Suppose that we start the chain at a common temperature $\beta > 0$ on two starting configurations $\sigma^0 \leq \sigma'^0$. Show that we can couple the two dynamics such that this ordering is preserved at each step of the Markov Chain, that is

$$\sigma^n \leq \sigma'^n$$

for all the time steps $n \in \mathbb{N}$.

Solution. We will define two Markov Chain σ^n and σ'^n starting from σ^0 and σ'^0 by using the Heat Bath Dynamics and:
(a) picking the same vertex x at random for the two Markov Chain,
(b) sampling the spin σ^{n+1}_x and σ'^{n+1}_x using the same underlying uniform random variable : we consider $U \sim Uni([0,1])$ and we define

$$\sigma^{n+1}_x = 1 \text{ if } U \leq \frac{e^{-\beta\mathcal{H}(\sigma^{n+})}}{e^{-\beta\mathcal{H}(\sigma^{n+})} + e^{-\beta\mathcal{H}(\sigma^{n-})}}$$

and $\sigma^{n+1}_x = -1$ if not,

$$\sigma'^{n+1}_x = 1 \text{ if } U \leq \frac{e^{-\beta\mathcal{H}(\sigma'^{n+})}}{e^{-\beta\mathcal{H}(\sigma'^{n+})} + e^{-\beta\mathcal{H}(\sigma'^{n-})}}$$

and $\sigma'^{n+1}_x = -1$ if not.

If we prove that at any time $\frac{e^{-\beta\mathcal{H}\left(\sigma^{n+}\right)}}{e^{-\beta\mathcal{H}(\sigma^{n+})}+e^{-\beta\mathcal{H}(\sigma^{n-})}} \leq \frac{e^{-\beta\mathcal{H}\left(\sigma'^{n+}\right)}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}+e^{-\beta\mathcal{H}(\sigma'^{n-})}}$ then by recursion we can conclude that $\sigma^n \leq \sigma'^n$. In order to prove the first inequality, we only need to prove that

$$\frac{e^{-\beta\mathcal{H}\left(\sigma^{n+}\right)}+e^{-\beta\mathcal{H}\left(\sigma^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma^{n+})}} \geq \frac{e^{-\beta\mathcal{H}\left(\sigma'^{n+}\right)}+e^{-\beta\mathcal{H}\left(\sigma'^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}}$$

or

$$\frac{e^{-\beta\mathcal{H}\left(\sigma^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma^{n+})}} \geq \frac{e^{-\beta\mathcal{H}\left(\sigma'^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}}.$$

Let us remark that for a configuration σ and any site x,

$$\frac{e^{-\beta\mathcal{H}\left(\sigma^{-}\right)}}{e^{-\beta\mathcal{H}(\sigma^{+})}} = e^{\beta\left(-\sum_{a\sim b}\sigma_a^+\sigma_b^+ + \sum_{a\sim b}\sigma_a^-\sigma_b^-\right)}$$

(Be carefull, the energy $\mathcal{H}$ is equal to $-\sum_{x\sim y}\sigma_x\sigma_y$. Do not forget the $-$ sign), yet σ^+ and σ^- only differs at x, thus it is equal to $e^{-2\beta\sum_{a\sim x}\sigma_a}$. This implies that

$$\frac{e^{-\beta\mathcal{H}\left(\sigma^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma^{n+})}} = e^{-2\beta\sum_{a\sim x}\sigma_a^n} \geq e^{-2\beta\sum_{a\sim x}\sigma_a'^n} = \frac{e^{-\beta\mathcal{H}\left(\sigma'^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}},$$

which allows us to conclude.

Exercise 3. *Monotonicity property for the boundary conditions*

Show that if $\mathfrak{b}_1, \mathfrak{b}_2 \in \{\pm 1\}^{\partial\mathbb{D}_\delta}$ are boundary conditions such that $\mathfrak{b}_1 \leq \mathfrak{b}_2$ (which means that for any element x of the boundary $\mathfrak{b}_1(x) \leq \mathfrak{b}_2(x)$). Then the corresponding Ising measures satisfy:

$$\mathbb{E}^{\beta}_{\mathbb{D}_\delta;\mathfrak{b}_1}(\sigma_a) \leq \mathbb{E}^{\beta}_{\mathbb{D}_\delta;\mathfrak{b}_2}(\sigma_a)$$

for any $a \in \mathbb{D}_\delta$. *Hint : Use the Markov chain dynamics seen in the previous exercise; the boundary spins remain unchanged.*

Solution. One just has to use the same coupling of Markov chains (where we never pick x on the boundary) as in the previous exercise, with similar initial condition except for the boundary where one begins with $\mathfrak{b}_1$ and $\mathfrak{b}_2$.The result follows from the general fact about Markov chains that the Glauber or Heat bath dynamics converge to the Ising measure over spin configurations.