Recall that the Green's function on a domain $A \subset \mathbb{Z}^d$ is the function

$$G_A(x,y) = \mathbb{E}^x\left[\#\{0 \le n < \tau_A :\ S_n = y\}\right]$$

where $(S_n)_{n\in\mathbb{N}}$ is the simple random walk started at x (under $\mathbb{P}^x$) and $\tau_A = \min\{n \ge 0 : S_n \notin A\}$.

We also recall that the Harmonic measure on A associated to a subset $B \subset \partial A$ is the function which for each $x \in A$ takes value

$$H_A(x,B) = \mathbb{P}^x\left[S_{\tau_A} \in B\right].$$

Exercise 1. *General knowledge*

(1) Recall what is the discrete PDE satisfied by the function $x \to G_A(x,y)$.
(2) Recall what is the discrete PDE satisfied by the harmonic measure $x \to H_A(x,\{y\})$ (where $y \in \partial A$)
(3) In this exercise sheet, a *salary* is a function $s : A \to \mathbb{R}$ and an *exit bonus* is a function $b : \partial A \to \mathbb{R}$. Given a path $\omega = [\omega_0, \cdots, \omega_n]$ in $A \sqcup \partial A$ such that only $\omega_n \in \partial A$, the *reward* associated with ω is $r_{s,b} = \sum_{k=0}^{n-1} s(\omega_k) + b(s_n)$. Give an interpretation of $G_A(x,y)$ and $H_A(x,\{y\})$as an *expected reward.*
(4) Give an explicit solution to

$$\begin{cases} \Delta f = 0 & \text{in } A \\ f = F & \text{in } \partial A \end{cases} \tag{0.1}$$

in terms of $\{H_A(x,\{y\})\}_{x,y}$, and give a interpretation of the solution as an *expected reward.*
(5) Solve

$$\begin{cases} \Delta f = \rho & \text{in } A \\ f = 0 & \text{in } \partial A \end{cases} \tag{0.2}$$

in terms of the Green function and give an interpretation of $f(x)$ as an *expected reward.*
(6) Explain why

$$G_A(x,x) = \sum_{\omega: x\to x, \omega\subset A} \left(\frac{1}{2d}\right)^{|\omega|}$$

where $|\omega|$ is the length of the path $\omega = \left[x = \omega_0, \cdots, \omega_{|\omega|} = x\right]$.

Exercise 2. *Discretisation of PDEs : the equilibrium case*

We want to study the discrete PDEs :

$$\begin{cases} \Delta f = \rho & \text{in } A \\ f = F & \text{in } \partial A \end{cases} \tag{0.3}$$

and to give an explicit formulation in terms of the given functions ρ, F, the Green's function G_A and the harmonic measure $H_A(x,y)$.

(1) Recall why there is at most one solution to the system (0.3).
(2) Solve the system (0.3) and give an interpretation of $f(x)$ as an *expected reward.*

Exercise 3. *Discretisation of PDEs: the evolution case*

We want to give an explicit formulation and a probabilistic interpretation of the solution to the discrete partial differential equation:

$$\begin{cases} \Delta f(x,t) = f(x,t+1) - f(x,t) & \text{for } (x,t) \in A \times \mathbb{N} \\ f(x,t) = F(x) & \text{for } (x,t) \in \partial A \times \mathbb{N} \cup A \times \{0\} \end{cases} \tag{0.4}$$

where $f : A \cup \partial A \to \mathbb{R}$.

(1) Suppose that $f(\cdot,t)$ converges to a function $g(\cdot)$ when t goes to infinity. What discrete partial differential equation does g satisfy ? Thus which function (or moditication of it) should appear in the explicit formulation : the Harmonic measure or the Green function ?
(2) Write the discrete PDE as $\Delta_t f(x,t) = 0$ where Δ_t is a linear operator.
(3) Find an explicit formulation of a solution. *Hint: think of a modification of the Harmonic measure when the random walk as a finite time to live.*

(4) Let us consider the graph $A^{\rightarrow} = A \times \mathbb{N}$ with the following "oriented notion" of neighbours: $(x_1, t_1) \rightsquigarrow (x_2, t_2)$ if and only if $t_1 - t_2 = -1$ and $x_1 \rightsquigarrow x_2$ in A (remark that this relation is not symmetric). Recall that the Laplacian on $A^{\rightarrow}$ is

$$\Delta f(\bar{x}) = \frac{1}{\#\{\bar{y} \rightsquigarrow \bar{x}\}} \sum_{\bar{y} \rightsquigarrow \bar{x}} (f(\bar{y}) - f(\bar{x})).$$

(a) Show that f is a solution to (0.4) if and only if f is harmonic on $A^{\rightarrow}$.
(b) Recall why the solution to (0.4) is unique.
(c) What is $\partial A^{\rightarrow}$? Show that the harmonic measure $H_{A^{\rightarrow}}((x,t), \{(y,s)\})$ is equal to

$$\begin{cases} \mathbb{P}^x (S_{\tau_A} = y \text{ and } \tau_A = t - s) & \text{if } s > 0 \\ \mathbb{P}^x (S_t = y \text{ and } \tau_A \geq t) & \text{if } s = 0 \end{cases}$$

where we recall that $(S_n)_n$ is the simple random walk on A starting at x.
(d) Using the point 4.(c), give the explicit formulation of (0.4).

Exercise 4. *Discretisation of PDEs: the time-dependant boundary condition.*

We want to give an explicit formulation and a probabilistic interpretation of the solution to the discrete partial differential equation:

$$\begin{cases} \Delta f(x,t) = f(x,t+1) - f(x,t) & \text{for } (x,t) \in A \times \mathbb{N} \\ f(x,t) = F(x,t) & \text{for } (x,t) \in \partial A \times \mathbb{N} \cup A \times \{0\} \end{cases}$$

where $f : A \cup \partial A \to \mathbb{R}$.

Following the same ideas used for the point 4. of Exercise 3, give an explicit formulation and a probabilistic interpretation of the solution discrete partial differential equation.