Recall that the Green's function on a domain $A \subset \mathbb{Z}^d$ is the function

$$G_A(x,y) = \mathbb{E}^x\left[\#\{0 \le n < \tau_A : S_n = y\}\right]$$

where $(S_n)_n$ is the simple random walk started at x (under $\mathbb{P}^x$) and $\tau_A = \min\{n \ge 0 : S_n \notin A\}$.

We also recall that the Harmonic measure on A associated to a subset $B \subset \partial A$ is the function which for each $x \in A$ takes value

$$H_A(x,B) = \mathbb{P}^x\left[S_{\tau_{\partial A}} \in B\right].$$

Exercise 1. *General knowledge*

(1) Recall what is the discrete PDE satisfied by the function $x \to G_A(x,y)$.

Solution. The function $f : x \to G_A(x,y)$ satisfies the discrete PDE $\Delta f(x) = -\delta_{x=y}$, with boundary conditions $f(x) = 0$ for $x \in \partial A$.

(2) Recall what is the discrete PDE satisfied by the harmonic measure $x \to H_A(x,\{y\})$ (where $y \in \partial A$)

Solution. The function $x \to H_A(x,\{y\})$ satisfies the discrete PDE $\Delta f(x) = 0$ with boundary conditions $f(x) = \delta_{x=y}$ for $x \in \partial A$.

(3) In this question, a *salary* is a function $s : A \to \mathbb{R}$ and an *exit bonus* is a function $b : \partial A \to \mathbb{R}$. Given a path $\omega = [\omega_0, \cdots, \omega_n]$ in $A \sqcup \partial A$ such that only $\omega_n \in \partial A$, the *reward* associated with ω is $r_{s,b} = \sum_{k=0}^{n-1} s(\omega_k) + b(s_n)$. Give an interpretation of $G_A(x,y)$ and $H_A(x,\{y\})$as an *expected reward.*

Solution. The number $G_A(x,y)$ is the expected reward of $(S_{n\wedge\tau_A})_n$, starting at x, with *salary* $\delta_{\cdot=y}$ and *exit bonus* 0:

$$G_A(x,y) = \mathbb{E}^x\left[\sum_{k=0}^{\tau_A-1} \delta_{S_k=y}\right].$$

The number $H_A(x,\{y\})$ is the expected reward of $(S_{n\wedge\tau_A})_n$, starting at x, with *salary* 0 and *exit bonus* $\delta_{\cdot=y}$:

$$H_A(x,y) = \mathbb{E}^x\left[\delta_{S_{\tau_A}=y}\right].$$

(4) Give an explicit solution to

$$\begin{cases} \Delta f = 0 & \text{in } A \\ f = F & \text{in } \partial A \end{cases} \tag{0.1}$$

in terms of $\{H_A(x,\{y\})\}_{x,y}$, and give a interpretation of the solution as an *expected reward.*

Solution. The solution is given by

$$f(x) = \sum_{y \in \partial A} H_A(x,\{y\}) F(y).$$

Indeed,

(a) if $x \in \partial A$, $f(x) = \sum_{y\in\partial A} H_A(x,\{y\}) F(y) = \sum_{y\in\partial A} \delta_{x=y} F(y) = F(x)$,
(b) if $x \in A$, $\Delta f(x) = \sum_{y\in\partial A} \Delta H_A(x,\{y\}) F(y) = \sum_{y\in \mathrm{A}} 0.F(y) = 0$,
(c) we have a unique solution since if f_1 and f_2 are solutions, then $h = f_1 - f_2$ is harmonic and null on ∂A: by the *maximum principle,* $h = 0$ and thus $f_1 = f_2$.

The solution f at x can be seen as the expected reward of $(S_{n\wedge\tau_A})_n$, starting at x, with *salary* 0 and *exit bonus* F:

$$f(x) = \mathbb{E}^x\left[F(S_{\tau_A})\right].$$

(5) Solve

$$\begin{cases} \Delta f = \rho & \text{in } A \\ f = 0 & \text{in } \partial A \end{cases} \tag{0.2}$$

in terms of the Green function and give an interpretation of $f(x)$ as an *expected reward.*

Solution. The unique solution is given by

$$f(x) = -\sum_{y\in A} \rho(y)\, G_A(x,y)\,.$$

Indeed,

(a) if $x \in A$, $\Delta f(x) = -\sum_{y\in A} \rho(y)\left(\Delta G_A(\cdot,y)\right)(x) = \sum_{y\in A} \rho(y)\,\delta_{x=y} = \rho(x)\,.$
(b) if $x \in \partial A$, $f(x) = \sum_{y\in A} \rho(y)\, G_A(x,y) = \sum_{y\in A} \rho(y)\, 0 = 0.$
The solution f at x can be seen as the expected reward of $\left(S_{n\wedge\tau_A}\right)_n$, starting at x, with *salary* $-\rho$ and *exit bonus* 0:

$$f(x) = -\mathbb{E}^x\left[\sum_{k=0}^{\tau_A-1} \rho(S_k)\right].$$

(6) Explain why

$$G_A(x,x) = \sum_{\omega:x\to x,\omega\subset A} \left(\frac{1}{2d}\right)^{|\omega|}$$

where $|\omega|$ is the length of the path $\omega = \left[x = \omega_0, \cdots, \omega_{|\omega|} = x\right]$.

Solution. We have

$$\begin{aligned} G_A(x,x) = \mathbb{E}^x\left[\#\{0\le n<\tau_A:\ S_n = x\}\right] = \mathbb{E}^x\left[\sum_{n<\tau_A} 1\!\!1_{S_n=x}\right] &= \mathbb{E}^x\left[\sum_n 1\!\!1_{\{S_n=x,n<\tau_A\}}\right] \\ &= \sum_n \mathbb{P}^x\left(S_n = x, n<\tau_A\right) \\ &= \sum_n \sum_{\omega:x\to x,\omega\subset A,|\omega|=n} \mathbb{P}(\omega) \\ &= \sum_{\omega:x\to x,\omega\subset A} \left(\frac{1}{2d}\right)^{|\omega|} \end{aligned}$$

Exercise 2. *Discretisation of PDEs : the equilibrium case*
We want to study the discrete PDEs :

$$\begin{cases} \Delta f = \rho & \text{in } A \\ f = F & \text{in } \partial A \end{cases} \tag{0.3}$$

and to give an explicit formulation in terms of the given functions ρ, F, the Green's function G_A and the harmonic measure $H_A(x,y)$

(1) Recall why there is at most one solution to the system (0.3).

Solution. If f_1 and f_2 are two solutions of the discrete PDEs (0.3), then $h = f_1 - f_2$ is harmonic and null on ∂A: by the *maximum principle,* $h = 0$ and thus $f_1 = f_2$.

(2) Solve the system (0.3) and give an interpretation of $f(x)$ as an *expected reward.*

Solution. If we consider a solution f_1 of the discrete PDE (0.1), and f_2 a solution of the discrete PDE (0.2), then $f_1 + f_2$ is a solution of the discrete PDE (0.3) (and actually the unique one by the point 1.)
Thus the unique solution of (0.3) is given by

$$f(x) = -\sum_{y\in A} \rho(y)\, G_A(x,y) + \sum_{y\in\partial A} H_A(x,\{y\})\, F(y)\,.$$

The solution f at x can be seen as the expected reward of $\left(S_{n\wedge\tau_A}\right)_n$, starting at x, with *salary* $-\rho$ and *exit bonus* F:

$$f(x) = \mathbb{E}^x\left[-\left[\sum_{k=0}^{\tau_A-1} \rho(S_k)\right] + F\left(S_{\tau_A}\right)\right].$$

Exercise 3. *Discretisation of PDEs: the evolution case*

We want to give an explicit formulation and a probabilistic interpretation of the solution to the discrete partial differential equation:

$$\begin{cases} \Delta f(x,t) = f(x,t+1) - f(x,t) & \text{for } (x,t) \in A \times \mathbb{N} \\ f(x,t) = F(x) & \text{for } (x,t) \in \partial A \times \mathbb{N} \cup A \times \{0\} \end{cases} \tag{0.4}$$

where $f : A \cup \partial A \to \mathbb{R}$.

(1) Suppose that $f(\cdot,t)$ converges to a function $g(\cdot)$ when t goes to infinity. What discrete partial differential equation does g satisfy ? Thus which function (or moditication of it) should appear in the explicit formulation : the Harmonic measure or the Green function ?

Solution. If $f(\cdot,t)$ converges to a function $g(\cdot)$ then taking $t \to \infty$ in the discrete PDE, we get : $\Delta g(x)=0$. Thus, we sould consider a modification of the Harmonic measure.

(2) Write the discrete PDE as $\Delta_t f(x,t) = 0$ where Δ_t is a linear operator.

Solution. The Laplacian is $\Delta f(x,t) = \left(\frac{1}{2d}\sum_{y\sim x} f(y,t)\right) - f(x,t)$. Thus we can write : $\Delta_t f(x,t) = \left(\frac{1}{2d}\sum_{y\sim x} f(y,t)\right) - f(x,t+1) = 0$.

(3) Find an explicit formulation of a solution. *Hint: think of a modification of the Harmonic measure when the random walk as a finite time to live.*

Solution. The last question implies that

$$f(x,t+1) = \frac{1}{2d}\sum_{y\sim x} f(y,t)$$

Thus, it is almost as if the function f was harmonic, but there is a new parameter t. Let us remark that when $t = 0$ then $f(x,0)$ is fixed : in the random walk image for harmonic functions, it is like if when $t = 0$ the random walk is killed. It is then natural to consider

$$f(x,t) = \mathbb{E}^x\left(F\left(S_{\tau_A \wedge t}\right)\right)$$

where the random walk gets a reward if either it exits the set A or it runs out of time.

Then $f(x,t) = \mathbb{E}^x\left(F\left(S_{\tau_A\wedge t}\right)\right) = \frac{1}{2d}\sum_{y\sim x} \mathbb{E}^x\left(F\left(S_{\tau_A\wedge t}\right)|S_1 = y\right)$, and

$$\mathbb{E}^x\left(F\left(S_{\tau_A\wedge t}\right)|S_1 = y\right) = \mathbb{E}^y\left(F\left(S_{\tau_A\wedge(t-1)}\right)\right)$$

by the Markov property thus

$$f(x,t) = \frac{1}{2d}\sum_{y\sim x} f(y,t-1).$$

(4) Let us consider the graph $A^{\rightarrow} = A \times \mathbb{N}$ with the following notion of neighbours: $(x_1,t_1) \rightsquigarrow (x_2,t_2)$ if and only if $t_1 - t_2 = -1$ and $x_1 \sim x_2$ in A (remark that this relation is not symmetric). Recall that the Laplacian on $A^{\rightarrow}$ is

$$\Delta f(\bar{x}) = \frac{1}{\#\{\bar{y} \rightsquigarrow \bar{x}\}}\sum_{\bar{y}\rightsquigarrow\bar{x}} \left(f(\bar{y}) - f(\bar{x})\right).$$

(a) Show that f is a solution to (0.4) if and only if f is harmonic on $A^{\rightarrow}$.

Solution. We have seen that the function f is a solution of (0.4) if and only if $f(x,t+1) = \frac{1}{2d}\sum_{y\rightsquigarrow x} f(y,t)$ and it satisfies the same boundary conditions. Let us remark that this last equation can be written as

$$f(x,t+1) = \frac{1}{2d}\sum_{(y,t)\rightsquigarrow(x,t+1)} f(y,t)$$

which is exactly equivalent to the fact that f is harmonic on $A^{\rightarrow}$.

(b) Prove that the solution to (0.4) is unique.

Solution. We can use again the maximum principle now on $A^{\rightarrow}$: the maximum of an harmonic function on $A^{\rightarrow}$ is on the boundary of $A^{\rightarrow}$. This gives us the unicity of the solution of the problem (0.4).

(c) What is $\partial A^{\rightarrow}$? Show that the harmonic measure $H_{A^{\rightarrow}}((x,t),\{(y,s)\})$ is equal to

$$\begin{cases}\mathbb{P}^x\left(S_{\tau_A}=y \text{ and } \tau_A=t-s\right) & \text{if } s>0\\ \mathbb{P}^x\left(S_t=y \text{ and } \tau_A\geq t\right) & \text{if } s=0\end{cases}$$

where we recall that $(S_n)_n$ is the simple random walk on A starting at x.

Solution. We have $\partial A^{\rightarrow}=(A\times\{0\})\sqcup(\partial A\times\mathbb{N})$. The harmonic measure $H_{A^{\rightarrow}}((x,t),\{(y,s)\})$ is equal to $\mathbb{P}^{(x,t)}\left(S^{\rightarrow}_{\tau_{A^{\rightarrow}}}=(y,s)\right)$, where $S^{\rightarrow}_{\tau_{A^{\rightarrow}}}$ the *inverse* simple random walk on $A^{\rightarrow}$ (i.e. it can only jump in the inverse direction to any oriented edge). If we consider the random walk $\left(S^{\rightarrow}_{n\wedge\tau^{\rightarrow}_A}\right)_{n\in\mathbb{N}}$ starting from (x,t) and stopped when it hits $\partial A^{\rightarrow}$, then has the same law as $\left(\left(S_{(n\wedge\tau_A)\wedge t},(t-(n\wedge\tau_A))^+\right)\right)_{n\in\mathbb{N}}$ where $(S_n)_n$ is the simple random walk starting at x and $n^+=\max(n,0)$. Thus if the simple random walk stopped when it hits $\partial A^{\rightarrow}$ goes out at (y,s) with $s>0$ it means that $\tau^{\rightarrow}_A=\tau_A$ and $\tau_A<t$ thus

$$\begin{aligned}\mathbb{P}^{(x,t)}\left(S^{\rightarrow}_{\tau_{A^{\rightarrow}}}=(y,s)\right) &= \mathbb{P}^{(x,t)}\left(\left(S_{(\tau^{\rightarrow}_A\wedge\tau_A)\wedge t},(t-(\tau^{\rightarrow}_A\wedge\tau_A))^+\right)=(y,s)\right)_{n\in\mathbb{N}}\\ &= \mathbb{P}^x\left((S_{\tau_A},t-\tau_A)=(y,s)\right)=\mathbb{P}^x\left(S_{\tau_A}=y \text{ and } \tau_A=t-s\right)\end{aligned}$$

and if the simple random walk stopped stopped when it hits $\partial A^{\rightarrow}$ goes out at (y,s) with $s=0$ it means that $\tau_{A^{\rightarrow}}=t$ and actually $\tau_A\geq t$ thus

$$\begin{aligned}\mathbb{P}^{(x,t)}\left(S^{\rightarrow}_{\tau_{A^{\rightarrow}}}=(y,s)\right) &= \mathbb{P}^{(x,t)}\left(\left(S_{(\tau^{\rightarrow}_A\wedge\tau_A)\wedge t},(t-(\tau^{\rightarrow}_A\wedge\tau_A))^+\right)=(y,s)\right)_{n\in\mathbb{N}}\\ &= \mathbb{P}^x\left(S_t=y \text{ and } \tau_A>t\right).\end{aligned}$$

(d) Using the last question, give the explicit formulation of (0.4).

Solution. We know that the unique harmonic function f on $A^{\rightarrow}$ with boundary conditions given by $f(x,t)=F(x)$ for $(x,t)\in\partial A^{\rightarrow}$ is given by

$$f((x,t))=\sum_{(y,s)\in\partial A^{\rightarrow}}H_{A^{\rightarrow}}((x,t),(y,s))\,F(y).$$

Using the last question, we can write it as

$$f((x,t))=\sum_{y\in A}\mathbb{P}^x\left(S_t=y \text{ and } \tau_A\geq t\right)F(y)+\sum_{y\in\partial A}\sum_{s=1}^{t}\mathbb{P}^x\left(S_{\tau_A}=y \text{ and } \tau_A=t-s\right)F(y).$$

Let us remark that $\sum_{s=1}^t\mathbb{P}^x\left(S_{\tau_A}=y \text{ and } \tau_A=t-s\right)=\mathbb{P}\left(S_{\tau_A}=y \text{ and } \tau_A<t\right)$ thus :

$$\begin{aligned}f((x,t)) &= \sum_{y\in A}\mathbb{P}^x\left(S_t=y \text{ and } \tau_A\geq t\right)F(y)+\sum_{y\in\partial A}\mathbb{P}^x\left(S_{\tau_A}=y \text{ and } \tau_A<t\right)F(y)\\ &= \sum_{y\in A\sqcup\partial A}\mathbb{P}^x\left(S_{t\wedge\tau_A}=y\right)F(y)\\ &= \mathbb{E}^x\left(F\left(S_{t\wedge\tau_A}\right)\right)\end{aligned}$$

and thus we recover the result of point 3.

Exercise 4. *Discretisation of PDEs: the time-dependant boundary condition.*

We want to give an explicit formulation and a probabilistic interpretation of the solution to the discrete partial differential equation:

$$\begin{cases}\Delta f(x,t)=f(x,t+1)-f(x,t) & \text{for } (x,t)\in A\times\mathbb{N}\\ f(x,t)=F(x,t) & \text{for } (x,t)\in\partial A\times\mathbb{N}\cup A\times\{0\}\end{cases}$$

where $f:A\cup\partial A\to\mathbb{R}$.

Following the same ideas used for the point 4. of Exercise 3, give an explicit formulation and a probabilistic interpretation of the solution to the latter discrete partial differential equation.

Solution. Using the same ideas used for the point 4. of Exercise 3, we get that the solution of this discrete PDE is given by:

$$f(x,t) = \mathbb{E}\left(F\left(S_{\tau_A \wedge t}, (t-\tau_A)^+\right)\right).$$

Indeed, we are still looking for an harmonic function on $A^{\rightarrow}$ but now the boundary conditions are different : if the walk starts at (x,t) and goes out at (y,s), then the reward is $F(y,s)$. But $s = t - \tau_A$ if $\tau_A < t$ and $s = 0$ if $\tau_A \geq 0$: thus $s = (t-\tau_A)^+$.