Exercise 1. *Coupling*

(1) Let $0 < p' < p < 1$, let X_p be a Bernoulli (p) random variable. How can you sample $X_{p'} \sim Ber(p')$ using X_p and one other Bernoulli random variable, such that $X_{p'} \leq X_p$ almost surely?

(2) Let us consider that you have an infinite random sequence of independent Bernoulli(p). How can you create an infinite random sequence of independent Bernouilli$\left(\frac{1}{2}\right)$?

Remark. This means that if you do not trust the coin of somebody, you can still create a fair "head/tail" process.

(3) Let U be a random uniform variable in $[0, 1]$. How can you sample a Bernoulli(p) ?

(4) Let us denote by $\mathbb{P}_p$ be the probability associated with the site percolation on some infinite lattice with probability p (i.e. a site is open independently from the other with probability p). Show that

$$\mathbb{P}_p(0 \rightsquigarrow \infty)$$

is increasing in p.

Exercise 2. *Connective constant of graphs*

In this exrecise we will only work on the graph $\mathbb{Z}_2$, but the result generalizes easily for any regular graph. We want to define a probability measure on the set of self-avoiding random walks (i.e. on the set of paths ω such that $\omega(i) \neq \omega(j)$ for any $i \neq j$) of the form:

$$P_\beta(\omega) = \frac{1}{Z_\beta} e^{-\beta|\omega|},$$

where $|\omega|$ is the length of ω. In order to do so, we need to understand Z_β: if it is infinite, we cannot define this probability measure, if it is finite, we can. We will admit the following lemma (that you can try to prove):

Lemma. *If $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that:*

(1) *there exists $c \geq 1$, $a_n \geq c^n$ for any n,*

(2) *for any $n, p \geq 1$, $a_{n+p} \leq a_n a_p$,*

then there exists $\mu \geq c$ such that $a_n^{\frac{1}{n}} \to \mu$ when $n \to \infty$. Besides, $\inf_n (a_n)^{\frac{1}{n}} = \mu$.

(1) What should be the value of Z_β ? *Hint : we want a probability measure.*

(2) Let us define by λ_N the number of simple walks of size N which start at 0. What is the limit of $(\lambda_N)^{\frac{1}{N}}$ as N goes to infinity?

(3) Let us define by μ_N the number of self-avoiding walks of size N which start at 0. Prove that $(\mu_N)^{\frac{1}{N}}$ converges as N goes to infinity to a number $\mu \geq 2$ which is *called the connective constant of the lattice.*

(4) Deduce that there exists β_c such that

$$\beta > \beta_c \iff Z_\beta < \infty.$$

Give the value of $\beta_c = \beta_c(\mu)$.

Remark. The connective constant of the honneycomb lattice has been computed in 2010 by H. Duminil-Copin and S. Smirnov with an elegant 6 pages proof (https://arxiv.org/pdf/1007.0575.pdf), using parafermionic observables.

Exercise 3. *Site and edge percolation*

For any graph $G = (V, E)$, the *edge percolation* is given by a collection $(X_e)_{e\in E}$ of independent Bernoulli (p) random variables.

(1) Show that for any edge percolation $(X_e)_{e\in E}$, there exists a graph $G' = (V', E')$ and a bijection $\phi : E \to V'$ such that

$$\left(X_{\phi^{-1}(x)}\right)_{x\in V'}$$

is a site percolation on G'. Moreover show that there exists an infinite componant for the edge percolation $(X_e)_{e\in E}$ if and only if there exists an infinite componant for the site percolation $\left(X_{\phi^{-1}(x)}\right)_{x\in V'}$. We say that every edge percolation is equivalent to a site percolation on a modified graph.

(2) What is the modified graph associated with $\mathbb{Z}^2$?

Remark. Let us remark that for any G without edges which are loops, the modified graph associated with G' has a special property : for any $x \in G'$, the set N_x of neighbours of x can be divided in two sets $N_x^{(1)} \sqcup N_x^{(2)}$ where for any $i \in \{1, 2\}$, any vertices $u, v \in N_x^{(i)}$ are linked by an edge (this is due to the fact that any edge $e \in G$ has two endpoints). This implies that one can in general *not* create an edge percolation which is equivalent to a given site percolation.