Let $G \subseteq \mathbb{Z}^d$ be a finite graph with boundary ∂G (i.e. the vertices in $\mathbb{Z}^d$ having one neighbor in G). The Laplacian on G with boundary conditions b is the linear operator $\Delta_G^b : \mathbb{R}^G \to \mathbb{R}^G$ defined on any function $f : G \to \mathbb{R}$ by the relation

$$\Delta f(x) = \frac{1}{2d} \sum_{y \sim x} [f(y) - f(x)],$$

where $x \in G$, $f(y) = b(y)$ for each $y \in \partial G$ and $y \sim x$ means that $|y - x| = 1$.

The *differential* df of a function f on $G \sqcup \partial G$ is a function defined on any oriented edge (x, y) of $G \sqcup \partial G$ as

$$df(x, y) := f(y) - f(x).$$

The scalar products on the set of functions $\mathbb{R}^G$ is defined as

$$\langle f, g\rangle_{\mathbb{R}^G} := \sum_{x \in G} f(x)g(x).$$

Similarly, if $\vec{\mathcal{E}}$ is a set of oriented edges in G, we define the scalar product on $\mathbb{R}^{\vec{\mathcal{E}}}$ as

$$\langle f, g\rangle_{\mathbb{R}^{\vec{\mathcal{E}}}} := \sum_{(x,y) \in \vec{\mathcal{E}}} f(x, y)g(x, y).$$

Exercise 1. *General knowledge.* Let G be a finite subgraph of $\mathbb{Z}^d$ with boundary ∂G.

(1) Show that $\Delta = \Delta_G^0$ is negative definite. *Hint: Show that* $\langle \Delta f, g\rangle_{\mathbb{R}^G} = -\frac{1}{4d}\langle df, dg\rangle_{\mathbb{R}^{\vec{\mathcal{E}}(G)}} - \frac{1}{2d}\langle df, dg\rangle_{\mathbb{R}^{\vec{\mathcal{E}}_{out}(\partial G)}}$, *where* $\vec{\mathcal{E}}(G)$ *is the set of oriented edges in* G *and* $\vec{\mathcal{E}_{out}}(\partial G)$ *is the set of oriented edges of the form* (x, y) *where* $x \in G$ $y \in \partial G$.

(2) Let ψ be an harmonic function such that ψ is null on ∂G. Show that ψ is null on G. Let h_1 and h_2 be two functions respectively on G and ∂G. Let f, g be solutions of

$$\begin{cases} \Delta\phi = h_1 & \text{in } G, \\ \phi = h_2 & \text{on } \partial G. \end{cases}$$

Show that $f = g$.

(3) Let $B \subset \partial G$. Recall the two definitions of the harmonic measure $H_B(\cdot)$ defined on G.

Exercise 2. *Large deviation estimate and recurrence for the simple random walk on the square lattice* $\mathbb{Z}^2$*.*

Consider the simple random walk $S_k = (X_k, Y_k)$ on $\mathbb{Z}^2$ starting at $(0, 0)$, where X_k and Y_k are the coordinates of S_k.

(1) Let $N_{2n}^{(x)}$ be the number of steps in direction x taken by $(S_k)_{k \le 2n}$. Let $\left(S_n^{(1)}\right)_{n\ge0}$ and $\left(\overline{S}_n^{(1)}\right)_{n\ge0}$ be two independant one dimensional SRW on $\mathbb{Z}$. Show that

$$\mathbb{P}\left(S_{2n} = 0 | \left\{N_{2n}^{(x)} = 2k\right\}\right) = \mathbb{P}\left(\left\{S_{2k}^{(1)} = 0\right\} \text{ and } \left\{\overline{S}_{2n-2k}^{(1)} = 0\right\}\right),$$

for all $k, n \in \mathbb{N}$ such that $k \le n$.

(2) Let us suppose that after $2n$ steps for n large enough, the number of steps the walk $(S_k)_{k\ge0}$ moved in the direction x is the even number in $\{n, n + 1\}$. Show that there is a constant $c > 0$ such that

$$\mathbb{P}_0 (S_{2n} = 0) > \frac{c}{n}.$$

(3) [*Large deviation estimate*] Show that if N is a $\mathcal{B}in(n, p)$ random variable and $q < p$ then there is a constant $c = c(p; q) > 0$ such that

$$\mathbb{P}(N \le qn) = O\left(e^{-cn}\right).$$

Hint : Think about an inequality which allows to bounds probability with expectations. And since there should be some exponential at the end of the day, try to think about what you can do before applying the inequality.

(4) Using the intuition built in question (2), the results about 1D SRW and the large deviation estimate, prove that

$$\mathbb{P}_0 (S_{2n} = 0) > \frac{c}{n}$$

and thus conclude that S_n is recurrent.

Hint : Use the bound $\mathbb{P}(S_{2n} = 0) \ge \sum_{k=n/2, even}^{3n/2} \mathbb{P}\left(S_{2n} = 0 \cap N_{2n}^{(x)} = k\right)$.

Remark. The point 3. of Exercise 2 can be improved and it yields the *Hoeffding's inequality* :

$$\mathbb{P}\left(N < (p-\epsilon)\, n\right) \leq e^{-2\epsilon^2 n},$$

whose proof is similar to the one proposed in the solution sheet for point (3).

Exercise 3. *Green function*

Let us consider A a finite subgraph of $\mathbb{Z}^d$, and let ∂A be the set of points in $\mathbb{Z}^d \setminus A$ that are adjacent to a point in A. Let $(S_n)_{n\geq 0}$ be a random walk started from a point in A. We define $\tau_A = \min\{n \geq 0, S_n \notin A\}$ which is the first time the random walk leaves A.

Recall that the *Green's function* G_A *defined on* $\mathbb{Z}^d \times \mathbb{Z}^d$ *is*

$$G_A(x,y) = \mathbb{E}^x\left[\#\{0 \leq n < \tau_A | S_n = y\}\right]$$

where we recall that $\mathbb{P}^x$ refers to the probability law of a simple random walk started at x.

(1) Prove that

$$\tau_A < \infty, \qquad \mathbb{E}(\tau_A) < \infty, \qquad G_A(x,y) < \infty$$

Remark. Be careful: if indeed $\tau_A < \infty$ still holds for $A \subset \mathbb{Z}^d$, $d \leq 2$ (were A is not necessarily finite, as a consequence of the recurrence), it can be true that $\mathbb{E}(\tau_A) = \infty$. (e.g. using point 5. of this exercise, one can see that $\mathbb{E}^1(\tau_0) = \infty$ for the random walk on $\mathbb{Z}$ starting at 0). Whereas for $d \geq 3$, we can have $\tau_A = \infty$ with positive probability and still $G_A(x,y) < \infty$ (because there will be a finite number of visits at a given point y).

(2) Show that $G_A(x,y) = \sum_{n=0}^{\infty} \mathbb{P}^x\{S_n = y \text{ and } n < \tau_A\}$.

(3) Let $y \in A$, and let $f(x) = G_A(x,y)$. Prove that f is a solution of

$$\begin{cases} \Delta f(x) = \begin{cases} -1 & x = y, \\ 0 & x \in A \setminus \{y\} \end{cases} \\ f(x) = 0 & \text{on } \partial A \end{cases}$$

(4) Prove that f is the unique solution to the above problem.

(5) Find an explicit formula for G_A when $d = 1$ and $A = \{1, \cdots, n-1\}$.

(6) Prove that the simple random walk on $\mathbb{Z}$ is recurrent.

(7) Show that G_A is the inverse of minus the Laplacian operator $-\Delta$ in the following sense: if $f : A \to \mathbb{R}$ is any function and $h(x) = \sum_{y\in A} G_A(x,y)\, f(y)$, then $-\Delta h = f$.