In this exercises sheet, when we talk about vertex percolation, we consider the vertex percolation on the triangular lattice. Recall that the vertex percolation on the triangular lattice can be seen equivalently as a face percolation on the hexagonal lattice : if you consider the dual graph of the triangular lattice, you get the hexagonal lattice, and every vertex becomes a face.

**Exercise 1.** *Duality*

(1) Show that the probability in bond percolation at $p = \frac{1}{2}$ on $\mathbb{Z}_2$ restricted to the rectangle $[0, n] \times [0; n + 1]$ that there is a crossing from top to bottom is exactly $\frac{1}{2}$.

**Solution.** This is almost what you have seen in the lesson, except that one has to use the dual percolation. Indeed, in the vertex percolation (or face percolation) you cannot have a black crossing from top to bottom and a white crossing from left to right. But in the bound percolation, you CAN have a crossing from top to bottom of black (labelled by 1/present/open) edges and a crossing from left to right of white edges (labelled by 0/absent/closed).

What can prevent us from using open edges to go from the top to the bottom : there must be a lot of closed edges that canot be used to go further down. These edges forms a sequence of closed edges going from left to right were two consecutive edges are contained in a common squared face: this gives a path of open edges in the dual percolation going from left to right. Thus we just saw that :

(a) either there exists a crossing from top to bottom of open edges
(b) or there exists a crossing from left to right of open edges in the dual percolation,

and the two events have no intersection.

But we saw that the dual percolation is a 1/2-percolation, and the graph is exactly the same as $\mathbb{Z}_2$ restricted to the rectangle $[0, n] \times [0; n + 1]$. Using the symmetry of the square we deduce that the two events have the same probability : the probability in bond percolation at $p = \frac{1}{2}$ on $\mathbb{Z}_2$ restricted to the rectangle $[0, n] \times [0; n + 1]$ that there is a crossing from top to bottom is exactly $\frac{1}{2}$.

(2) Suppose we have a discretisation of a simply connected domain $\Omega$ with smooth boundary. Mark three distinct points $a,\ b,\ c$ on $\partial\Omega$ (in clockwise order). Using duality for the vertex percolation at $p = \frac{1}{2}$, show that the probability that there is an open cluster which connects all three boundary segments $[a, b]\,, [b, c]\,, [c, a]$ is $\frac{1}{2}$.

*Hint : think about how we proved the first point... (but now in the context of vertex percolation)*

**Solution.** We need to find two events $\mathcal{E}$ and $\mathcal{F}$ such that :

(a) $\mathcal{E}$ is the event that there is an open cluster connecting all three boundary segments $[a, b]\,, [b, c]\,, [c, a]$,
(b) for any realisation of the percolation, either $\mathcal{E}$ or $\mathcal{F}$ happens,
(c) $\mathcal{E} \cap \mathcal{F} = \emptyset$ ,
(d) $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{F})$ because of some symmetry property.

You should then think that maybe since we are considering the vertex percolation (and not the bond percolation), we do not think about the dual graph, we will find $\mathcal{F}$ directly by considering the percolation on the discretisation of $\Omega$. Then thinking about the lesson (either black crossing up down of white crossing left right), we should consider

$$\mathcal{F} = \text{ there is a white cluster which connects all three boundary segments } [a, b]\,[b, c]\,[c, a]\,.$$

And actually either $\mathcal{E}$ or $\mathcal{F}$ happens, and they can not happen at the same time by the following observation: if there is no open cluster connecting all three boundary segments, then there is one segment, say $[a, b]$, for which all the clusters of open vertices connected to it do not touch both $[b, c]$ and $[c, a]$. Considering the outermost such open cluster $C$ (relative to $[a, b]$), and assume without loss of generality that it doesn't connect to $[c, a]$. Then the white cluster bording it away from $[a, b]$ must touch $[a, b]$ and $[b, c]$ (since so does the cluster $C$) as well as the segment $[c, a]$ (since otherwise $C$ wouldn't be outermost relative to $[a, b]$). Besides, since we are considering the $p = \frac{1}{2}$ case, we can flip all the white in black without changing the probabilities : $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{F})$. Thus $\mathbb{P}(\mathcal{E}) = \frac{1}{2}$.

(3) In the same discretisation, prove that the two following events (in the vertex/face percolation) have the same probability (black lines mean black connection, red lines mean white connection)

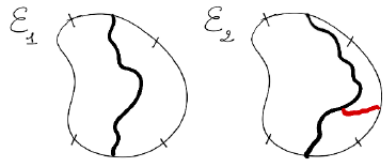
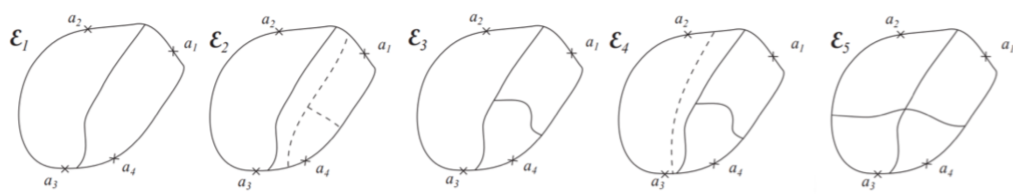


FIGURE 0.1. The events $\mathcal{E}_1$ and $\mathcal{E}_2$

**Solution.** Actually the two events are equal $\mathcal{E}_1 = \mathcal{E}_2$. Indeed, trivially $\mathcal{E}_2 \subset \mathcal{E}_1$. Now let us suppose that there exists a crossing (i.e. we are in $\mathcal{E}_1$). Let us consider the rightmost black crossing : it defines a right part of $\Omega$, denoted by $\Omega_r$. In $\Omega_r$, either there exists a top-bottom black crossing or a left-right white crossing. But we took the rightmost crossing : there can not exists a top-bottom black crossing ! Thus there exists a left-right white crossing in $\Omega_r$: this proves that $\mathcal{E}_1 \subset \mathcal{E}_2$.

(4) For the vertex percolation at $p = \frac{1}{2}$ on a plane, define events $\mathcal{E}_k$, $k = 1, \dots, 5$ as in the figure below (solid lines mean white connection, dashed lines mean black connection). Show that

$$\mathbb{P}(\mathcal{E}_2) = \mathbb{P}(\mathcal{E}_4) = \frac{\mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_5)}{2} \text{ and } \mathbb{P}(\mathcal{E}_3) = \frac{\mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_5)}{2}$$



*Hint: Prove that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$ and then $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_4 \sqcup \mathcal{E}_5$.*

**Solution.** Let us follow the steps proposed in the hint :

(1) Let us prove that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$: in order to do so, we use the same argument as in the previous question. Indeed, we consider the right most black top-bottom crossing : it defines $\Omega_r$ a right part of $\Omega$. In $\Omega_r$ either there exists a black left-right crossing or a white top-bottom crossing (this proves the first equality in the hints). Now in the case that there exists a white top-bottom crossing, this white top-bottom crossing defines a new right part of $\Omega$, denoted by $\overline{\Omega}_r$. In $\overline{\Omega}$, either there exists a top-bottom black crossing, or a left-right white crossing. Yet since we considered at the begining the right most black top-bottom crossing, the first case can not occur : this proves the second equality given in the hint. Putting everything together, this proves that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$.

(2) Let us prove that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_4 \sqcup \mathcal{E}_5$. Let us remark that $\mathcal{E}_3 = \mathcal{E}_4 \sqcup \mathcal{E}_5$ since in the left part, either there is a white crossing from up to bottom, of there is a white crossing from left to right : this is exactly what represent $\mathcal{E}_4$ and $\mathcal{E}_5$.

(3) Let us remark that by self-duality argument (ie we exchange black and white), $\mathbb{P}(\mathcal{E}_2) = \mathbb{P}(\mathcal{E}_4)$.

(4) Thus $\mathbb{P}(\mathcal{E}_1) = \mathbb{P}(\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_4) + \mathbb{P}(\mathcal{E}_5) = 2\mathbb{P}(\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_5)$ : this implies that

$$\mathbb{P}(\mathcal{E}_2) = \frac{\mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_5)}{2}$$

and since $\mathbb{P}(\mathcal{E}_1) = \mathbb{P}(\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_3)$, we have that

$$\mathbb{P}(\mathcal{E}_3) = \mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_2) = \mathbb{P}(\mathcal{E}_1) - \frac{\mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_5)}{2} = \frac{\mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_5)}{2}$$

which allows us to conclude.

**Exercise 2.** *Arzela-Ascoli theorem*
We will use the same notation as in the lectures :

$$\mathcal{H}_\delta^1(z) = \mathbb{P}_{\frac{1}{2}}\left(\text{a black path disconnects } \{a_1, z\} \text{ from } \{a_2, a_3\}\right)$$

where we consider the vertex percolation at $p = \frac{1}{2}$ on the triangular graph $\Omega_\delta$, and $a_1, a_2, a_3$ are anti-clockwise ordered points on the boundary $\partial\Omega$ (and $z$ is here the middle of a face since, in the lesson, $z$ is a vertex of the hexagonal lattice). You have seen in the lesson how to deduce the following Hölder estimate:

$$|\mathcal{H}_\delta^1(x) - \mathcal{H}_\delta^1(y)| \leq C d_{\Omega_\delta}(x, y)^\alpha,$$

where $d_{\Omega_\delta}$ is calculated by taking the length of the shortest path between $x$ and $y$ in $\Omega_\delta$. Our goal is to use this estimate to extract a limit for $\left(\mathcal{H}_\delta^1\right)_\delta$. For this, we extend the discrete function $\mathcal{H}_\delta^1$ defined on faces of $\Omega_\delta$ to a continuous function defined on $\Omega \cup \partial\Omega$ by piecewise linear interpolation.

(1) Recall Arzela-Ascoli theorem.

**Solution.** We will only give the version we need for the exercise : let $(f_n)_{n\in\mathbb{N}}$ defined on a compact subset $\overline{\Omega}$ of $\mathbb{R}^2$. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence $(f_{k_n})_{n\in\mathbb{N}}$ that converges uniformly.

(2) Show that $\left(\mathcal{H}_\delta^1\right)_\delta$ is uniformly bounded. *Hint : this can be done using the estimate above, but it makes more sense to go back to the definition.*

**Solution.** Let us remark that $\mathcal{H}_\delta^1$ is a probability (to be more precise, it is a probability on the vertices of the hexagonal lattice) : it is thus bounded between 0 and 1.

(3) Show that the above Hölder estimate implies that the family $\left(\mathcal{H}_\delta^1\right)_\delta$ is equicontinuous.

**Solution.** It is a well known fact that any family of functions which are Hölder continuous with a constant which does not depend on the function is equicontinuous. Yet, there are two small points which have to be tackled :

(a) we are considering functions which are piecewise linear interpolation, but the Hölder inequality was proven for $x$ and $y$ on the vertices of dual hexagonal lattice (i.e. on the faces of the triangle graph).

(b) In the Hölder estimate, we used the distance $d_\Omega$ which is the length of the shortest path between $x$ and $y$ in $\Omega$ and not the usual distance.

*Remark.* Which linear interpolation do we use ? In some sense we do not care, the only important thing is that the value of $\mathcal{H}_\delta^1$ on the vertices of the hexagonal dual lattice stay the same. For exemple, each vertex $v$ of the triangular lattice is a face $f$ of the dual hexagonal dual lattice : we define the value on this vertex as the mean of the values on the vertices of $f$ in the hexagonal latice. If you consider the dual vertices and the vertices of the triangular graph, you see that the domain is now discretized by a smaller triangular lattice and we know the values of the function ${\mathcal{H}_\delta}^1$ on each vertex : we can now linear interpolate in each small triangle.

**Solution.** Yet, the goal of the lecture is not to be too technical : for the first point, if you can control a function at a lot of points you control its linear interpolation. For the second point let us remark that given a fixed $x$, $d_\Omega(x, y) \to 0$ as $|y - x| \to 0$ : thus for any vertex $x$ of the hexagonal graph, you can find a neighbourhood of $x$ such that $d_\Omega(x, y)$ is small on this neighbourhood, and thus a neighbourhood of $x$ such that for any vertices $y$ of the hexagonal graph $|\mathcal{H}_\delta^1(x) - \mathcal{H}_\delta^1(y)|$ is (uniformly in $\delta$) small.

(4) Deduce that we can extract a limit in $\left(\mathcal{H}_\delta^1\right)_\delta$.

**Solution.** The family is equicontinuous and thus we can extract a limit using Arzela-Ascoli.