Exercise 1. *High-temperature expansion and positive correlations*

(1) Recall the high-temperature expansion of the Ising model.
(2) Consider the Ising model on a finite connected graph $\mathbb{G}$ without a boundary (i.e. with free boundary conditions). Show that for any inverse temperature $\beta \in]0, \infty[$, we have

$$\forall x, y \in \mathbb{G}, \ \mathbb{E}\left[\sigma_x \sigma_y\right] > 0.$$

Exercise 2. *Kramers-Wannier duality*

Consider the Ising model on the lattice $V = \mathbb{Z}^2 \cap \Omega$ with edge set E at the self-dual inverse temperature $\beta_c = \frac{1}{2} \ln\left(1+\sqrt{2}\right)$ with free boundary conditions, where Ω is a bounded, connected open subset of the plane. Fix two neighbouring vertices x and $y \in V$ in the lattice, connected by the edge $e = \{x, y\} \in E$. Write $C \subset 2^E$ for the collection of subsets $\mathcal{E} \subset E$ such that every vertex is incident to an even (possibly zero) number of edges in $\mathcal{E}$ (informally, $\mathcal{E}$ is a set of loops formed by elements of E). Similarly, write $C_{x,y}$ for the collection of $\mathcal{E}_{x,y}$ such that every vertex except for x, y is incident to an even number of edges in $\mathcal{E}_{x,y}$, while x and y are both incident to an odd number of edges in $\mathcal{E}_{x,y}$. Write

$$Z(C) = \sum_{\mathcal{E} \in C} \exp\left(-2\beta_c \left|\mathcal{E}\right|\right) = \sum_{\mathcal{E} \in C} \left(\tanh \beta_c\right)^{|\mathcal{E}|}, \quad Z\left(C_{x,y}\right) = \sum_{\mathcal{E}_{x,y} \in C_{x,y}} \exp\left(-2\beta_c \left|\mathcal{E}_{x,y}\right|\right).$$

(1) Express the spin correlation $\mathbb{E}^{free}\left[\sigma_x \sigma_y\right]$ of two neighbouring vertices x, y in terms of $Z(C)$ and $Z\left(C_{x,y}\right)$.
(2) Recall Kramers-Wannier duality.
(3) Now, write $C = C^e \cup C^{-e}$ where C^e is the collection of $\mathcal{E} \in C$ with $e \in \mathcal{E}$ and $C^{-e} = C \setminus C^e$, then accordingly decompose the sum $Z = Z\left(C^{-e}\right) + Z\left(C^e\right)$. By Kramers-Wannier duality, we have a dual Ising model on the *faces* of the lattice with *plus* boundary conditions. Suppose the two faces separated by e are denoted f_1,f_2. Recall the low-temperature expansion: what are the probabilities

$$\mathbb{P}^+\left[\sigma_{f_1} = \sigma_{f_2}\right], \ \mathbb{P}^+\left[\sigma_{f_1} \neq \sigma_{f_2}\right]$$

in terms of $Z(C)$,$Z\left(C^e\right)$,$Z\left(C^{-e}\right)$? What is $\mathbb{E}^+\left[\sigma_{f_1}\sigma_{f_2}\right]$?
(4) Note that there is a bijection from C to $C_{x,y}$: given $\mathcal{E} \in C^e$, $\mathcal{E} \setminus \{e\} \in C_{x,y}$, and give $\mathcal{E} \in C^{-e}$, $\mathcal{E} \cup \{e\} \in C_{x,y}$. This also means there is a one-to-one correspondence between the terms of $Z(C) = Z\left(C^e\right) + Z\left(C^{-e}\right)$ and $Z\left(C_{x,y}\right)$. Express $Z\left(C_{x,y}\right)$ in terms of $Z\left(C^e\right)$ and $Z\left(C^{-e}\right)$.
(5) We know that, as we take bigger and bigger $\Omega \in \mathbb{R}^2$, $\mathbb{E}^{free}\left(\sigma_x\sigma_y\right)$ and $\mathbb{E}^+\left(\sigma_{f_1}\sigma_{f_2}\right)$ both tend to a single positive number μ. Compute μ by using above results.

Exercise 3. *$\beta \to 0$ and boundary conditions*

Consider the Ising model on the lattice $\mathbb{Z}^2 \cap [0, N]^2$. We impose respectively plus and minus spins on the boundary vertices $\{-1\} \times [0, N] \cup [0, N] \times \{N+1\}$ and $\{N+1\} \times [0, N] \cup [0, N] \times \{-1\}$. Describe the $\beta \to \infty$ limit of the model.