

Exercise 1. *Neumann boundary conditions and Dirichlet boundary conditions*

Let us consider the square $A = [[1, n] \times [1, m]]$ in \mathbb{Z}^2 , and its boundary ∂A which is the set of vertices in $\mathbb{Z}^2 \setminus A$ adjacent to a vertex of A . We define the normal derivative at $y \in \partial A$ as :

$$\partial_n f(y) = f(x) - f(y),$$

where x is the unique neighbour of y in A . We denote by $\underline{\partial A}, \partial A, \overline{\partial A}$ and $|\partial A$ the 4 parts of the boundary, respectively the lower horizontal, the right most vertical, the upper horizontal and the left-most vertical parts of ∂A .

- (1) Prove that the problem

$$\begin{cases} \Delta f(x) = 0 & \text{in } A \\ f(y) = 0 & \text{on } \underline{\partial A} \\ f(y) = 1 & \text{on } \overline{\partial A} \\ \partial_n f(y) = 0 & \text{on } \partial A \setminus |\partial A \end{cases}$$

has a unique solution if any.

- (2) Let us consider the simple random walk on $A \cup \partial A$ ($x \in A$ and $y \in \partial A$ are linked by an edge if y is adjacent to x in \mathbb{Z}^2 , two points of ∂A are not linked by an edge). Is $\tau_{\underline{\partial A} \cup \overline{\partial A}}$, the hitting time for $\underline{\partial A} \cup \overline{\partial A}$, finite almost surely ?
- (3) Give an explicit formulation of the unique solution of the discrete PDE in terms of random walks. What is the difference with the pure Dirichlet conditions ?

Remark. We used a mix of conditions in order to have a finite stopping time. Yet, one can solve the Neumann problem with pure Neumann boundary conditions. In this case, the boundary conditions must satisfy some additional conditions for a solution to exist, and the solution is unique only up to a constant. This is slightly more technical. (Section 6.7 of <https://www.math.uchicago.edu/~lawler/srwbook.pdf>)

Remark. Actually we considered a rectangle for simplicity, but one can do the same with any discretisation of any domain Ω with 4 points marked on the boundary in counterclockwise order a, b, c and d and with Dirichlet boundary conditions on $[a, b], [c, d]$ and Neumann conditions on $[b, c], [d, e]$. Then the solution would be the imaginary part of the discretisation of the conformal mapping which sends the domain Ω to a rectangle $[0, L] \times [0, i]$ (for some L) which sends a, b, c, d to the corners of the rectangle.

Exercise 2. *Green's function represented with determinants*

The goal of this exercise is to prove that if $x_1, \dots, x_n \in A$ and $A_k = A \setminus \{x_1, \dots, x_k\}$ then

$$G_A(x_1, x_1) G_{A_1}(x_2, x_2) \cdots G_{A_{n-1}}(x_n, x_n)$$

is independent of the order of x_1, \dots, x_n .

Remark. For this exercise sheet, we will consider :

$$\Delta f(x) = f(x) - \frac{1}{\deg(x)} \sum_{y \sim x} f(y)$$

in order to avoid minus signs (this is the opposite of the definition in the lesson).

- (1) Recall that we can consider Δ to be a linear operator $\Delta_A : \mathbb{R}^A \rightarrow \mathbb{R}^A$, where \mathbb{R}^A is the space of real functions on $A \cup \partial A$ such that $f|_{\partial A} = 0$. Show that :

$$G_A(x, x) = \frac{\det \Delta_{A \setminus \{x\}}}{\det \Delta_A}.$$

Hint. Think about Cramer's rule for describing inverse matrices.

- (2) If $x_1, \dots, x_n \in A$ and $A_k = A \setminus \{x_1, \dots, x_k\}$, give the value of

$$G_A(x_1, x_1) G_{A_1}(x_2, x_2) \cdots G_{A_{n-1}}(x_n, x_n)$$

and prove that it is independent of the order of x_1, \dots, x_n .

Exercise 3. *Determinant of Laplacian and uniform spanning trees : Kirchhoff's theorem*

Remark. If M is a matrix, we denote by $M^{i,j}$ the matrix obtained by deleting the i -th row and j -column of M . We also denote by $M^{i,\cdot}$ the matrix obtained by deleting only the i -th row of M . The cofactor $\det^{i,j} M$ is $(-1)^{i+j} \det M^{i,j}$.

Let G be a connected graph with n vertices and m edges (here an edge is a couple of vertices, in particular, we do not consider the case where two vertices are related by two or more edges). Recall that a spanning tree of G is a connected subgraph of G with no loop and which covers all vertices of G . Let us give a unique number between 1 and n to each vertex and a unique number between 1 and m to each edge. For this exercise, we denote by $\tilde{\Delta}_G$ the matrix defined by:

$$\tilde{\Delta}_G(i, j) = \delta_{i,j} \deg(i) + (\delta_{i,j} - 1) \delta_{j \sim i},$$

where $1 \leq i, j \leq n$ denote vertices of G and $\deg(i)$ is the degree of i .

We will prove Kirchhoff's theorem :

$$\text{Kirchhoff's theorem : } \# \{\text{spanning trees of } G\} = \det^{i,j} \left(\tilde{\Delta}_G \right)$$

Actually, we will only show that $\# \{\text{spanning trees of } G\} = \det^{1,1} \left(\tilde{\Delta}_G \right)$, the general case can be deduced using elementary linear algebra arguments.

- (1) Let the $n \times m$ incidence matrix E such that the only non zero elements are given by the following: if the k -th edge goes between i and j and $i < j$ then $E_{ik} = 1$ and $E_{jk} = -1$. Show that $\tilde{\Delta}_G = EE^t$, where E^t is the transpose of E .
- (2) Show that $\tilde{\Delta}_G^{1,1} = E^{1,\cdot} (E^{1,\cdot})^t$.
- (3) Prove that $m \geq n - 1$ i.e. the matrix $E^{1,\cdot}$ has a horizontal shape more than a vertical shape.
- (4) Recall the Cauchy-Binet formula which says that if A and B are two matrices of size $l \times k$ and $k \times l$ then

$$\det (AB) = \sum_{S \subset [k], \#S=l} \det (A_{[l],S}) \det (B_{S,[l]})$$

where $[k] = \{1, \dots, k\}$, $A_{S,[k]}$ is the matrix obtained by choosing the rows in S and the columns in $[k]$. Use this formula to show that

$$\det \tilde{\Delta}_G^{1,1} = \sum_{S \subset [m], \#S=n-1} \det \left((E^{1,\cdot})_{[n-1],S} \right)^2.$$

- (5) What represents a choice of $S \subset [m]$ in the original graph G ? We want to show that S forms a spanning tree of G if and only if $\det \left((E^{1,\cdot})_{[n-1],S} \right) = \pm 1$, and if it does not form a spanning tree then $\det \left((E^{1,\cdot})_{[n-1],S} \right) = 0$.
 - (a) Show that if S is not a spanning tree, then there exists a cycle in S .
 - (b) Show that if S is not a spanning tree then $\det \left((E^{1,\cdot})_{[n-1],S} \right) = 0$.
 - (c) Let us suppose that S is a spanning tree. Consider the vertex 1 and an edge e in S connected to 1 and to a vertex i . Prove that

$$\det \left((E^{1,\cdot})_{[n-1],S} \right) = \pm \det \left((E^{1,\cdot})_{[n-1] \setminus \{i-1\}, S \setminus \{e\}} \right).$$
 - (d) Conclude that if S is a spanning tree, then $\det \left((E^{1,\cdot})_{[n-1],S} \right) = \pm 1$.
- (6) Prove Kirchhoff's theorem.
- (7) How do you write the number $\# \{\text{spanning trees of } G\}$ using the Laplacian Δ_G ?