APPROXIMATION OF THE QUASIGEOSTROPHIC SYSTEM WITH THE PRIMITIVE SYSTEMS

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Abstract. In this paper we show that the quasigeostrophic system is well approximated by the primitive systems. More precisely, we prove that if the initial data are weakly well-prepared then the maximal time existence of the regular solution of the primitive system goes to infinity and the regular solution goes to the solution of the quasigeostrophic system, strongly on an arbitrary time interval. By weakly well-prepared initial data we mean that the initial data of the primitive systems is converging to an initial data with zero oscillating part, without any assumptions on the speed.

Résumé. Dans cet article on montre que le système quasigéostrophique est bien approximé par les systèmes primitifs. Plus précisément, on montre que, dans le cas des données initiales faiblement bien préparées, le temps maximal d'existence de la solution régulière du système primitif tend vers l'infini et la solution régulière du système primitif tend vers la solution du système quasigéostrophique, et ce fortement sur tout intervalle de temps borné. Par données initiales faiblement bien préparées, on comprend des données initiales qui convergent vers une donnée initiale avec la partie oscillante nulle, sans aucune hypothèse sur la vitesse.

Introduction

The well-known quasigeostrophic system (QG) has been extensively used in oceanography and meteorology for modeling and forecasting mid-latitude oceanic and atmospheric circulation. This system is obtained by taking the limit on ε in a family of primitive systems. The primitive models are given by

$$(PE_\varepsilon)\begin{cases} \partial_t U + v\cdot\nabla U + \dfrac{1}{\varepsilon}AU &= \dfrac{1}{\varepsilon}(-\nabla\Phi, 0) \\ \operatorname{div} v &= 0 \\ U\big|_{t=0} &= U_0 \end{cases}$$

where $U(t,x) = (v(t,x), T(t,x))$, v is a vector field on $\mathbb{R}^3$ depending on the time, T is a scalar function and

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Physically, v is the velocity, T is the potential temperature and ε is proportional to the Rossby number. When the Rossby number is small, the fluid is highly rotating. For further details about the physical significance of these systems, see [14].

Before taking the limit on ε in (PE_ε) we need to define the potential vorticity

$$\Omega = \partial_1 v_2 - \partial_2 v_1 - \partial_3 T$$

and the oscillating (or ageostrophic) part

$$U_{osc} = (v_{osc}, T_{osc})$$

given by

$$\begin{aligned} v^1_{osc} &= v^1 + \partial_2 \Delta^{-1}\Omega \\ v^2_{osc} &= v^2 - \partial_1 \Delta^{-1}\Omega \\ v^3_{osc} &= v^3 \\ T_{osc} &= T + \partial_3 \Delta^{-1}\Omega. \end{aligned}$$

Some formal calculus show that taking the limit on ε implies that the oscillating part of the limit is vanishing, therefore the limit of the primitive systems for $\varepsilon \to 0$ may be written as the following quasigeostrophic system (see [3], [6]):

$$(QG) \left\{ \begin{array}{rcl} \partial_t \Omega + v \cdot \nabla \Omega &=& 0 \\ v &=& (-\partial_2 \Delta^{-1}\Omega, \partial_1 \Delta^{-1}\Omega, 0) \\ T &=& -\partial_3 \Delta^{-1}\Omega \\ U_{0,osc} &=& 0. \end{array} \right.$$

It is easy to see that the first equation in (QG) implies the conservation of the L^∞-norm of Ω and that, in general, the (QG) system behaves like the Euler system in $\mathbb{R}^2$. The methods used to prove the well-posedness of the Euler system in $\mathbb{R}^2$ are easily adjustable in order to prove that the (QG) system has a global strong solution, too. For the (PE_ε) system the classical theory for quasi-linear, symmetric hyperbolic systems shows the local existence of strong solutions. The problem which appears is whether the solution of (PE_ε) converges to the solution of (QG). T. Beale and A. Bourgeois considered in [3] the same problem with periodic boundary conditions in the horizontal directions and rigid boundary conditions in the vertical direction. They proved that if U_0^ε goes to U_0 where $U_{0,osc} = 0$ with the O(ε)-speed (which means that $\partial_t U^\varepsilon\big|_{t=0}$ is bounded independently of ε) then the maximal time existence of the regular solution goes to infinity when $\varepsilon \to 0$ and that U^ε, the solution of (PE_ε), goes to the solution U_{QG} of (QG) with the $O(\varepsilon)$-speed on any bounded time interval. Here we show that, in order to obtain that the maximal time existence of the regular solution goes to infinity, it suffices to assume that $U^\varepsilon_{0,osc} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$ and if, in addition, $U_0^\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} U_0$ then the solution U_ε goes to U_{QG} on any bounded time interval. Hence, we allow oscillations in time of the initial data. Also, our regularity assumptions on the initial data are more general than those of T. Beale and A. Bourgeois, that is we use the space H^s, $s > \frac{5}{2}$, instead of H^5.

J.-Y. Chemin in [6] and E. Grenier in [8] are studying the same problem of convergence for models which are modified in the sense that in the equation satisfied by U appears a viscosity term $-\nu \Delta U$, $\nu > 0$. The ideas of [6] and the methods used to solve the Euler equations are the foundation of this paper.

S. Schochet proved in [15] a general theorem of convergence for a class of systems of the same type but with arbitrary convergent initial data non-necessarily well-prepared. He derived a limit system which, in the well-prepared case for the primitive systems that we consider, is the quasigeostrophic system. However, his theorem is valid only on the torus and this hypothesis seems to be important. In the same case of periodic boundary conditions, P. Embid and A. Majda in [7] and A. Babin, A. Mahalov, B. Nicolaenko, Y. Zhou in [1], [2] considered, between other problems, the particular case of primitive systems with an arbitrary convergent initial data non-necessarily well-prepared. They studied in detail the limit system, who was already deduced by S. Schochet. In [1] and [2] a study of small divisors leads to similar results as proved in this paper but in the periodic case and not well-prepared initial data.

For the mathematical modeling which leads to these systems we refer to [9], [10], [11] and [12].

1. Notations and assertions

We denote by Δ^s the operator given by $\Delta^s U = \mathcal{F}^{-1}(|\xi|^{2s}\hat{U}(\xi))$. We work in $\mathbb{R}^3$ and we shall use H^s, the space of tempered distributions valued in $\mathbb{R}^3$ which satisfy

$$\|U\|_s \stackrel{def}{=} (\int_{\mathbb{R}^3} (1+|\xi|^2)^s |\hat{U}(\xi)|^2 d\xi)^{\frac{1}{2}} < \infty,$$

where $|\cdot|$ is the usual length. It is easy to see that the two norms $\|U - U_{osc}\|_0 + \|\Omega\|_{s-1} + \|U_{osc}\|_s$ and $\|U\|_s$ are equivalent. From now on, s will denote a real number strictly greater than $\frac{5}{2}$.

Definition 1.1. *We say that U is a regular solution on $[0, \mathcal{T}[$ if $U \in C([0, \mathcal{T}[; H^s)$.*

Theorem 1.1. *Let $(U_0^\varepsilon)_{\varepsilon>0}$ be a bounded family of divergence free vector fields in H^s such that*

$$\lim_{\varepsilon\to 0} U^\varepsilon_{0,osc} = 0 \text{ in } L^2.$$

Then there exists $\mathcal{T}_\varepsilon > 0$ such that

i) there exists U^ε a regular solution of (PE_ε) on $[0, \mathcal{T}_\varepsilon)$ with initial data U_0^ε,

ii) $\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon = +\infty$.

Theorem 1.2. *Let $(U_0^\varepsilon)_{\varepsilon>0}$ be a bounded family of divergence free vector fields in H^s such that there exists $U_{0,QG} \in H^s$ with $(U_{0,QG})_{osc} = 0$ and*

$$\lim_{\varepsilon\to 0} U^\varepsilon_0 = U_{0,QG} \text{ in } L^2.$$

Then the solutions U^ε of (PE_ε) with initial data U_0^ε converge to U_{QG}, the solution of (QG), strongly in $L^\infty(]0, \mathcal{T}[; H^\sigma)$, for all $\mathcal{T} < \infty$ and $\sigma < s$.

We shall constantly use the fact that if $t > \frac{3}{2}$ then $H^t \subset L^\infty$, H^t is an algebra and $\|uv\|_t \leq C(\|u\|_t\|v\|_{L^\infty} + \|v\|_t\|u\|_{L^\infty})$. In the following we denote by C a constant which depends only on s. It is not difficult to see that the system (PE_ε) is equivalent to the system

$$(\widetilde{PE_\varepsilon})\begin{cases} \partial_t U + v\cdot\nabla U + \dfrac{1}{\varepsilon}AU_{osc} &= GP(U,U)\\ U\Big|_{t=0} &= U_0\end{cases}$$

where

$$GP(U,U) = (-\nabla p, 0) = \left(\nabla\sum_{i,j}\partial_i\partial_j\Delta^{-1}(v^iv^j), 0\right)$$

or, equivalently,

$$(DPE_\varepsilon)\begin{cases} \partial_t\Omega + v\cdot\nabla\Omega &= q(U_{osc},U)\\ \partial_t U_{osc} + v\cdot\nabla U_{osc} + \dfrac{1}{\varepsilon}AU_{osc} &= Q(U,U)\\ U\Big|_{t=0} &= U_0\end{cases}$$

where

$$q(U_{osc},U) = \partial_3 v^3_{osc}(\partial_1 v_2 - \partial_2 v_1) - \partial_1 v^3_{osc}\partial_3 v^2 + \partial_2 v^3_{osc}\partial_3 v^1 + \partial_3 v_{osc}\cdot\nabla T + \partial_3(v - v_{osc})\cdot\nabla T_{osc}$$

and

$$Q(U,U) = \begin{pmatrix} -\partial_2\Delta^{-1}q(U_{osc},U) + [v\cdot\nabla,\partial_2\Delta^{-1}]\Omega - \partial_1 p\\ \partial_1\Delta^{-1}q(U_{osc},U) - [v\cdot\nabla,\partial_1\Delta^{-1}]\Omega - \partial_2 p\\ -\partial_3 p\\ \partial_3\Delta^{-1}q(U_{osc},U) + [v\cdot\nabla,\partial_3\Delta^{-1}]\Omega\end{pmatrix},$$

(see [6] for the detailed computations). In the estimates on q we shall use that $q(U_{osc},U)$ is a sum of products of derivatives of U and U_{osc} plus derivatives of U_{osc} multiplied by derivatives of $U - U_{osc}$.

2. Proofs

Proof of theorem 1.1

Let $M = \sup_{\varepsilon>0}\|U^\varepsilon_o\|_s$. In the following we shall denote by $C(M)$ a constant which depends only on s and M and by $o_M(\varepsilon)$ a constant which depends on s, M and ε with the property that $\lim\limits_{\varepsilon\to 0} o_M(\varepsilon) = 0$ for each fixed $M < \infty$. Let U be a regular solution of system (PE_ε). Since the matrix A is antisymmetric, the singular term vanishes while making energy estimates. Hence, applying the operator $(Id-\Delta)^{\frac{s}{2}}$ to the equation verified by U and multiplying by $(Id-\Delta)^{\frac{s}{2}}U$ yields

$$\partial_t\|U\|^2_s \leq |<(Id-\Delta)^{\frac{s}{2}}(v\cdot\nabla U)|(Id-\Delta)^{\frac{s}{2}}U>| \leq C\,\|\nabla U\|_{L^\infty}\,\|U\|^2_s.$$

(for the proof of the last inequality see, for instance, [13]). Gronwall's lemma yields

$$\|U(t)\|_s \leq \|U(0)\|_s \exp\left(C\int_0^t \|\nabla U(\tau)\|_{L^\infty}\,d\tau\right). \tag{1}$$

Let $f(t) = C \int_0^t \|\nabla U(\tau)\|_{L^\infty} \, d\tau$. We find that

$$\|U(t)\|_s \leq M \exp f(t). \tag{2}$$

Hence, in order to control U, we need the control of f. This will be done by deriving a differential inequality for f. We start by writing

$$\|\nabla U(t)\|_{L^\infty} \leq \|\nabla U_{osc}(t)\|_{L^\infty} + \|\nabla (U - U_{osc})(t)\|_{L^\infty}. \tag{3}$$

From the definition of U_{osc} we see that $\nabla(U - U_{osc})$ is a sum of terms of the type $\partial_i \partial_j \Delta^{-1} \Omega$. But it is well-known that

$$\|\partial_i \partial_j \Delta^{-1} \Omega\|_{L^\infty} \leq C \|\Omega\|_{L^\infty} \log \left(e + \frac{\|\Omega\|_{s-1}}{\|\Omega\|_{L^\infty}} \right) + C \|\Omega\|_{L^2},$$

(see, for instance, [4], [5]). We also have

$$\|\Omega\|_{s-1} \leq C \|U\|_s \leq CM \exp f.$$

The definition of the oscillating part shows that $\nabla(U - U_{osc})$ has as components functions which are vanishing or are of the type $\partial_i \partial_j \Delta^{-1} \Omega$. Therefore, the two inequalities above yield

$$\|\nabla(U - U_{osc})\|_{L^\infty} \leq C \|\Omega\|_{L^\infty} \log \left(e + \frac{CM \exp f}{\|\Omega\|_{L^\infty}} \right) + C \|\Omega\|_{L^2}. \tag{4}$$

If $\frac{CM}{\|\Omega\|_{L^\infty}} \leq 1$, then we find from the relation above

$$\|\nabla(U - U_{osc})\|_{L^\infty} \leq C \|\Omega\|_{L^\infty} \log(e + \exp f) + C \|\Omega\|_{L^2}.$$

If $\frac{CM}{\|\Omega\|_{L^\infty}} \geq 1$, we use the fact that, for all $\alpha \geq 0$, the function

$$x \to x \log \left(e + \frac{\alpha}{x} \right)$$

is increasing to deduce from (4) that

$$\|\nabla(U - U_{osc})\|_{L^\infty} \leq CM \log(e + \exp f) + C \|\Omega\|_{L^2}.$$

In both cases, the following inequality is true

$$\|\nabla(U - U_{osc})\|_{L^\infty} \leq C \max\{M, \|\Omega\|_{L^\infty}, \|\Omega\|_{L^2}\} \log(e + \exp f).$$

Now, let $\sigma \in]\frac{3}{2} + 1, s[$. Returning to inequality (3), using that $H^{\sigma-1} \subset L^\infty$ and the above estimate, we obtain

$$\|\nabla U(t)\|_{L^\infty} \leq \|U_{osc}(t)\|_\sigma + C \max\{M, \|\Omega(t)\|_{L^\infty}, \|\Omega(t)\|_{L^2}\} \log(e + \exp f(t)). \tag{5}$$

We need to control the L^∞ norm of Ω. But this quantity is almost conserved. Indeed, the equation of Ω

$$\partial_t \Omega + v \cdot \nabla \Omega = q(U_{osc}, U),$$

together with the particular form of q imply

$$\begin{aligned}\|\Omega(t)\|_{L^\infty} &\le \|\Omega_0\|_{L^\infty} + \int_0^t \|q\Big(U_{osc}(\tau), U(\tau)\Big)\|_{L^\infty}\, d\tau \\ &\le \|\Omega_0\|_{L^\infty} + C\int_0^t \|\nabla U_{osc}(\tau)\|_{L^\infty}(\|\nabla U_{osc}(\tau)\|_{L^\infty} + \|\nabla(U-U_{osc})(\tau)\|_{L^\infty})\, d\tau \\ &\le \|\Omega_0\|_{L^\infty} + C\int_0^t \|U_{osc}(\tau)\|_\sigma \|U(\tau)\|_s\, d\tau.\end{aligned}$$

Using again the basic estimate (2) and the monotonicity of f, we obtain

$$\|\Omega(t)\|_{L^\infty} \le \|\Omega_0\|_{L^\infty} + CMt\exp(f(t)) \sup_{0\le\tau\le t} \|U_{osc}(\tau)\|_\sigma.$$

A similar estimate holds for $\|\Omega\|_{L^2}$. Since $\|\Omega_0\|_{L^\infty} \le CM$ and $\log(e+\exp f) \le C(1+f) \le C\exp f$, inserting the above inequality in (5) gives

$$\|\nabla U(t)\|_{L^\infty} \le \|U_{osc}(t)\|_\sigma + CM(1+f(t)) + CMt\exp(2f(t)) \sup_{0\le\tau\le t}\|U_{osc}(\tau)\|_\sigma. \tag{6}$$

We shall now estimate U_{osc}. By differentiation of (PE_ε) we get

$$\partial_t\partial_t U + v\nabla\partial_t U + \frac{1}{\varepsilon}A\partial_t U = \frac{1}{\varepsilon}(-\nabla\partial_t\Phi, 0) - \partial_t v\nabla U.$$

Taking the scalar product with $\partial_t U$ implies

$$\frac{1}{2}\partial_t\|\partial_t U\|_{L^2}^2 \le |<\partial_t v\nabla U|\partial_t U>| \le \|\nabla U\|_{L^\infty}\|\partial_t U\|_{L^2}^2.$$

Thus

$$\|\partial_t U(t)\|_{L^2} \le \|\partial_t U\big|_{t=0}\|_{L^2}\exp f(t).$$

The L^2-norm applied to $(\widetilde{PE_\varepsilon})$ at time $t=0$ gives

$$\|\partial_t U\big|_{t=0}\|_{L^2} \le \frac{1}{\varepsilon}\|U_{0,osc}\|_{L^2} + \|v_0\cdot\nabla U_0\|_{L^2} + \|GP(U_0,U_0)\|_{L^2} \le \frac{1}{\varepsilon}\|U_{0,osc}\|_{L^2} + C(M).$$

Taking the L^2-norm in $(\widetilde{PE_\varepsilon})$ and using that $H^{s-1}\subset L^\infty$ we get

$$\begin{aligned}\frac{1}{\varepsilon}\|AU_{osc}(t)\|_{L^2} &\le \|v(t)\cdot\nabla U(t)\|_{L^2} + \|GP(U(t),U(t))\|_{L^2} + \|\partial_t U(t)\|_{L^2} \\ &\le C\|U(t)\|_s^2 + \|\partial_t U(t)\|_{L^2} \\ &\le C(M)\exp(2f(t))\left(1+\frac{1}{\varepsilon}\|U_{0,osc}\|_{L^2}\right) \\ &\le \frac{1}{\varepsilon}C(M)\exp(2f(t))(\varepsilon + \|U_{0,osc}\|_{L^2}).\end{aligned}$$

Since the matrix A is invertible and $\|U_{0,osc}\|_{L^2}\to 0$ we have

$$\|U_{osc}(t)\|_{L^2} \le o_M(\varepsilon)\exp(2f(t)).$$

Interpolating H^σ between L^2 and H^s and using that $\|U_{osc}\|_s \leq C\|U\|_s$ implies

$$\begin{aligned}\|U_{osc}(t)\|_\sigma &\leq \|U_{osc}(t)\|_{L^2}^{1-\frac{\sigma}{s}} \|U_{osc}(t)\|_s^{\frac{\sigma}{s}} \\ &\leq o_M(\varepsilon) \exp\Big(2\Big(1-\frac{\sigma}{s}\Big)f(t)\Big) C(M) \exp\Big(\frac{\sigma}{s}f(t)\Big) \\ &\leq o_M(\varepsilon)\exp(2f(t)).\end{aligned} \tag{7}$$

Using this in relation (6) we find

$$f'(t) \leq CM(1+f(t)) + o_M(\varepsilon)(1+t)\exp(4f(t)).$$

We have almost finished, it remains to apply the same method which proves that if f satisfies $f' \leq C(1+f)$ then $f \in L^\infty_{loc}(\mathbb{R}_+)$. Indeed, by Gronwall's lemma we get

$$f(t) \leq \Big(tCM + o_M(\varepsilon)\int_0^t (1+\tau)\exp(4f(\tau))\, d\tau\Big)\exp(tCM). \tag{8}$$

Let $\mathcal{T}_\varepsilon$ be the maximal time existence of the regular solution U_ε and $\mathcal{T}$ be fixed. We choose $K > 2\mathcal{T}CM\exp(\mathcal{T}CM)$. If ε is small enough the following inequality is true

$$\Big(\mathcal{T}CM + o_M(\varepsilon)(\mathcal{T}^2 + 2\mathcal{T})\exp(4K)\Big)\exp(\mathcal{T}CM) < \frac{K}{2}.$$

It follows from this, from $f(0) = 0$ and from relation (8) that

$$\forall\ t \in [0, \min(\mathcal{T}_\varepsilon, \mathcal{T})[, \quad f(t) < K, \tag{9}$$

so the regular solution of (PE_ε) satisfies

$$\|U_\varepsilon\|_{L^\infty([0,\min(\mathcal{T}_\varepsilon,\mathcal{T})[;H^s)} < M\exp(K).$$

But $\lim\limits_{t\to\mathcal{T}_\varepsilon} \|U_\varepsilon(t)\|_s = \infty$. Hence $\mathcal{T}_\varepsilon \geq \mathcal{T}$ if ε is small enough. We proved that

$$\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon = \infty.$$

Proof of theorem 1.2

We have

$$\lim_{\varepsilon\to 0}\|U^\varepsilon_{0,osc}\|_{L^2} = \lim_{\varepsilon\to 0}\|(U^\varepsilon_0 - U_{0,QG})_{osc}\|_{L^2} \leq \lim_{\varepsilon\to 0}\|U^\varepsilon_0 - U_{0,QG}\|_{L^2} = 0.$$

This means that we can apply Theorem 1.1. Let $\mathcal{T}$ be fixed. It follows from (7) and (9) that for any η there exists ε_0 such that

$$\forall\ \varepsilon < \varepsilon_0,\ \forall\ t \in [0, \mathcal{T}], \quad \|U^\varepsilon_{osc}(t)\|_\sigma < \eta.$$

It follows that

$$\forall\ \sigma < s, \quad U^\varepsilon_{osc} \to 0 \text{ in } L^\infty(]0,\mathcal{T}[; H^\sigma).$$

It remains to show that

$$\Omega^\varepsilon \to \Omega_{QG} \text{ in } L^\infty(]0,\mathcal{T}[; H^{\sigma-1}).$$

We have

$$\partial_t \Omega^\varepsilon + v^\varepsilon \cdot \nabla \Omega^\varepsilon = q(U^\varepsilon_{osc}, U^\varepsilon)$$
$$\partial_t \Omega_{QG} + v_{QG} \cdot \nabla \Omega_{QG} = 0.$$

Subtracting these equations yields

$$\partial_t(\Omega^\varepsilon - \Omega_{QG}) + v^\varepsilon \cdot \nabla(\Omega^\varepsilon - \Omega_{QG}) = q(U^\varepsilon_{osc}, U^\varepsilon) - (v^\varepsilon - v_{QG}) \cdot \nabla \Omega_{QG}. \tag{10}$$

From the definition of q we infer

$$\begin{aligned}\|q(U^\varepsilon_{osc}, U^\varepsilon)\|_{L^2} \leq & C\,\|\nabla U^\varepsilon_{osc}\|_{L^2}\,\|\nabla U^\varepsilon\|_{L^\infty} + C\,\|\nabla U^\varepsilon_{osc}\|_{L^2}\,\|\nabla(U^\varepsilon - U^\varepsilon_{osc})\|_{L^\infty} \\ & \leq C\|U^\varepsilon_{osc}\|_{s-1}\|U^\varepsilon\|_s + C\|U^\varepsilon_{osc}\|_{s-1}\|U^\varepsilon - U^\varepsilon_{osc}\|_s \\ & \leq C\|U^\varepsilon_{osc}\|_{s-1}\|U^\varepsilon\|_s.\end{aligned}$$

Using the classical product theorem in Sobolev spaces implies

$$\|(v^\varepsilon - v_{QG}) \cdot \nabla \Omega_{QG}\|_{L^2} \leq C\|(v^\varepsilon - v_{QG})|_1 \|\nabla \Omega_{QG}\|_{\frac{1}{2}} \leq C\|U^\varepsilon - U_{QG}|_1 \|\Omega_{QG}\|_{s-1},$$

where $\|\cdot|_s$ denotes the homogeneous version of the Sobolev norm. But the oscillating part of $U^\varepsilon - U_{QG}$ is the oscillating part of U^ε and the quasigeostrophic part of $U^\varepsilon - U_{QG}$ is $\Omega^\varepsilon - \Omega_{QG}$. It follows that

$$\|U^\varepsilon - U_{QG}|_1 \leq C(\|U^\varepsilon_{osc}\|_1 + \|\Omega^\varepsilon - \Omega_{QG}\|_{L^2}).$$

Taking the scalar product of (10) with $\Omega^\varepsilon - \Omega_{QG}$ and using the three above inequalities gives

$$\begin{aligned}\partial_t \|\Omega^\varepsilon - \Omega_{QG}\|^2_{L^2} &\leq C\Big(\|q(U^\varepsilon_{osc}, U^\varepsilon)\|_{L^2} + \|(v^\varepsilon - v_{QG}) \cdot \nabla \Omega_{QG}\|_{L^2}\Big)\|\Omega^\varepsilon - \Omega_{QG}\|_{L^2} \\ &\leq C\Big(\|U^\varepsilon_{osc}\|_{s-1}\|U^\varepsilon\|_s + (\|U^\varepsilon_{osc}\|_1 + \|\Omega^\varepsilon - \Omega_{QG}\|_{L^2})\|\Omega_{QG}\|_{s-1}\Big) \\ &\qquad \times \|\Omega^\varepsilon - \Omega_{QG}\|_{L^2}.\end{aligned}$$

By Gronwall's lemma it follows that

$$\begin{aligned}\|\Omega^\varepsilon - \Omega_{QG}\|_{L^2} \leq & \exp\left(\int_0^t \|\Omega_{QG}(\tau)\|_{s-1}\, d\tau\right) \\ & \times \left(\|\Omega^\varepsilon_0 - \Omega_{0,QG}\|_{L^2} + \int_0^t \|U^\varepsilon_{osc}(\tau)\|_{s-1}\Big(\|U^\varepsilon(\tau)\|_s + \|\Omega_{QG}(\tau)\|_{s-1}\Big)\, d\tau\right).\end{aligned} \tag{11}$$

Now it is obvious that

$$\Omega^\varepsilon \to \Omega_{QG} \text{ in } L^\infty(]0, \mathcal{T}[; L^2)$$

if we keep in mind that

$$U^\varepsilon_{osc} \to 0 \text{ in } L^\infty(]0, \mathcal{T}[; H^{s-1})$$

and that U^ε is bounded in $L^\infty(]0, \mathcal{T}[; H^s)$. We use again that Ω^ε is bounded in $L^\infty(]0, \mathcal{T}[; H^{s-1})$ and by interpolation we find that

$$\forall\, \sigma < s, \quad \Omega^\varepsilon \to \Omega_{QG} \text{ in } L^\infty(]0, \mathcal{T}[; H^{\sigma-1}),$$

so

$$\forall\ \sigma < s, \quad U^\varepsilon \to U_{QG} \text{ in } L^\infty(]0,\mathcal{T}[;H^\sigma).$$

This completes the proof.

References

1. A. Babin, A. Mahalov, and B. Nicolaenko, *Regularity and integrability of* 3*D Euler and Navier-Stokes equations for rotating fluids*, Asymptot. Anal. **15** (1997), no. 2, 103–150.
2. A. Babin, A. Mahalov, B. Nicolaenko, and Y. Zhou, *On the asymptotic regimes and the strongly stratified limit of rotating Boussinesq equations*, Theor. Comput. Fluid Dyn. **9** (1997), no. 3–4, 223–251.
3. J. T. Beale and A. J. Bourgeois, *Validity of the quasigeostrophic model for large-scale flow in the atmosphere and ocean*, SIAM J. Math. Anal. **25** (1994), no. 4, 1023–1068.
4. J. T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions for the* 3*-D Euler equations*, Comm. Math. Phys. **94** (1984), no. 1, 61–66.
5. J.-Y. Chemin, *Fluides parfaits incompressibles*, Astérisque (1995), no. 230, 177.
6. ———, *À propos d'un problème de pénalisation de type antisymétrique*, J. Math. Pures Appl. (9) **76** (1997), no. 9, 739–755.
7. P. F. Embid and A. J. Majda, *Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity*, Comm. Partial Differential Equations **21** (1996), no. 3-4, 619–658.
8. E. Grenier, *Oscillatory perturbations of the Navier-Stokes equations*, J. Math. Pures Appl. (9) **76** (1997), no. 6, 477–498.
9. J.-L. Lions, R. Temam, and S. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity **5** (1992), no. 2, 237–288.
10. ———, *On the equations of the large-scale ocean*, Nonlinearity **5** (1992), no. 5, 1007–1053.
11. ———, *Models for the coupled atmosphere and ocean*, Comput. Mech. Adv. **1** (1993), no. 1, 120.
12. ———, *Mathematical theory for the coupled atmosphere-ocean models*, J. Math. Pures Appl. (9) **74** (1995), no. 2, 105–163.
13. G. Métivier, *Problèmes de Cauchy et ondes non linéaires*, Journées "Équations aux dérivées partielles" (Saint Jean de Monts, 1986), École Polytech., Palaiseau, 1986.
14. J. Pedlosky, *Geophysical fluid dynamics*, Springer–Verlag, New York, 1987, 2nd ed.
15. S. Schochet, *Fast singular limits of hyperbolic PDEs*, J. Differential Equations **114** (1994), no. 2, 476–512.

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