Exercise 1. *Coupling*

(1) Let $0 < p' < p < 1$, let X_p be a Bernoulli (p) random variable. How can you sample $X_{p'} \sim Ber\left(p'\right)$ using X_p and one other Bernoulli random variable, such that $X_{p'} \leq X_p$ almost surely?

Solution. Let X_p be a Bernoulli(p) random variable. We need to discard the result with a certain probability when $X_p = 1$. Let us consider Y a Bernoulli random variable of parameter q. Let us consider X_pY: it is also a Bernoulli random variable since it takes only $0-1$ values. Let us compute its parameter :

$$\mathbb{E}\left(X_pY\right) = \mathbb{E}\left(X_p\right)\mathbb{E}\left(Y\right) = pq$$

Thus, if $Y \sim Ber\left(\frac{p'}{p}\right)$ then $X_pY \sim Ber\left(p'\right)$ and by definition $X_pY \leq X_p$ almost surely.

(2) Let us consider that you have an infinite random sequence of independent Bernoulli(p). How can you create an infinite random sequence of independent Bernouilli$\left(\frac{1}{2}\right)$?

Remark. This means that if you do not trust the coin of somebody, you can still create a fair "head/tail" process.

Solution. Let us consider two random variables X_1 and X_2 which are two independant Bernoulli(p) random variables. Let us write the probabilities for the couple (X_1, X_2):

$$\begin{array}{ccccc} X_1, X_2: & 1,1 & 0,1 & 1,0 & 0,0 \\ \mathbb{P} & p^2 & p-p^2 & p-p^2 & 1-2p+p^2 \end{array}$$

The probabilities to get $(0,1)$ or $(1,0)$ are equal : thus we condition on the fact that $(X_1, X_2) \in \{(0,1), (1,0)\}$ and we set

$$\begin{cases} Y = 0 & if \;\; (X_1, X_2) = (0,1) \\ Y = 1 & if \;\; (X_1, X_2) = (1,0) \end{cases}$$

If we consider an infinite number of independant pairs of independant Bernoulli(p), $\left(\left(X_1^i, X_2^i\right)_{i\in\mathbb{N}}\right)$ then we consider τ_1 the first time where $(X_1^{\tau_1}, X_2^{\tau_1}) \in \{(0,1), (1,0)\}$ and we set

$$\begin{cases} Y_1 = 0 & if \;\; (X_1^{\tau_1}, X_2^{\tau_1}) = (0,1) \\ Y_1 = 1 & if \;\; (X_1^{\tau_1}, X_2^{\tau_1}) = (1,0) \end{cases}$$

and we define similarly Y_i for any $i \geq n$. The sequence $(Y_i)_{i\in\mathbb{N}}$ is an infinite random sequence of independent Bernouilli$\left(\frac{1}{2}\right)$.

(3) Let U be a random uniform variable in $[0,1]$. How can you sample a Bernoulli(p) ?

Solution. We consider U a random uniform variable in $[0,1]$. The random variable $X = \mathbb{1}_{U\leq p}$ is a Bernoulli(p) .

(4) Let us denote by $\mathbb{P}_p$ be the probability associated with the site percolation on some infinite lattice with probability p (i.e. a site is open independently from the other with probability p). Show that

$$\mathbb{P}_p\left(0 \rightsquigarrow \infty\right)$$

is increasing in p.

Solution. The idea is to use a coupling as for example defined in point 1. or, as used here, using point 3. and a collection of independent uniform random variable associated to each vertices with mutual probability $\mathbb{P}$.

Let us therefore consider $(U_x)_{x\in V}$ a family of independent uniform random variables on $[0,1]$. If we define

$$\left(X_x^p = \mathbb{1}_{U_x\leq p}\right)_{x\in V}$$

we recover a site percolation of parameter p. If p increases, then almost surely (in fact for all events) for any $x \in V$ we have that X_x^p increases and thus

$$\left\{\omega \in \Omega, 0 \rightsquigarrow \infty \text{ for } \left(X_x^p\right)_{x\in V}\right\} \subset \left\{\omega \in \Omega, 0 \rightsquigarrow \infty \text{ for } \left(X_x^{p'}\right)_{x\in V}\right\}$$

if $p \leq p'$. Thus $p \mapsto \mathbb{P}_p(0 \rightsquigarrow \infty) = \mathbb{P}\left(0 \rightsquigarrow \infty \text{ for } (X_x^p)_{x\in V}\right)$ is increasing.

Exercise 2. *Connective constant of graphs*

In this exrecise we will only work on the graph $\mathbb{Z}_2$, but the result generalizes easily for any regular graph. We want to define a probability measure on the set of self-avoiding random walks (i.e. on the set of paths ω such that $\omega(i) \neq \omega(j)$ for any $i \neq j$) of the form:

$$P_\beta(\omega) = \frac{1}{Z_\beta} e^{-\beta|\omega|},$$

where $|\omega|$ is the length of ω. In order to do so, we need to understand Z_β: if it is infinite, we cannot define this probability measure, if it is finite, we can. We will admit the following lemma (that you can try to prove):

Lemma. *If $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that:*

(1) *there exists $c \geq 1$, $a_n \geq c^n$ for any n,*
(2) *for any $n, p \geq 1$, $a_{n+p} \leq a_n a_p$,*

then there exists $\mu \geq c$ such that $a_n^{\frac{1}{n}} \to \mu$ when $n \to \infty$. Besides, $\inf_n (a_n)^{\frac{1}{n}} = \mu$.

(1) What should be the value of Z_β ? *Hint : we want a probability measure.*

Solution. We want P_β to be a probability measure : thus

$$\sum_\omega P_\beta(\omega) = \frac{1}{Z_\beta}\sum_\omega e^{-\beta|\omega|} = 1,$$

hence $Z_\beta = \sum_\omega e^{-\beta|\omega|}$.

(2) Let us define by λ_N the number of simple walks of size N which start at 0. What is the limit of $(\lambda_N)^{\frac{1}{N}}$ as N goes to infinity?

Solution. The number of simple random walks of size N which start at 0 is equal to 4^N, hence $(\lambda_N)^{\frac{1}{N}} = 4$, which in particular converges to 4 as N goes to infinity.

(3) Let us define by μ_N the number of self-avoiding walks of size N which start at 0. Prove that $(\mu_N)^{\frac{1}{N}}$ converges as N goes to infinity to a number $\mu \geq 2$ which is called the connective constant of the lattice.

Solution. We will use the lemma given at the beginning of the exercise, we need to prove that for any $n, p \geq 1$ that

$$\mu_{n+p} \leq \mu_n \mu_p.$$

Hence, we need to prove that

$$\#\{\omega, |\omega| \leq n+p\} \leq \#\{\omega, |\omega| \leq n\}.\#\{\omega, |\omega| \leq p\}$$

where any ω is a self-avoiding walk.

But if ω is a self-avoiding walk of length $n+p$, then $\omega_{[1\ldots n]}$, and $\omega_{[n+1,\ldots,n+p]}$ are two self-avoiding walks of length n and p. This implies the previous inequality. Besides, if we consider paths which only goes on the right or goes up, we see that $\mu_n \geq 2^n$. Using the lemma at the beginning of the exercise, this implies,: $\mu_n^{\frac{1}{n}}$ converges as n goes to infinity towards a number $\mu \geq 2$.

(4) Deduce that there exists β_c such that

$$\beta > \beta_c \iff Z_\beta < \infty.$$

Give the value of $\beta_c = \beta_c(\mu)$.

Solution. The number $Z_\beta = \sum_\omega e^{-\beta|\omega|} = \sum_n \mu_n e^{-\beta n} = \sum_n \left((\mu_n)^{\frac{1}{n}} e^{-\beta}\right)^n$ is finite if and only if the limit of $(\mu_n)^{\frac{1}{n}} e^{-\beta}$ is strictly less than 1. Indeed, if $\lim_n (\mu_n)^{\frac{1}{n}} e^{-\beta} > 1$ then the sum is clearly infinite, and if the limit is equal to 1, then recall the lemma at the beginning of the exercise : we know that

$$\lim_n \left((\mu_n)^{\frac{1}{n}} e^{-\beta}\right) = \inf_n \left((\mu_n)^{\frac{1}{n}} e^{-\beta}\right)$$

thus it means that $\inf_n (\mu_n)^{\frac{1}{n}} e^{-\beta} \geq 1$ and thus the sum is also infinite.

Thus the sum is finite if and only if $\mu e^{-\beta} < 1$, this implies that $\beta_c = \ln \mu$.

Remark. The connective constant of the honneycomb lattice has been computed in 2010 by H. Duminil-Copin and S. Smirnov with an elegant 6 pages proof (https://arxiv.org/pdf/1007.0575.pdf), using parafermionic observables.

Exercise 3. *Site and edge percolation*

For any graph $G = (V, E)$, the *edge percolation* is given by a collection $(X_e)_{e \in E}$ of independent RV $X_e \sim Ber\,(p)$.

(1) Show that for any edge percolation $(X_e)_{e \in E}$, there exists a graph $G' = (V', E')$ and a bijection $\phi : E \to V'$ such that

$$\left(X_{\phi^{-1}(x)}\right)_{x \in V'}$$

is a site percolation on G'. Moreover show that there exists an infinite componant for the edge percolation $(X_e)_{e \in E}$ if and only if there exists an infinite componant for the site percolation $\left(X_{\phi^{-1}(x)}\right)_{x \in V'}$. We say that every edge percolation is equivalent to a site percolation on a modified graph.

Solution. We consider the graph $G' = (V', E')$ with $V' = E$ and $(e, e') \in E'$ for any $e \in E$ and $e' \in E$ which share an endpoint. We can consider the canonical bijection $\phi : E \to V' = E$. If there exists an infinite open path in the edge percolation $e_1, e_2, ...e_n, ...$ then since e_{i+1} must share a vertex with e_i, this means that $\phi\,(e_{i+1})$ is linked to $\phi\,(e_i)$ in G'. Thus $\phi\,(e_1)\,, \ldots, \phi\,(e_n)\,, \ldots$ is an open path for the site percolation. It is also true that if there exists an open path $x_1, \ldots, x_n, \ldots$ for the site percolation, then $\phi^{-1}\,(x_1)\,, \ldots, \phi^{-1}\,(x_n)\,, \ldots$ is also an open path for the edge percolation for G.

(2) What is the modified graph associated with $\mathbb{Z}^2$?

Solution. Any edge e is replaced by a vertex (we consider the midpoint of e). Two edges (i.e. two midpoints) are connected either if the two edges are adjacent and orthogonal, or adjacent and parallel. In the first case, the links that we need to add give a graph similar to $\mathbb{Z}^2$ and rotated by $\frac{\pi}{4}$. The second types of edges are the diagonals of one out of two squares.

Remark. Let us remark that for any G without edges which are loops, the modified graph associated with G' has a special property : for any $x \in G'$, the set N_x of neighbours of x can be divided in two sets $N_x^{(1)} \sqcup N_x^{(2)}$ where for any $i \in \{1, 2\}$, any vertices $u, v \in N_x^{(i)}$ are linked by an edge (this is due to the fact that any edge $e \in G$ has two endpoints). This implies that one can in general not create an edge percolation which is equivalent to a given site percolation.