Global existence and uniqueness of solutions for the equations of third grade fluids

Valentina Busuioc[a] and Dragoş Iftimie[b]

[a]Section de mathématiques, Institut Bernoulli, École Polytechnique Fédérale de Lausanne.
CH–1015 Lausanne. Switzerland.
Email: adriana.busuioc@epfl.ch

[b]IRMAR, Université de Rennes 1, Campus de Beaulieu,
35042 Rennes cedex, France.
Email: iftimie@maths.univ-rennes1.fr

Abstract. We consider the equations governing the motion of third grade fluids in $\mathbb{R}^n$, $n = 2, 3$. We show global existence of solutions without any smallness assumption, by assuming only that the initial velocity belongs to the Sobolev space H^2. The uniqueness of such solutions is also proven in dimension two.

Introduction

The fluids of grade n, introduced by Rivlin and Ericksen [1], are the fluids for which the stress tensor is a polynomial of degree n in the first n Rivlin-Ericksen tensors defined recursively by

$$A_1 = A = 2D, \quad A_{k+1} = \frac{\mathrm{d}}{\mathrm{dt}} A_k + L^t A_k + A_k L,$$

where $\frac{\mathrm{d}}{\mathrm{dt}} = \partial_t + u \cdot \nabla$ denotes the material derivative and

$$L = (\partial_j u_i)_{i,j}, \quad L^t = (\partial_i u_j)_{i,j}, \quad D = \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{i,j}.$$

For third grade fluids, physical considerations were taken into account by Fosdick and Rajagopal [2] in order to obtain the following form for the constitutive law:

$$T = -pI + \nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta |A_1|^2 A_1,$$

which, introduced in the equation of conservation of momentum leads to the following equation:

$$(1) \quad \partial_t(u - \alpha_1 \triangle u) - \nu \triangle u + u \cdot \nabla u - \alpha_1 \operatorname{div}(u \cdot \nabla A + L^t A + AL) \\ - \alpha_2 \operatorname{div} A^2 - \beta \operatorname{div}(|A|^2 A) = f - \nabla p, \quad \operatorname{div} u = 0.$$

Moreover, the coefficients ν, α_1, α_2 and β must satisfy the following hypotheses:

$$\nu \geq 0, \quad \alpha_1 > 0, \quad \beta \geq 0 \quad \text{and} \quad |\alpha_1 + \alpha_2| \leq (24\nu\beta)^{1/2}.$$

The equation of third grade fluids is usually given under the form

$$\partial_t(u-\alpha_1\triangle u)-\nu\triangle u+\operatorname{curl}(u-\alpha_1\triangle u)\wedge u$$
$$-(\alpha_1+\alpha_2)\Big(A\triangle u+2\operatorname{div}\big(\nabla u(\nabla u)^t\big)\Big)-\beta\operatorname{div}(|A|^2A)=f-\nabla p.$$

These two different forms are, of course, equivalent. Here, the form (1) is more useful.

Notice that if $\beta=0$ then $\alpha_1+\alpha_2=0$ and (1) becomes the equation of second grade fluids (see [3]) which is studied by many authors (see, for instance, [4], [5], [6], [7], [8] and [9]). Therefore, for third grade fluids, one can assume that $\beta>0$. We will make this hypothesis in the sequel.

The mathematical results available in the literature consider these equations in a domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ and show local existence and uniqueness of solutions for arbitrary size of initial data, or global existence and uniqueness if $\nu>0$ and if the initial data is small compared with the viscosity ν. The regularity of the initial data needed in order to obtain these results is at least H^3 (see [10], [11], [12]) or $W^{2,r}$, $r>3$ (see [13]).

Here, we will prove that global solutions exist without any smallness assumption and with less regularity requirements as before. These solutions are also less regular and uniqueness is much more difficult to prove. We can prove uniqueness of bidimensional solutions only. More precisely, we will prove the following theorem:

Theorem 1. *Consider the equation* (1) *in* $\mathbb{R}^n$, $n=2,3$, *with* $f\in L^\infty_{loc}\big([0,\infty);L^2\big)$ *and* $u_0\in H^2$, $\operatorname{div}u_0=0$. *There exists a global solution* $u\in C_w\big([0,\infty);H^2\big)\cap C\big([0,\infty);H^s\big)$ *for all* $s<2$. *Moreover, if* $n=2$ *then this solution is unique.*

The starting point of the proof is the remark that the equation contains a term which is more "regularizing" then the viscosity one; it is the term

$$\beta K(u)=-\beta\operatorname{div}(|A|^2A).$$

For example, making H^1 energy estimates, the viscosity term yields $\int_{\mathbb{R}^n}|\nabla u|^2$ while the term $K(u)$ gives $\int_{\mathbb{R}^n}|D(u)|^4$; therefore, the viscosity term gives H^1 estimates while K implies $W^{1,4}$ estimates, which are better then the H^1 estimates.

It is clear that H^1 estimates are not sufficient to pass to the limit in the equation and one has to look for another a priori estimates. The problem is that the "regularizing" term $K(u)$ is non-linear and it may behave dangerously in higher order energy estimates. For instance, in H^3 estimates, we could not even show a sign for this term and it also becomes the most troublesome one. Here we will show that, when making H^2 energy estimates, the term coming from $K(u)$ has the good sign. Furthermore, this term will be used to cancel other bad terms. This is the basis in obtaining the global existence of an H^2 solution; it is detailed in the second section.

The last section contains the proof of uniqueness of 2D solutions which is in three steps. First, we write the term $\langle K(u)-K(\widetilde u),u-\widetilde u\rangle$ under a form which has the good sign and can be used to simplify bad terms (we denoted by $\langle\cdot,\cdot\rangle$ the scalar product of $\big(L^2(\mathbb{R}^n)\big)^n$). Secondly, we write the other terms under a form where the bad part simplifies with terms from $\langle K(u)-K(\widetilde u),u-\widetilde u\rangle$. However, all the terms can not be treated in this way. For the other terms, there is a third step which is the only one to use that the space dimension is equal to two. It consists in remarking that these terms can be estimated by using H^s norms of u and $\widetilde{u}$ with $s > 2$ instead of $s = 2$. We conclude by using a sort of limit argument.

Let us remark that the inequality $|\alpha_1 + \alpha_2| \leq (24\nu\beta)^{1/2}$ will not be used in the proofs. Moreover, the viscosity ν plays no role in our results; unlike the previous results where the global existence is achieved under the assumption that $\nu > 0$ (and a smallness assumption for the initial data), in our proof ν may vanish and even be negative. Indeed, the only places in the proofs where "we use" that $\nu \geq 0$ are in the passages from relation (16) to (17) and from (37) to (38). But it is trivial to see that, in each of these relations, the viscosity term may be incorporated in the right-hand side: for relation (16), $\nu\|\nabla u\|_{L^2}^2$ may be incorporated in $\frac{C}{\varepsilon}\|v\|_{L^2}^2$ and for relation (37), $\nu\|\nabla w\|_{L^2}^2$ may be incorporated in $\frac{C}{\varepsilon}\int_{\mathbb{R}^2}|\nabla w|^2$.

Before going to the proofs, let us make some comments on the boundary conditions required for the equations of motion of a third grade fluid. In contrast with the Navier-Stokes equations, the nonlinear terms are higher order (order three) than the linear terms (order two). This suggests that the no-slip boundary conditions may be inadequate to fully determine the solution. Even though existence and uniqueness of solutions is known on domains with boundaries by assuming only the no-slip boundary condition, existence and uniqueness is proved only under some smallness assumption (small time or small data) which allows to consider the nonlinear terms negligible compared to the linear terms; the viscosity term is viewed as a regularizing term and is used to control the other terms. Here, we consider a nonlinear term, $K(u)$, as "regularizing" term instead of the viscosity one. Consequently, the boundary conditions issue becomes even more difficult when trying to extend the results of this paper to the case of domains with boundaries. We refer to [14], [15] and [16] for a more detailed discussion, some examples and further references on the problem of finding the right boundary conditions for fluids of differential type.

1. Notations and preliminary results

We denote by H^s the following Sobolev space:

$$H^s = \Big\{ g : \mathbb{R}^n \to \mathbb{C}\,; \quad \|g\|_{H^s}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{g}(\xi)|^2 \, d\xi < +\infty \Big\},$$

where $\widehat{g}$ denotes the Fourier transform of g and $|\cdot|$ the Euclidean norm. For vector-valued functions $h : \mathbb{R}^n \to \mathbb{R}^m$, we say that $h \in H^s$ if each component h_i belongs to H^s and we define

$$\|h\|_{H^s}^2 = \sum_{i=1}^m \|h_i\|_{H^s}^2.$$

We denote by $\langle \cdot, \cdot \rangle$ the L^2 scalar product, or also the duality parenthesis between H^s and H^{-s}. For vector valued functions $g, h : \mathbb{R}^n \to \mathbb{R}^m$, we define

$$\langle g, h \rangle = \sum_{i=1}^m \langle g_i, h_i \rangle.$$

For a vector field $u:\mathbb{R}^n\to\mathbb{R}^n$ we introduce the following matrices:

$$L(u)=(\partial_j u_i)_{i,j},\quad A(u)=2D(u)=L(u)+L^t(u)=(\partial_i u_j+\partial_j u_i)_{i,j}.$$

The scalar product of two matrices $A=(a_{ij})_{i,j}$ and $B=(b_{ij})_{i,j}$ is defined by $A\cdot B=\sum_{i,j}a_{ij}b_{ij}$. We denote by $|A|^2$ the quantity $A\cdot A$. We also introduce the divergence of a matrix $(\operatorname{div}A)_i=\sum_j\partial_j a_{ij}$. One can check the following identities:

$$(AB)\cdot C=(CB^t)\cdot A=(A^tC)\cdot B, \tag{2}$$

which hold for arbitrary square matrices A, B and C.

The Leray projector $\mathbb{P}$ denotes the L^2 orthogonal projection on the space of divergence free vector fields.

The following product theorem is classical (see, for instance, [17]):

Theorem 2. *Let s and t be two real numbers such that $s+t>0$, $s<n/2$ and $t<n/2$. There exists a constant $C>0$ such that if $u\in H^s$ and $v\in H^t$ then $uv\in H^{s+t-n/2}$ and*

$$\|uv\|_{H^{s+t-n/2}}\leq C\|u\|_{H^s}\|v\|_{H^t}.$$

We also have the following easy lemma:

Lemma 1. *Let H be a Hilbert space and $A\subset H$ a dense subset. If u_n is a sequence bounded in H such that $\langle u_n,a\rangle_H\to\langle v,a\rangle_H$ for every $a\in A$, then u_n converges weakly to v.*

We write the equation of third grade fluids (1) under the form

$$\partial_t(u-\alpha_1\triangle u)+u\cdot\nabla u-\nu\triangle u+\operatorname{div}N(u)+\beta K(u)=f-\nabla p, \tag{3}$$

where

$$K(u)=-\operatorname{div}(|A|^2A),\quad N(u)=-\alpha_1(u\cdot\nabla A+L^tA+AL)-\alpha_2A^2.$$

Another equivalent form, useful in the proof of the existence, is the following:

$$\begin{cases}\partial_t v-\nu\triangle u+u\cdot\nabla v+\sum_j v_j\nabla u_j-(\alpha_1+\alpha_2)\operatorname{div}A^2+\beta K(u)=f-\nabla p,\\ v=u-\alpha_1\triangle u.\end{cases} \tag{4}$$

The fact that equations (3) and (4) are equivalent is a simple computation which is given in the appendix.

2. Global existence of solutions

The regularity of the solutions we consider is $L^\infty(0,T;H^2)$. But, in equation (4) there are derivatives of order 3 of u and also a trilinear term. One has to check that these terms are defined in the sense of distributions. The term

$$u\cdot\nabla v=\operatorname{div}(u\otimes v),$$

is a derivative of an L^1 function, so it defines a distribution. The term

$$K(u)=-\operatorname{div}(|A|^2A)$$

is a sum of terms of the type

$$\mathcal{D}u\,\mathcal{D}u\,\mathcal{D}^2u,$$

where $\mathcal{D}^k$ denotes a space derivative of order k. The time regularity of the above term is L^∞. As for the space regularity, since the dimension is less then 3, one has $\mathcal{D}^2u \in L^2$ and $\mathcal{D}u \in H^1 \subset L^p$ for all $p \in [2,6]$. Hölder's inequality now implies that $\mathcal{D}u\,\mathcal{D}u\,\mathcal{D}^2u \in L^q$, for all $q \in [1,6/5]$. Consequently, the term $K(u)$ is (locally) integrable and defines a distribution. The remaining terms are easily seen to define a distribution.

The rigorous definition of an H^2 solution is the following:

Definition 1. *Let $u_0 \in H^2(\mathbb{R}^n)$, $f \in L^\infty(0,T;L^2(\mathbb{R}^n))$, $n \in \{2,3\}$. We say that u is a H^2 solution of* (1) *on $[0,T)$ if u is a divergence free vector field*

$$u \in L^\infty\big(0,T;H^2(\mathbb{R}^n)\big)$$

verifying

$$\int_0^T\int_{\mathbb{R}^n}\Big(-v\cdot\partial_t\varphi - \nu\triangle u\cdot\varphi - (u\cdot\nabla\varphi)\cdot v + \sum_j v_j\nabla u_j\cdot\varphi - (\alpha_1+\alpha_2)\operatorname{div}(A^2)\cdot\varphi$$

$$+\beta K(u)\cdot\varphi\Big) = \int_0^T\int_{\mathbb{R}^n} f\cdot\varphi + \int_{\mathbb{R}^n} v_0(x)\cdot\varphi(0,x)\,dx,$$

for every divergence free vector field $\varphi \in C_0^\infty(\mathbb{R}^n\times[0,T);\mathbb{R}^n)$.

A solution which belongs to $L^\infty_{loc}\big([0,\infty);H^2(\mathbb{R}^n)\big)$ automatically satisfies the additional regularity given in theorem 1. More precisely, one has the following lemma:

Lemma 2. *Let u be a solution of* (3) *on $[0,T)$ such that $u \in L^\infty(0,T;H^2)$ and $f \in L^\infty(0,T;L^2)$. Then*

$$u \in C_w\big([0,T];H^2\big)\cap C\big([0,T];H^s\big) \quad \textit{and} \quad \partial_t u \in L^\infty(0,T;L^2)$$

for all $s < 2$.

Proof. Theorem 2 easily implies that the term

$$u\cdot\nabla u - \nu\triangle u + \operatorname{div} N(u) + \beta K(u)$$

belongs to $L^\infty(0,T;H^{-2})$. Applying the Leray projector $\mathbb{P}$ to (3) implies that

$$\partial_t(u-\alpha_1\triangle u) \in L^\infty(0,T;H^{-2}).$$

We deduce that

$$\partial_t u \in L^\infty(0,T;L^2), \tag{5}$$

so $u \in C\big([0,T];L^2\big)$. The strong continuity $u \in C\big([0,T];H^s\big)$, $0 \le s < 2$, follows from the interpolation inequality

$$\|u(t,\cdot)-u(t_0,\cdot)\|_{H^s} \le C\|u(t,\cdot)-u(t_0,\cdot)\|_{L^2}^{1-s/2}\|u(t,\cdot)-u(t_0,\cdot)\|_{H^2}^{s/2}.$$

Finally, the density of smooth functions in H^2 and lemma 1 implies the assertion on the weak continuity of u. □

The proof of the existence consists in proving some H^2 a priori estimates and passing to the limit in the equation.

2.1. **A priori estimates.** Multiplying equation (4) by v and integrating in space yields

$$(6)\quad \frac{1}{2}\partial_t\|v\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 + \alpha_1\nu\|\triangle u\|_{L^2}^2 + \beta\int |A|^2A\cdot\nabla v \\ = -\int(u\cdot\nabla v)\cdot v - \int(v\cdot\nabla u)\cdot v + (\alpha_1+\alpha_2)\int \operatorname{div}(A^2)\cdot v + \int f\,v.$$

The fact that A is a symmetric matrix implies that

$$\begin{aligned}\int |A|^2A\cdot\nabla v =& \frac{1}{2}\Big(\int |A|^2A\cdot\nabla v + \int |A|^2A\cdot(\nabla v)^t\Big)\\ =&\frac{1}{2}\int |A|^2A\cdot A(v)\\ =&\frac{1}{2}\int |A|^2A\cdot(A-\alpha_1\triangle A)\\ =&\frac{1}{2}\int |A|^4 - \frac{\alpha_1}{2}\int |A|^2A\cdot\triangle A.\end{aligned}$$

On the other hand, some integrations by parts show that

$$\begin{aligned}-\int |A|^2A\cdot\triangle A &= -\sum_{i,j,k,l,m}\int a_{ij}^2a_{kl}\partial_m^2a_{kl}\\ &= 2\sum_{i,j,k,l,m}\int a_{ij}\partial_m a_{ij}a_{kl}\partial_m a_{kl} + \sum_{i,j,k,l,m}\int a_{ij}^2(\partial_m a_{kl})^2\\ &= \int |A|^2|\nabla A|^2 + 2\sum_m\int (A\cdot\partial_m A)^2.\end{aligned}$$

We deduce from the above two relations that

$$\int |A|^2A\cdot\nabla v = \frac{1}{2}\int |A|^4 + \frac{\alpha_1}{2}\int |A|^2|\nabla A|^2 + \alpha_1\sum_m\int (A\cdot\partial_m A)^2.$$

Using this relation in (6) yields

$$(7)\quad \frac{1}{2}\partial_t\|v\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 + \alpha_1\nu\|\triangle u\|_{L^2}^2 + \frac{\beta}{2}\int |A|^4 + \frac{\alpha_1\beta}{2}\int |A|^2|\nabla A|^2 \\ + \alpha_1\beta\sum_m\int(A\cdot\partial_m A)^2 = -\int(v\cdot\nabla u)\cdot v + (\alpha_1+\alpha_2)\int\operatorname{div}(A^2)\cdot v + \int f\,v.$$

We will now estimate the terms from the right-hand side. First, note that

$$(8)\qquad \int(v\cdot\nabla u)\cdot v = \sum_{i,j}\int v_iv_j\partial_i u_j = \frac{1}{2}\sum_{i,j}\int v_iv_j(\partial_i u_j + \partial_j u_i) = \frac{1}{2}\int Av\cdot v.$$

But $v = u - \alpha_1 \triangle u$. So, using again the symmetry of A, we get

$$\int (v \cdot \nabla u) \cdot v = \frac{1}{2} \int Au \cdot u + \int A \triangle u \cdot u + \frac{1}{2} \int A \triangle u \cdot \triangle u. \tag{9}$$

Reversing the argument of (8) gives

$$\int Au \cdot u = 2 \int (u \cdot \nabla u) \cdot u = 0. \tag{10}$$

To estimate the other terms, remark that

$$\operatorname{div} A = \triangle u.$$

Therefore

$$\frac{1}{2} \Big| \int A \triangle u \cdot \triangle u \Big| = \frac{1}{2} \Big| \int A \operatorname{div} A \cdot \triangle u \Big| \le C \int |A||\nabla A||\triangle u| \le \varepsilon \int |A|^2 |\nabla A|^2 + \frac{C}{\varepsilon} \|\triangle u\|_{L^2}^2, \tag{11}$$

where $\varepsilon > 0$ will be chosed later. We obtain in the same way that

$$\Big| \int A \triangle u \cdot u \Big| \le \varepsilon \int |A|^2 |\nabla A|^2 + \frac{C}{\varepsilon} \|u\|_{L^2}^2. \tag{12}$$

We now use relations (10), (11) and (12) in (9) to obtain that

$$\Big| \int (v \cdot \nabla u) \cdot v \Big| \le 2\varepsilon \int |A|^2 |\nabla A|^2 + \frac{C}{\varepsilon} \|v\|_{L^2}^2. \tag{13}$$

Since $\operatorname{div} A^2$ is a term of the form $A \nabla A$, we can further estimate as in (11)

$$\Big| \int \operatorname{div}(A^2) \cdot v \Big| \le \varepsilon \int |A|^2 |\nabla A|^2 + \frac{C}{\varepsilon} \|v\|_{L^2}^2. \tag{14}$$

The forcing term is very easy to estimate:

$$\Big| \int f\, v \Big| \le \|f\|_{L^2} \|v\|_{L^2} \le \frac{\|v\|_{L^2}^2}{2} + \frac{\|f\|_{L^2}^2}{2}. \tag{15}$$

Finally, using (13), (14) and (15) in (7) gives

$$\frac{1}{2} \partial_t \|v\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 + \alpha_1 \nu \|\triangle u\|_{L^2}^2 + \frac{\beta}{2} \int |A|^4 + \frac{\alpha_1 \beta}{2} \int |A|^2 |\nabla A|^2 + \alpha_1 \beta \sum_m \int (A \cdot \partial_m A)^2 \le 2\varepsilon (1 + |\alpha_2 + \alpha_2|) \int |A|^2 |\nabla A|^2 + \frac{C}{\varepsilon} \|v\|_{L^2}^2 + \frac{\|f\|_{L^2}^2}{2}. \tag{16}$$

The choice

$$\varepsilon = \frac{\alpha_1 \beta}{4(1 + |\alpha_1 + \alpha_2|)},$$

now implies

$$\partial_t \|v\|_{L^2}^2 \le C \|v\|_{L^2}^2 + \|f\|_{L^2}^2. \tag{17}$$

Gronwall's lemma gives

$$\|v(t)\|_{L^2}^2 \le \big(\|v_0\|_{L^2}^2 + \frac{1}{C}\|f\|_{L^\infty(0,t;L^2)}^2\big)e^{Ct}, \tag{18}$$

where $C = C(\alpha_1, \alpha_2)/\beta$ is a constant depending on the material coefficients α_1, α_2 and β but not on the viscosity ν.

2.2. **Regularization and passage to the limit.** The fact that allows us to pass to the limit is that the equation does not contain any term with products of second order derivatives of u with other second order derivatives. With this remark, passing to the limit is classical, so we will only sketch the proof.

We use Friedrichs method, also known as "modified Galerkin method", which can be found for example, with minor modifications, in [18], [19] and [20]. We denote by J_ε a Friedrichs mollifier, that is the convolution operator

$$J_\varepsilon u = j_\varepsilon * u,$$

where

$$j_\varepsilon(x) = \varepsilon^{-n} j(\varepsilon^{-1}x),$$

and j is a positive radial compactly supported smooth function whose integral is equal to 1. Let us briefly recall the main properties of these mollifiers:

(1) J_ε commutes with derivatives.
(2) For all $s \ge 0$, J_ε is continuous from H^s to $\bigcap\limits_{l<\infty} H^l$.
(3) For all $s \ge 0$ and $u \in H^s$, one has that $\|J_\varepsilon u\|_{H^s} \le \|u\|_{H^s}$ and $J_\varepsilon u \to u$ strongly in H^s as $\varepsilon \to 0$.
(4) J_ε is selfadjoint for the L^2 scalar product.
(5) If u is compactly supported and $\varepsilon \le 1$, then the support of $J_\varepsilon u$ is included in a compact independent of ε.

Approximating equation. Consider the following approximating system:

$$\begin{cases} \partial_t v_\varepsilon - \nu J_\varepsilon \triangle J_\varepsilon u_\varepsilon + J_\varepsilon(J_\varepsilon u_\varepsilon \nabla J_\varepsilon v_\varepsilon) + \sum\limits_j J_\varepsilon(J_\varepsilon v_{\varepsilon,j} \nabla J_\varepsilon u_{\varepsilon,j}) - (\alpha_1+\alpha_2) J_\varepsilon \operatorname{div} A_\varepsilon^2 \\ \qquad\qquad\qquad\qquad\qquad + \beta J_\varepsilon K(J_\varepsilon u_\varepsilon) = J_\varepsilon f - \nabla p_\varepsilon \\ u_\varepsilon(0) = J_\varepsilon u(0) \\ \operatorname{div} u_\varepsilon = 0 \\ v_\varepsilon = u_\varepsilon - \alpha_1 \triangle u_\varepsilon, \end{cases}$$

where $A_\varepsilon = A(J_\varepsilon u_\varepsilon)$. To eliminate the pressure, we apply the Leray projector $\mathbb{P}$ to obtain

$$(S_\varepsilon)\quad \begin{cases} \partial_t v_\varepsilon - \nu \mathbb{P} J_\varepsilon \triangle J_\varepsilon u_\varepsilon + \mathbb{P} J_\varepsilon (J_\varepsilon u_\varepsilon \nabla J_\varepsilon v_\varepsilon) + \sum_j \mathbb{P} J_\varepsilon (J_\varepsilon v_{\varepsilon,j} \nabla J_\varepsilon u_{\varepsilon,j}) \\ \qquad\qquad - (\alpha_1+\alpha_2)\mathbb{P} J_\varepsilon \operatorname{div} A_\varepsilon^2 + \beta \mathbb{P} J_\varepsilon K(J_\varepsilon u_\varepsilon) = \mathbb{P} J_\varepsilon f \\ u_\varepsilon(0) = J_\varepsilon u(0) \\ v_\varepsilon = u_\varepsilon - \alpha_1 \triangle u_\varepsilon. \end{cases}$$

These two systems are equivalent. Thanks to the smoothing properties of J_ε, the second system can be viewed as an ordinary differential equation (with values in a Banach space). Therefore, Cauchy's theorem gives the existence of a short-time smooth solution.

Estimates independent of ε. The H^2 estimates (18) proved above are true for the approximating solution u_ε. Indeed, multiplying (S_ε) by v_ε, using that u_ε is divergence free and J_ε self-adjoint, we get

$$\frac{1}{2}\partial_t \|v_\varepsilon\|_{L^2}^2 + \nu \|\nabla J_\varepsilon u_\varepsilon\|_{L^2}^2 + \alpha_1 \nu \|\triangle J_\varepsilon u_\varepsilon\|_{L^2}^2 + \beta \int |A_\varepsilon|^2 A_\varepsilon \cdot \nabla J_\varepsilon v_\varepsilon$$
$$= -\int (J_\varepsilon u_\varepsilon \cdot \nabla J_\varepsilon v_\varepsilon)\cdot J_\varepsilon v_\varepsilon - \int (J_\varepsilon v_\varepsilon \cdot \nabla J_\varepsilon u_\varepsilon)\cdot J_\varepsilon v_\varepsilon + (\alpha_1+\alpha_2)\int \operatorname{div}(A_\varepsilon^2)\cdot J_\varepsilon v_\varepsilon + \int f\, J_\varepsilon v_\varepsilon.$$

The same estimates as above yield

$$\partial_t \|v_\varepsilon\|_{L^2}^2 \le C \|J_\varepsilon v_\varepsilon\|_{L^2}^2 + \|f\|_{L^2}^2.$$

According to the properties of J_ε, it follows that

$$\partial_t \|v_\varepsilon\|_{L^2}^2 \le C \|v_\varepsilon\|_{L^2}^2 + \|f\|_{L^2}^2,$$

whence, by Gronwall's lemma,

$$\|v_\varepsilon(t)\|_{L^2}^2 \le \big(\|v_\varepsilon(0)\|_{L^2}^2 + \|f\|_{L^\infty(0,t;L^2)}^2/C\big)e^{Ct}.$$

The constant C is of course independent of ε. Therefore u_ε exists globally and

$$(19)\qquad u_\varepsilon \quad \text{is bounded in} \quad L^\infty\big(0,T;H^2\big) \quad \text{for all } T>0.$$

We now have to obtain estimates on $\partial_t u_\varepsilon$. According to the properties of J_ε and using theorem 2, it is an exercise to check that all the terms of (S_ε), except $\partial_t v_\varepsilon$, are bounded in $L^\infty(0,T;H^{-2})$ independently of ε for all $T>0$. Therefore, the same holds for $\partial_t v_\varepsilon$ and we finally get that

$$(20)\qquad \partial_t u_\varepsilon \quad \text{is bounded in} \quad L^\infty\big(0,T;L^2\big) \quad \text{for all } T>0.$$

Extraction of a convergent sequence. Relation (20) implies that u_ε are equicontinuous in $C([0,T];L^2)$. Ascoli's theorem allows to extract a sequence u_m corresponding to $\varepsilon_m \to 0$, such that u_m converges strongly in $C([0,T];H^{-1}_{loc})$, for all $T>0$. A standard interpolation inequality next shows that u_m converges strongly in $C([0,T];H^s_{loc})$ for all $s<2$. Using also (19), we obtain the existence of u such that

$$u_m \rightharpoonup u \text{ in } L^\infty(0,T;H^2) \text{ weak}*, \tag{21}$$

and

$$u_m \to u \text{ strongly in } C([0,T];H^s_{loc}), \tag{22}$$

for all $T>0$ and $s<2$.

Relations (19) and (20) remain true for $J_\varepsilon u_\varepsilon$ instead of u_ε. Therefore, assertions (21) and (22) remain true with $J_{\varepsilon_m}u_m$ instead of u_m and some $\widetilde{u}$ instead of u. It is easy to see that $u=\widetilde{u}$, since for $\varphi\in C_0^\infty((0,\infty)\times\mathbb{R}^n)$ and by the properties of the Friedrichs mollifier one has that

$$\int_0^\infty\int_{\mathbb{R}^n}\widetilde{u}\,\varphi=\lim_{m\to\infty}\int_0^\infty\int_{\mathbb{R}^n}J_{\varepsilon_m}u_m\,\varphi=\lim_{m\to\infty}\int_0^\infty\int_{\mathbb{R}^n}u_m\,J_{\varepsilon_m}\varphi=\int_0^\infty\int_{\mathbb{R}^n}u\,\varphi.$$

Consequently, we also have that

$$J_{\varepsilon_m}u_m \rightharpoonup u \text{ in } L^\infty(0,T;H^2) \text{ weak}*, \tag{23}$$

and

$$J_{\varepsilon_m}u_m \to u \text{ strongly in } C([0,T];H^s_{loc}), \tag{24}$$

for all $T>0$ and $s<2$.

Passage to the limit. Consider a divergence free vector field $\varphi\in C_0^\infty(\mathbb{R}^n\times[0,\infty);\mathbb{R}^n)$. Multiplying (S_ε) by φ and integrating we get after some integrations by parts

$$\int_0^\infty\int_{\mathbb{R}^n}\Big(-v_m\cdot\partial_t\varphi-\nu\triangle J_{\varepsilon_m}u_m\cdot J_{\varepsilon_m}\varphi-(J_{\varepsilon_m}u_m\cdot\nabla J_{\varepsilon_m}\varphi)\cdot J_{\varepsilon_m}v_m$$
$$+\sum_j J_{\varepsilon_m}v_{m,j}\nabla J_{\varepsilon_m}u_{m,j}\cdot J_{\varepsilon_m}\varphi-(\alpha_1+\alpha_2)(\operatorname{div}A^2(J_{\varepsilon_m}u_m))\cdot J_{\varepsilon_m}\varphi+\beta K(J_{\varepsilon_m}u_m)\cdot J_{\varepsilon_m}\varphi\Big)$$
$$=\int_0^\infty\int_{\mathbb{R}^n}f\cdot J_{\varepsilon_m}\varphi+\int_{\mathbb{R}^n}v_0(x)\cdot J_{\varepsilon_m}\varphi(0,x)\,dx.$$

With the informations at hand, it is not difficult to pass to the limit in all the terms. For instance, the term

$$\int_0^\infty\int_{\mathbb{R}^n}K(J_{\varepsilon_m}u_m)\cdot J_{\varepsilon_m}\varphi,$$

is a sum of terms of the type

$$\int_0^T\int_M\mathcal{D}^2J_{\varepsilon_m}u_m\;\mathcal{D}J_{\varepsilon_m}u_m\,\mathcal{D}J_{\varepsilon_m}u_m\,J_{\varepsilon_m}\varphi,$$

where $[0,T]\times M$ is a bounded open set (containing $\operatorname{supp}J_{\varepsilon_m}\varphi$) independent of m. We now use (23) and the strong convergences of $\mathcal{D}J_{\varepsilon_m}u_m$ in $L^\infty(0,T;H^{3/4}(M))\subset L^\infty(0,T;L^4(M))$ and of $J_{\varepsilon_m}\varphi$ in $L^1(0,T;L^\infty(M))$ to pass to the limit.

3. Uniqueness of bidimensional solutions

We denote by C a constant which depends solely on the material coefficients α_1, α_2, β and which may change from one relation to another; in particular, the constant C does not depend on the viscosity ν nor on the quantities p, ε, η, $\dots$, which will be introduced in the course of the proof. Let u and $\widetilde{u}$ be two solutions of (3) belonging to $C_w\big([0,\infty);H^2\big)$ with the same initial data $u(0)=\widetilde{u}(0)\in H^2$. Let $w=u-\widetilde{u}$. To simplify the presentation, we introduce

$$A=A(u),\quad \widetilde{A}=A(\widetilde{u}),\quad L=L(u),\quad \widetilde{L}=L(\widetilde{u}).$$

We have that $w\in L^\infty_{loc}\big([0,\infty);H^2\big)$ and, by (5), $\partial_t(w-\alpha_1\triangle w)\in L^\infty_{loc}\big([0,\infty);H^{-2}\big)$. It follows that

$$\langle \partial_t(w-\alpha_1\triangle w),w\rangle=\frac12\partial_t(\|w\|^2_{L^2}+\alpha_1\|\nabla w\|^2_{L^2})\in L^\infty_{loc}\big([0,\infty)\big),$$

in the distributional sense of $(0,\infty)$. Rigorously, the above relation is verified for smooth functions and may be obtained in general by density of smooth functions and passage to the limit.

Subtracting the equations of u and $\widetilde{u}$, multiplying by w and integrating in space gives, after some integrations by parts,

$$\begin{aligned}(25)\quad \frac12\partial_t(\|w\|^2_{L^2}+\alpha_1\|\nabla w\|^2_{L^2})+\nu\|\nabla w\|^2_{L^2}+\beta\langle |A|^2A-|\widetilde{A}|^2\widetilde{A},L(w)\rangle&\\ =-\langle u\cdot\nabla u-\widetilde{u}\cdot\nabla\widetilde{u},w\rangle+\langle N(u)-N(\widetilde{u}),L(w)\rangle&,\end{aligned}$$

where the equality holds in the sense of distributions or almost everywhere since all the terms are locally bounded. As usual, the integration by parts may be rigorously justified by multiplying with an approximation of w and passing to the limit. This argument is classical and we will omit it.

As the matrix $|A|^2A-|\widetilde{A}|^2\widetilde{A}$ is symmetric, we have that

$$(26)\quad \langle |A|^2A-|\widetilde{A}|^2\widetilde{A},L(w)\rangle=\frac12\langle |A|^2A-|\widetilde{A}|^2\widetilde{A},A(w)\rangle=\frac12\int_{\mathbb{R}^2}\big(|A|^2A-|\widetilde{A}|^2\widetilde{A}\big)\cdot A(w).$$

But one can check the following identity

$$(|x|^2x-|y|^2y)\cdot(x-y)=\frac12\big(|x|^2-|y|^2\big)^2+\frac12|x-y|^2(|x|^2+|y|^2),$$

which holds for arbitrary vectors $x,y\in\mathbb{R}^k$. Consequently, we get from (26) that

$$(27)\quad \langle |A|^2A-|\widetilde{A}|^2\widetilde{A},L(w)\rangle=\frac14\int_{\mathbb{R}^2}\big(|A|^2-|\widetilde{A}|^2\big)^2+\frac14\int_{\mathbb{R}^2}|A(w)|^2(|A|^2+|\widetilde{A}|^2).$$

Next,

$$\begin{aligned}(28)\quad -\langle u\cdot\nabla u-\widetilde{u}\cdot\nabla\widetilde{u},w\rangle&=-\int_{\mathbb{R}^2}w\cdot\nabla u\,w-\int_{\mathbb{R}^2}\widetilde{u}\cdot\nabla w\,w\\ &=-\int_{\mathbb{R}^2}w\cdot\nabla u\,w\le\|\nabla u\|_{H^1}\|w\otimes w\|_{H^{-1}}\le C\|w\|^2_{H^1}\|u\|_{H^2},\end{aligned}$$

where we have used theorem 2.

It remains to estimate the last term of (25). The fact that the matrix $N(u)-N(\widetilde{u})$ is symmetric implies that

$$
\begin{aligned}
(29)\quad \langle N(u)-N(\widetilde{u}),L(w)\rangle &= \frac{1}{2}\int_{\mathbb{R}^2}\big(N(u)-N(\widetilde{u})\big)\cdot A(w)\\
&= -\frac{\alpha_2}{2}\underbrace{\int_{\mathbb{R}^2}(A^2-\widetilde{A}^2)\cdot A(w)}_{I_1} - \frac{\alpha_1}{2}\underbrace{\int_{\mathbb{R}^2}(u\cdot\nabla A-\widetilde{u}\cdot\nabla\widetilde{A})\cdot A(w)}_{I_2}\\
&\qquad -\frac{\alpha_1}{2}\underbrace{\int_{\mathbb{R}^2}\big(L^tA+AL-\widetilde{L}^t\widetilde{A}-\widetilde{A}\widetilde{L}\big)\cdot A(w)}_{I_3}.
\end{aligned}
$$

We now study each of the last three terms. One has after some integrations by parts

$$
(30)\qquad I_1=\int_{\mathbb{R}^2}\big((A-\widetilde{A})A+\widetilde{A}(A-\widetilde{A})\big)\cdot A(w)=\int_{\mathbb{R}^2}\big(A(w)A+\widetilde{A}A(w)\big)\cdot A(w),
$$

$$
(31)\qquad I_2=\int_{\mathbb{R}^2}(u\cdot\nabla(A-\widetilde{A})+(u-\widetilde{u})\cdot\nabla\widetilde{A})\cdot A(w)=\int_{\mathbb{R}^2}(w\cdot\nabla\widetilde{A})\cdot A(w).
$$

To estimate I_3, we first use the identities (2) to deduce that

$$
\begin{aligned}
(L^tA+AL)\cdot A(w)&=(L^tA)\cdot A(w)+(AL)\cdot A(w)\\
&=\big(A(w)A\big)\cdot L^t+\big(AA(w)\big)\cdot L=\big(A(w)A\big)^t\cdot L+\big(AA(w)\big)\cdot L=2\big(AA(w)\big)\cdot L.
\end{aligned}
$$

It follows that

$$
(32)\qquad I_3=2\int_{\mathbb{R}^2}\Big(\big(AA(w)\big)\cdot L-\big(\widetilde{A}A(w)\big)\cdot\widetilde{L}\Big)=2\int_{\mathbb{R}^2}\Big(A^2(w)\cdot L+\big(\widetilde{A}A(w)\big)\cdot L(w)\Big).
$$

Thanks to relations (30) and (32) we can bound

$$
(33)\quad \Big|\frac{\alpha_2}{2}I_1+\frac{\alpha_1}{2}I_3\Big|\le C\int_{\mathbb{R}^2}|A(w)|(|A|+|\widetilde{A}|)|\nabla w|\le\varepsilon\int_{\mathbb{R}^2}|A(w)|^2(|A|^2+|\widetilde{A}|^2)+\frac{C}{\varepsilon}\int_{\mathbb{R}^2}|\nabla w|^2,
$$

where $\varepsilon>0$ will be chosed later.

The key point of the proof consists in the estimate of I_2. It is the only place where we use that the space dimension is equal to two. Let $\gamma\in(0,1)$ be arbitrary. Thanks to (31), one can write

$$
(34)\qquad
\begin{aligned}
\Big|\frac{\alpha_1}{2}I_2\Big|&\le C\int_{\mathbb{R}^2}|w||\nabla\widetilde{A}||A(w)|\\
&=C\int_{\mathbb{R}^2}|w||A(w)|^\gamma|\nabla\widetilde{A}||A(w)|^{1-\gamma}\\
&\le C\|w\|_{L^p}\,\big\||A(w)|^\gamma\big\|_{L^{2/\gamma}}\,\|\nabla\widetilde{A}\|_{L^2}\,\big\||A(w)|^{1-\gamma}\big\|_{L^{4/(1-\gamma)}}\\
&=C\|w\|_{L^p}\,\|A(w)\|_{L^2}^\gamma\,\|\nabla\widetilde{A}\|_{L^2}\,\|A(w)\|_{L^4}^{1-\gamma},
\end{aligned}
$$

where we have applied Hölder's inequality and set

$$p = \frac{4}{1-\gamma}. \tag{35}$$

But, in dimension 2, the Sobolev embedding $H^1 \hookrightarrow L^p$ holds and there exists a constant C which is independent of p such that

$$\|w\|_{L^p} \leq Cp\|w\|_{H^1},$$

(see, for instance, the computations of [21], pages 723–724). Actually, the optimal constant is $Cp^{1/2}$ but Cp is sufficient for our purposes. We now obtain from (34) that

$$\left|\frac{\alpha_1}{2} I_2\right| \leq Cp\|w\|_{H^1}^{1+\gamma}\|\widetilde{u}\|_{H^2}\|A(w)\|_{L^4}^{1-\gamma}.$$

Applying Young's inequality yields

$$\left|\frac{\alpha_1}{2} I_2\right| \leq \frac{1-\gamma}{4}\eta^{4/(1-\gamma)}\|A(w)\|_{L^4}^4 + \frac{\gamma+3}{4}\left(\frac{Cp}{\eta}\right)^{4/(\gamma+3)}\|w\|_{H^1}^{4(1+\gamma)/(\gamma+3)}\|\widetilde{u}\|_{H^2}^{4/(\gamma+3)},$$

where η is such that

$$\frac{1-\gamma}{4}\eta^{4/(1-\gamma)} = \varepsilon$$

which, according to (35), leads to

$$\left(\frac{p}{\eta}\right)^{4/(\gamma+3)} = \frac{4}{1-\gamma}\varepsilon^{(\gamma-1)/(\gamma+3)}.$$

Consequently

$$\begin{aligned}
\left|\frac{\alpha_1}{2} I_2\right| &\leq \varepsilon\|A(w)\|_{L^4}^4 + \frac{C}{1-\gamma}\varepsilon^{(\gamma-1)/(\gamma+3)}\|w\|_{H^1}^{4(1+\gamma)/(\gamma+3)}\|\widetilde{u}\|_{H^2}^{4/(\gamma+3)} \\
&= \varepsilon\int_{\mathbb{R}^2}|A(w)|^2|A-\widetilde{A}|^2 + \frac{C}{1-\gamma}\varepsilon^{(\gamma-1)/(\gamma+3)}\|w\|_{H^1}^{4(1+\gamma)/(\gamma+3)}\|\widetilde{u}\|_{H^2}^{4/(\gamma+3)} \\
&\leq 2\varepsilon\int_{\mathbb{R}^2}|A(w)|^2(|A|^2+|\widetilde{A}|^2) + \frac{C}{1-\gamma}\varepsilon^{(\gamma-1)/(\gamma+3)}\|w\|_{H^1}^{4(1+\gamma)/(\gamma+3)}\|\widetilde{u}\|_{H^2}^{4/(\gamma+3)}.
\end{aligned} \tag{36}$$

We finally obtain from relations (25), (27), (28), (29), (33) and (36) the following inequality:

(37)

$$\begin{aligned}
\frac{1}{2}\partial_t(\|w\|_{L^2}^2 + \alpha_1\|\nabla w\|_{L^2}^2) + \nu\|\nabla w\|_{L^2}^2 + \frac{\beta}{4}\int_{\mathbb{R}^2}\left(|A|^2-|\widetilde{A}|^2\right)^2 + \frac{\beta}{4}\int_{\mathbb{R}^2}|A(w)|^2(|A|^2+|\widetilde{A}|^2) \\
\leq 3\varepsilon\int_{\mathbb{R}^2}|A(w)|^2(|A|^2+|\widetilde{A}|^2) + \frac{C}{\varepsilon}\int_{\mathbb{R}^2}|\nabla w|^2 + C\|w\|_{H^1}^2\|u\|_{H^2} \\
+ \frac{C}{1-\gamma}\varepsilon^{(\gamma-1)/(\gamma+3)}\|w\|_{H^1}^{4(1+\gamma)/(\gamma+3)}\|\widetilde{u}\|_{H^2}^{4/(\gamma+3)}.
\end{aligned}$$

As $u, \widetilde{u} \in L^\infty(0,1;H^2)$, the choice $\varepsilon = \beta/12$ implies the existence of a constant $M = M(\alpha_1, \alpha_2, \beta, u, \widetilde{u})$ independent of γ such that

$$f'(t) \leq \frac{M}{1-\gamma}\big(f(t) + f(t)^{2(\gamma+1)/(\gamma+3)}\big), \quad \forall t \in [0,1], \tag{38}$$

where we defined

$$f(t) = \|w(t)\|_{L^2}^2 + \alpha_1 \|\nabla w(t)\|_{L^2}^2.$$

As $f(0) = 0$, we have by the continuity of f the existence of a time $T_0 \leq 1$ such that $f(t) \leq 1$ for all $t \in [0, T_0]$. Using also that $2(\gamma+1)/(\gamma+3) < 1$, we deduce from (38) that, for all $t \in [0, T_0]$,

$$f' \leq \frac{2M}{1-\gamma} f^{2(\gamma+1)/(\gamma+3)}$$

which, in turn, implies

$$\big(f^{(1-\gamma)/(\gamma+3)}\big)' = \frac{1-\gamma}{\gamma+3} f' f^{-2(\gamma+1)/(\gamma+3)} \leq \frac{2M}{\gamma+3}.$$

It comes after integration

$$f(t) \leq \left(\frac{2Mt}{\gamma+3}\right)^{(\gamma+3)/(1-\gamma)} \qquad \forall t \in [0, T_0].$$

If $t \leq \frac{3}{4M}$ one has that $\frac{2Mt}{\gamma+3} \leq \frac{1}{2}$; we deduce that

$$f(t) \leq \left(\frac{1}{2}\right)^{(\gamma+3)/(1-\gamma)} \qquad \forall t \in \Big[0, \min\big(T_0, \frac{3}{4M}\big)\Big].$$

Passing to the limit $\gamma \nearrow 1$ now yields

$$f(t) = 0 \quad \forall t \in \Big[0, \min\big(T_0, \frac{3}{4M}\big)\Big],$$

which implies local uniqueness. Global uniqueness follows by a connexity argument. The set

$$\big\{t\,;\, f(s) = 0 \ \forall s \in [0,t]\big\} \subset [0, +\infty)$$

is nonvoid, open and closed; therefore it coincides with $[0, +\infty)$.

Appendix

The purpose of this appendix is to show that equations (3) and (4) are equivalent. To do that, it is sufficient to check that the expression

$$B = u \cdot \nabla u + \operatorname{div}\big(-\alpha_1(u \cdot \nabla A + L^t A + AL) - \alpha_2 A^2\big) \\ - u \cdot \nabla v - \sum_j v_j \nabla u_j + (\alpha_1 + \alpha_2) \operatorname{div} A^2$$

is a gradient. From the definition of v, one has

$$u \cdot \nabla u - u \cdot \nabla v = \alpha_1 u \cdot \nabla \triangle u.$$

The i-th component of B can now be written under the form

$$B_i = \alpha_1 \sum_{j,k} u_j \partial_j \partial_k^2 u_i - \sum_j (u_j - \alpha_1 \triangle u_j)\partial_i u_j - \alpha_1 \operatorname{div}\big(u \cdot \nabla A + L^t A + AL - A^2\big),$$

or, equivalently,

$$\frac{1}{\alpha_1}\big(B_i + \frac{1}{2}\partial_i(|u|^2)\big) = \sum_{j,k} \Big(u_j \partial_j \partial_k^2 u_i + \partial_k^2 u_j \partial_i u_j - \partial_j \Big(u_k \partial_k (\partial_j u_i + \partial_i u_j) + \partial_i u_k (\partial_j u_k + \partial_k u_j) + (\partial_i u_k + \partial_k u_i)\partial_j u_k - (\partial_i u_k + \partial_k u_i)(\partial_k u_j + \partial_j u_k)\Big)\Big).$$

Expanding further, simplifying the obvious terms and using that u is divergence free yields

$$\frac{1}{\alpha_1}\big(B_i + \frac{1}{2}\partial_i(|u|^2)\big) = \sum_{j,k} \big(\underbrace{u_j \partial_j \partial_k^2 u_i}_{D_1} + \underbrace{\partial_k^2 u_j \partial_i u_j}_{D_2} - \underbrace{\partial_j u_k \partial_k \partial_j u_i}_{D_3} - \partial_j u_k \partial_k \partial_i u_j - \underbrace{u_k \partial_k \partial_j^2 u_i}_{D_4} - \partial_j \partial_i u_k \partial_j u_k - \underbrace{\partial_i u_k \partial_j^2 u_k}_{D_5} + \underbrace{\partial_j \partial_k u_i \partial_k u_j}_{D_6}\big).$$

Interchanging j and k, the term D_1 simplifies with D_4, D_2 with D_5 and D_3 with D_6. On the other hand, we have that

$$\partial_j \partial_i u_k \partial_j u_k = \frac{1}{2}\partial_i\big((\partial_j u_k)^2\big),$$

and, again interchanging j and k,

$$\sum_{j,k} \partial_j u_k \partial_k \partial_i u_j = \partial_i \sum_{j,k} \partial_j u_k \partial_k u_j - \sum_{j,k} \partial_i \partial_j u_k \partial_k u_j = \partial_i \sum_{j,k} \partial_j u_k \partial_k u_j - \sum_{j,k} \partial_j u_k \partial_k \partial_i u_j,$$

so that

$$\frac{1}{\alpha_1}\big(B_i + \frac{1}{2}\partial_i(|u|^2)\big) = \frac{1}{2}\partial_i \sum_{j,k} \big(\partial_j u_k \partial_k u_j + (\partial_j u_k)^2\big) = \frac{1}{2}\partial_i |\nabla u|^2.$$

We deduce that

$$B = \frac{1}{2}\nabla(\alpha_1 |\nabla u|^2 - |u|^2),$$

which completes the proof.

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