For exercises 1, 2 and 3, we consider the Ising model with + boundary conditions on the square lattice inside the open unit disc $\mathbb{D} \subset \mathbb{R}^2$. We denote by $\mathbb{D}_\delta$ the discretisation $\mathbb{D} \cap \delta\mathbb{Z}^2$.

Exercise 1. *Low-temperature expansion*

(1) Recall the partition function of the Ising model on $\mathbb{D}_\delta$ with + boundary conditions:

$$Z_{\mathbb{D}_\delta,+} = \sum_{\sigma \in \{\pm 1\}^{\mathbb{D}_\delta,+}} e^{\beta \sum_{xy \in \mathcal{E}} \sigma_x \sigma_y}.$$

Express Z using the low-temperature expansion, i.e. expand Z using the relation

$$e^{\beta \sigma_x \sigma_y} = e^{\beta} \left(\delta_{\sigma_x \sigma_y = 1} + e^{-2\beta} \delta_{\sigma_x \sigma_y = -1} \right)$$

.

(2) What is the expectation of a spin at a given site in terms of the low-temperature expansion ?

(3) Show that there is $\beta > 0$ such that

$$\liminf_{\delta \to 0} \mathbb{E}^{\beta}_{\mathbb{D}_\delta,+} \left(\sigma_{(0,0)} \right) \geq 0.99.$$

Hint : We are looking at large β, hence you can use the previous question to obtain a lower bound on $\mathbb{E}^{\beta}_{\mathbb{D}_\delta,+} \left(\sigma_{(0,0)} \right)$.

Additional hint if needed: $\mathbb{E}^{\beta}_{\mathbb{D}_\delta,+} \left(\sigma_{(0,0)} \right) \geq 1 - 2\mathbb{P}(N > 0)$ is big because $\mathbb{P}(N > 0)$ is small, where N is the number of loops in the low temperature expansion which surrounds 0. It remains to count how many loops of size l surround $(0, 0)$.

Exercise 2. *Coupling and stochastic domination*

(1) Recall the Markov Chain for the Ising model that you have seen in class (the Glauber dynamics).

(2) Consider the following Heat Bath Dynamics :
   (a) Pick a vertex x at random,
   (b) Sample the spin σ_x at random by giving probability

$$\mathbb{P}(\sigma_x = 1) = \frac{e^{-\beta \mathcal{H}(\sigma^+)}}{e^{-\beta \mathcal{H}(\sigma^+)} + e^{-\beta \mathcal{H}(\sigma^-)}}$$

   where σ^+ and σ^- denote the configuration σ with the spin σ_x forced to be $+1$ and -1 respectively.

Prove that the Ising measure is the invariant probability measure of this dynamics. *Hint : check the detailed balance equation.*

(3) There is a partial ordering between spin configurations $\sigma \in \{\pm 1\}^{\mathbb{D}_\delta}$: $\sigma \leq \sigma'$ if $\sigma_a \leq \sigma'_a$ for all $a \in \mathbb{D}_\delta$. Suppose that we start the chain at a common temperature $\beta > 0$ on two starting configurations $\sigma^0 \leq \sigma'^0$. Show that we can couple the two dynamics such that this ordering is preserved at each step of the Markov Chain, that is

$$\sigma^n \leq {\sigma'}^n$$

for all the time steps $n \in \mathbb{N}$.

Exercise 3. *Monotonicity property for the boundary conditions*

Show that if $\mathfrak{b}_1, \mathfrak{b}_2 \in \{\pm 1\}^{\partial \mathbb{D}_\delta}$ are boundary conditions such that $\mathfrak{b}_1 \leq \mathfrak{b}_2$ (which means that for any element x of the boundary $\mathfrak{b}_1(x) \leq \mathfrak{b}_2(x)$). Then the corresponding Ising measures satisfy:

$$\mathbb{E}^{\beta}_{\mathbb{D}_\delta;\mathfrak{b}_1}(\sigma_a) \leq \mathbb{E}^{\beta}_{\mathbb{D}_\delta;\mathfrak{b}_2}(\sigma_a)$$

for any $a \in \mathbb{D}_\delta$. *Hint : Use the Markov chain dynamics seen in the previous exercise; the boundary spins remain unchanged.*