In this exercises sheet, when we talk about vertex percolation, we consider the vertex percolation on the triangular lattice. Recall that the vertex percolation on the triangular lattice can be seen equivalently as a face percolation on the hexagonal lattice : if you consider the dual graph of the triangular lattice, you get the hexagonal lattice, and every vertex becomes a face.

Exercise 1. *Duality*

(1) Show that the probability in bond percolation at $p = \frac{1}{2}$ on $\mathbb{Z}_2$ restricted to the rectangle $[0, n] \times [0; n + 1]$ that there is a crossing from top to bottom is exactly $\frac{1}{2}$.

(2) Suppose we have a discretisation of a simply connected domain Ω with smooth boundary. Mark three distinct points a, b, c on $\partial\Omega$ (in clockwise order). Using duality for the vertex percolation at $p = \frac{1}{2}$, show that the probability that there is an open cluster which connects all three boundary segments $[a, b]$ $[b, c]$ $[c, a]$ is $\frac{1}{2}$.

Hint : think about how we proved the first point... (but now in the context of vertex percolation)

(3) In the same discretisation, prove that the two following events (in the vertex/face percolation) have the same probability (black lines mean black connection, red lines mean white connection)

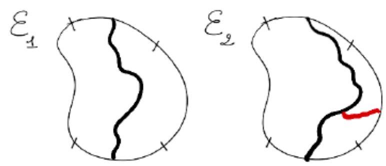
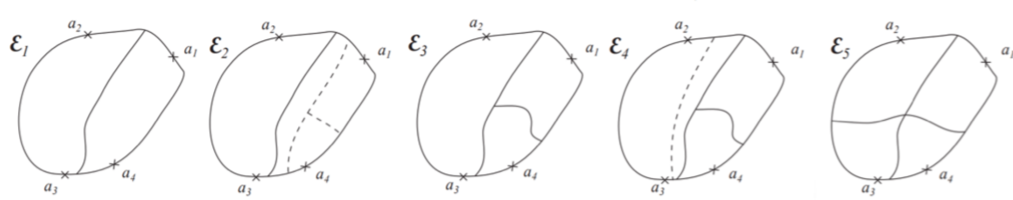


FIGURE 0.1. The events $\mathcal{E}_1$ and $\mathcal{E}_2$

(4) For the vertex percolation at $p = \frac{1}{2}$ on a plane, define events $\mathcal{E}_k$, $k = 1, \ldots, 5$ as in the figure below (solid lines mean white connection, dashed lines mean black connection). Show that

$$\mathbb{P}(\mathcal{E}_2) = \mathbb{P}(\mathcal{E}_4) = \frac{\mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_5)}{2} \text{ and } \mathbb{P}(\mathcal{E}_3) = \frac{\mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_5)}{2}$$



Hint: Prove that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$ and then $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_4 \sqcup \mathcal{E}_5$.

Exercise 2. *Arzela-Ascoli theorem*

We will use the same notation as in the lectures :

$$\mathcal{H}_\delta^1(z) = \mathbb{P}_{\frac{1}{2}}(\text{a black path disconnects } \{a_1, z\} \text{ from } \{a_2, a_3\})$$

where we consider the vertex percolation at $p = \frac{1}{2}$ on the triangular graph Ω_δ, and a_1, a_2, a_3 are anti-clockwise ordered points on the boundary $\partial\Omega$ (and z is here the middle of a face since, in the lesson, z is a vertex of the hexagonal lattice). You have seen in the lesson how to deduce the following Hölder estimate:

$$|\mathcal{H}_\delta^1(x) - \mathcal{H}_\delta^1(y)| \le C d_{\Omega_\delta}(x, y)^\alpha,$$

where d_{Ω_δ} is calculated by taking the length of the shortest path between x and y in Ω_δ. Our goal is to use this estimate to extract a limit for $(\mathcal{H}_\delta^1)_\delta$. For this, we extend the discrete function $\mathcal{H}_\delta^1$ defined on faces of Ω_δ to a continuous function defined on $\Omega \cup \partial\Omega$ by piecewise linear interpolation.

(1) Recall Arzela-Ascoli theorem.
(2) Show that $\left(\mathcal{H}_\delta^1\right)_\delta$ is uniformly bounded.
(3) Show that the above Hölder estimate implies that the family $\left(\mathcal{H}_\delta^1\right)_\delta$ is equicontinuous.
(4) Deduce that we can extract a limit in $\left(\mathcal{H}_\delta^1\right)_\delta$.