Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions

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Abstract

We consider the Navier-Stokes equations with Navier friction boundary conditions and prove two results. First, in the case of a bounded domain we prove that weak Leray solutions converge (locally in time in dimension ≥ 3 and globally in time in dimension 2) as the viscosity goes to 0 to a strong solution of the Euler equations provided that the initial data converges in L^2 to a sufficiently smooth limit. Second, we consider the case of a half-space and anisotropic viscosities: we fix the horizontal viscosity, we send the vertical viscosity to 0 and prove convergence to the expected limit system under weaker hypothesis on the initial data.

Introduction

We consider in this paper the vanishing viscosity limit for the incompressible Navier-Stokes equations in a domain Ω:

$$\begin{aligned} \partial_t u - \nu \triangle u + u \cdot \nabla u = -\nabla p, \quad \text{in} \quad \Omega \times (0, +\infty), \\ \operatorname{div} u = 0, \quad \text{in} \quad \Omega \times [0, +\infty), \\ u\big|_{t=0} = u_0, \quad \text{in} \quad \Omega. \end{aligned} \tag{1}$$

The vanishing viscosity limit for the incompressible Navier-Stokes equations, in the case where there exist physical boundaries, is a challenging problem due to the formation of a boundary layer which is caused by the classical no-slip boundary condition. A partial result, in the case of half-space, was given in [17, 18] by imposing analyticity on the initial data. The authors proved in these papers that the Navier-Stokes solution goes to an Euler solution outside a boundary layer, and it is close to a solution of the Prandtl equations within the boundary layer. Concerning the anisotropic Navier-Stokes equations, in some particular domains such as the half-space, it was showed in [10] that if the ratio of vertical viscosity to horizontal viscosity also goes to zero, then the weak solutions converge to the solution of the Euler system.

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From a physical point of view, the no-slip condition is only justified where the molecular viscosity is concerned. In [14] Navier claimed that the tangential component of the viscous stress at the boundary should be proportional to the tangential velocity. We call Navier friction boundary conditions the following conditions:

$$u \cdot n = 0, \quad [D(u)n + \alpha u]_{tan} = 0, \quad \text{on } \partial\Omega \times (0, +\infty). \tag{2}$$

Here, $D(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{1 \le i,j \le n}$ denotes the deformation tensor, n is the exterior normal to Ω, $\alpha \ge 0$ is a material constant (the friction coefficient) and $[D(u)n + \alpha u]_{tan}$ is the tangential component of the vector $D(u)n + \alpha u$.

The Navier friction condition was rigorously justified as a homogenization of the no-slip condition on a rough boundary, see [6]. The Navier-Stokes equations, and also other equations in fluid mechanics, with Navier and other similar boundary conditions were studied in the mathematical literature, see for example [1, 5, 13] and the references therein. Recently, in [3] and [9] was studied the inviscid limit of the two dimensional incompressible Navier-Stokes equations in a bounded domain subject to Navier friction-type boundary conditions, see also [12] for the case of permeable boundary. These works show that the boundary layers arising from the inviscid limit can be controlled in dimension two, thus proving convergence to solutions of the Euler equations.

In the first part of this paper, we extend these bidimensional convergence results to higher dimensions by proving convergence to solutions of the Euler equations on the time interval where (strong) solutions are known to exist. More precisely, we have the following theorem:

Theorem 1 *Let Ω be a bounded smooth open domain in $\mathbb{R}^n$, $n \ge 2$, and $v_0 \in H^s(\Omega)$, $s > 1 + \frac{n}{2}$, a divergence free vector field tangent to the boundary. For each $\nu > 0$, we consider $u_0^\nu \in L^2(\Omega)$ a divergence free vector field tangent to the boundary such that $u_0^\nu \to v_0$ strongly in $L^2(\Omega)$ as $\nu \to 0$. Let u^ν be a (global) weak Leray solution of the Navier-Stokes equations* (1)-(2) *with initial data u_0^ν and $v \in C^0\big([0,T); H^s\big)$ the local strong solution of the Euler equations defined up to the maximal time-existence T. Then, u^ν converges to v strongly in $L^\infty_{loc}\big([0,T); L^2\big)$ as $\nu \to 0$.*

Let us comment this result. First of all, the hypothesis that the limit velocity belongs to H^s seems hard to be improved. Indeed, since we expect to obtain the Euler equations in the limit, we should have a limit initial velocity compatible to known solutions of the Euler equation. But, in general n-domains with $n \ge 3$, only strong solutions (which belong to H^s, $s > 1 + \frac{n}{2}$) are known to exist. Next, even in dimension two this result says something new over the results of [3] and [9]. Indeed, these articles assume that the initial velocity is fixed and belongs to $W^{1,p}$, with $p > 2$ in [9] and $p = \infty$ in [3]. The bidimensional case of Theorem 1 assumes stronger regularity on the limit solution than [3] and [9], but applies to weaker solutions of the Navier-Stokes equations.

The proofs of [3, 9, 12] consist in making *a priori* estimates and pass to the limit with compactness methods. A similar approach does not seem to work in dimension three. The reason is that the *a priori* estimates to prove should hold true for the limit system. But on the limit system, one has in general only H^s estimates with $s = 0$ or $s > 5/2$. L^2 estimates are not sufficient to make a compactness method work while *a priori* estimates in H^s with $s > 5/2$ cannot be true. Indeed, that would imply that the limit equation also verifies the Navier boundary conditions. We shall see in the last section that in general this is not the case. Our proof of Theorem 1 consist in a direct estimate of the L^2 norm of the difference which turns out to be surprisingly easy.

The second part of this paper is our main contribution. Here, we will consider the anisotropic inviscid limit for the following anisotropic Navier-Stokes equations in the half-space $\mathbb{H} = \{x \in \mathbb{R}^3;\ x_3 > 0\}$,

$$\begin{aligned} \partial_t u - \nu(\partial_1^2 + \partial_2^2)u - \varepsilon\partial_3^2 u + u \cdot \nabla u = -\nabla p, \quad \text{in} \quad \mathbb{H} \times (0, +\infty),\\ \operatorname{div} u = 0, \quad \text{in} \quad \mathbb{H} \times [0, +\infty),\\ u\big|_{t=0} = u_0, \quad \text{in} \quad \mathbb{H}, \end{aligned} \tag{3}$$

supplemented with the Navier boundary conditions that can be written in this particular case under the following form:

$$u_3 = 0, \quad \partial_3 u_1 = \alpha' u_1, \quad \partial_3 u_2 = \alpha' u_2 \quad \text{on} \quad \partial\mathbb{H} \times (0, +\infty), \tag{4}$$

where $\alpha' = 2\alpha$. The constants $\nu > 0$ and $\varepsilon \geq 0$ represent respectively the horizontal and vertical viscosities.

The anisotropic Navier-Stokes equations are widely used in geophysical fluid dynamics as a mathematical model for water flow in lakes and oceans, and also in the study of the Ekman boundary layers for rotating fluids, see for instance [16, 4]. These equations appear when the domain in use has very different horizontal and vertical dimensions; the turbulent viscosity coefficients may not be isotropic in this case.

In the absence of physical boundary, i.e., when the fluid occupies the whole space, the Navier-Stokes equations with vanishing vertical viscosity were primarily studied in [2]. The authors proved results of local existence for large data in anisotropic Sobolev spaces $H^{0,s}$, $s > 1/2$, and of global existence for small initial data, compared with the horizontal viscosity, in the same space. The uniqueness was showed for $s > 3/2$. The gap between the existence result and the uniqueness result was closed in [8]. The anisotropic space $H^{0,s}$ is a space with L^2 regularity in the horizontal variable and H^s regularity in the vertical variable. In the case of null vertical viscosity, similar results were obtained by [15] in anisotropic Besov-Sobolev spaces which contain the spaces $H^{0,s}$, $s > 1/2$. These results are very similar to results known for the isotropic Navier-Stokes equations, we refer to [8] for a discussion of this subject.

In this part, our aim is to prove that the limit when the vertical viscosity ε goes to 0 is what one should expect, *i.e.* the solution of the system of equations obtained by setting

$\varepsilon = 0$ in (3) and taking only the first boundary condition in (4):

$$\begin{aligned} \partial_t u - \nu(\partial_1^2 + \partial_2^2)u + u \cdot \nabla u = -\nabla p, &\quad \text{in} \quad \mathbb{H} \times (0,T), \\ \operatorname{div} u = 0, &\quad \text{in} \quad \mathbb{H} \times [0,T), \\ u\big|_{t=0} = u_0, &\quad \text{in} \quad \mathbb{H}, \\ u_3 = 0 &\quad \text{on} \quad \partial\mathbb{H} \times [0,T). \end{aligned} \tag{5}$$

We observe that the boundary condition above is sufficient to solve (5). Indeed, the second order derivatives are only in the tangential direction and can be integrated by parts. Moreover, this system of equations can be reduced to the case of full plane. Indeed, if we extend u_1 and u_2 to $\mathbb{R}^3$ by even reflection and u_3 by odd reflection with respect to the plane $x_3 = 0$, then the resulting vector field verifies the first two equations of (5) with $\mathbb{H} = \mathbb{R}^3$. Conversely, a solution of the first two equations of (5) in $\mathbb{R}^3$ preserves this special symmetry structure, so if the initial data has this structure, then the restriction to $\mathbb{H}$ is a solution of (5). Therefore the study of (5) reduces to the one of the anisotropic Navier-Stokes equations studied in [2, 8, 15]. Exactly the same observation holds true for the system of equations (3)-(4) in the case $\alpha' = 0$. Using precisely the same reflection extension, we observe that the study of (3)-(4) with $\alpha' = 0$ reduces to the study of the Navier-Stokes equations in the full space with an initial data having the above special structure.

Our main result on the second part is stated in the following theorem.

Theorem 2 *Let $u_0 \in L^2(\mathbb{H})$ be a divergence free vector field, independent of ε, tangent to the boundary and such that $\partial_3 u_0 \in L^2(\mathbb{H})$. For $\varepsilon \in (0,1]$, there exist a positive time T independent of ε and a solution u^ε of system (3)-(4) up to time T such that*

$$u^\varepsilon,\ \partial_3 u^\varepsilon \in L^\infty(0,T;L^2), \qquad \partial_i u^\varepsilon,\ \partial_i\partial_3 u^\varepsilon \in L^2\big((0,T)\times\mathbb{H}\big),\ i = 1,2.$$

Moreover, u^ε converges up to a subsequence to a solution of the limit system (5) as $\varepsilon \to 0$.

Furthermore, there exists a constant K independent of ε and ν such that if the smallness assumption $\|u_0\|_{L^2} + \|u_0\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_0\|_{L^2}^{\frac{1}{2}} \leq K\nu$ holds true, then the existence and convergence of u^ε hold true globally in time, i.e. one can take $T = +\infty$.

The key point of the proof is the use of the special structure of the incompressible Navier-Stokes system in obtaining *a priori* estimates.

The second part is organized as follows. In the first section, after introducing the basic notation, we prove some anisotropic inequalities and a result for a linearized problem which we will use in later proofs. The second section is devoted to obtaining *a priori* estimates independent of ε, which are the core of the proof of our main Theorem. In the third section we prove Theorem 2, which is a trivial consequence of the *a priori* estimates. The last section contains a final remark.

Part I
The classical inviscid limit

Throughout this part, Ω denotes a bounded smooth domain of $\mathbb{R}^n$. We denote by $H^1_\sigma(\Omega)$, respectively $L^2_\sigma(\Omega)$, the space of H^1, respectively L^2, divergence free vector fields tangent to the boundary. We recall the following formula:

Lemma 3 *Let f be a divergence free vector field verifying the Navier boundary conditions and g a divergence free vector field tangent to the boundary. Then*

$$-\int_\Omega \triangle f \cdot g \; dx = 2\alpha \int_{\partial\Omega} f \cdot g \; dx + 2 \int_\Omega D(f) \cdot D(g) \; dx.$$

The proof of this identity consists in an integration by parts, see for instance [20] or [19]. We next give the definition of a weak Leray solution.

Definition 4 *We call weak Leray solution of* (1)-(2) *a time dependent vector field $u(t,x)$: $[0,\infty) \times \Omega \to \Omega$ verifying:*

- $u \in C_w\big([0,\infty); L^2_\sigma\big) \cap L^2_{loc}\big([0,\infty); H^1_\sigma\big)$*;*
- *u verifies the system of equations* (1)-(2) *under the following weak form:*

$$-\int_0^\infty \int_\Omega u \cdot \partial_t \varphi + 2\alpha\nu \int_0^\infty \int_{\partial\Omega} u \cdot \varphi + 2\nu \int_0^\infty \int_\Omega D(u) \cdot D(\varphi) + \int_0^\infty \int_\Omega u \cdot \nabla u \cdot \varphi = \int_\Omega u(0) \cdot \varphi(0),$$

for all divergence free test vector fields $\varphi \in C_0^\infty\big([0,\infty) \times \overline{\Omega}\big)$ tangent to the boundary;

- *u verifies the energy inequality*

$$\|u(t)\|^2_{L^2} + 4\alpha\nu \int_0^t \int_{\partial\Omega} |u|^2 + 4\nu \int_0^t \int_\Omega |D(u)|^2 \leq \|u(0)\|^2_{L^2}, \qquad \text{for all } t \geq 0. \tag{6}$$

Let us remark that a standard density argument allows to take less smooth test vector fields φ in the above weak formulation. We next observe that the definition above encodes the information contained in both (1) and (2). Indeed, choosing first $\varphi \in C_0^\infty\big((0,\infty) \times \Omega\big)$ we deduce that (1) is verified in the distributional sense of $(0,\infty) \times \Omega$. The boundary condition is also verified in a weak sense. Indeed, if we assume u more regular, then one can make an integration by parts in the weak formulation and use Lemma 3 to obtain that

$$\int_0^\infty \int_{\partial\Omega} [D(u)n - \alpha u] \cdot \varphi = 0.$$

Since φ is an arbitrary test function tangent to the boundary, we must necessarily have that $D(u)n-\alpha u$ is normal to the boundary. Next, we note that the energy inequality (6) follows after multiplying formally (1) by u, integrating on Ω and from 0 to t and using Lemma 3 to integrate by parts the Laplacian term. Finally, we observe that global solutions in the sense of Definition 4 are well-known to exist in the case of Dirichlet boundary conditions for any divergence free square integrable initial data tangent to the boundary. The extension of this classical existence result to the case of Navier boundary conditions is straightforward.

Let v be the local strong solution of the Euler equation

$$\begin{aligned} \partial_t v + v\cdot\nabla v &= -\nabla q, \quad \text{in} \quad \Omega\times(0,T),\\ \operatorname{div} v &= 0, \quad \text{in} \quad \Omega\times[0,T),\\ v\big|_{t=0} &= v_0, \quad \text{in} \quad \Omega,\\ v\cdot n &= 0 \quad \text{on} \quad \partial\Omega\times[0,T). \end{aligned} \tag{7}$$

We prove now Theorem 1. With the notations of Theorem 1, the difference $w=w^\nu=u^\nu-v$ verifies the equation

$$\partial_t w + u^\nu\cdot\nabla w + w\cdot\nabla v - \nu\triangle u^\nu = -\nabla(p-q) \quad \text{in} \quad \Omega. \tag{8}$$

We assume in the sequel that $t\in[0,T)$. Let us multiply the above equation by w, integrate in space and time from 0 to t and finally integrate by parts the Laplacian term using Lemma 3 to obtain

$$\frac12\|w(t)\|_{L^2}^2+\int_0^t\int_\Omega w\cdot\nabla v\cdot w+2\nu\alpha\int_0^t\int_{\partial\Omega}u^\nu\cdot w+2\nu\int_0^t\int_\Omega D(u^\nu)\cdot D(w)\leq\frac12\|w(0)\|_{L^2}^2. \tag{9}$$

We used above that w is divergence and tangent to the boundary to deduce that the pressure term vanishes and that $\int_0^t\int_\Omega u^\nu\cdot\nabla w\cdot w=0$. In fact, a little discussion is required here. Indeed, if the space dimension is ≥ 3, then the a priori regularity of u^ν and w does not allow to deduce that the integral $\int_0^t\int_\Omega u^\nu\cdot\nabla w\cdot w=0$ converges, and therefore to infer that it vanishes. In fact, one cannot multiply directly (8) by w since the regularity at hand is not sufficient. However, there is a classical trick that allows to deduce that (9) is still true at the price of assuming the energy inequality for u^ν, see for example [21]. The idea is the following. Let us denote by $E(u^\nu)$, respectively $E(v)$, the left-hand side of (1), respectively (7). Formally multiplying (8) by w is the same as writing $\int_0^t\int_\Omega[E(u^\nu)-E(v)]\cdot(u^\nu-v)=0$, that is

$$\int_0^t\int_\Omega E(u^\nu)\cdot u^\nu+\int_0^t\int_\Omega E(v)\cdot v-\int_0^t\int_\Omega E(u^\nu)\cdot v-\int_0^t\int_\Omega E(v)\cdot u^\nu=0. \tag{10}$$

Since v has sufficient regularity, we see that, for $t \in [0,T)$, all the terms above make sense except for $\int_0^t \int_\Omega E(u^\nu) \cdot u^\nu$. The observation is that multiplying the equation of u^ν by itself formally yields the energy inequality (6). Since we assumed that the energy inequality holds true, we can therefore say that $\int_0^t \int_\Omega E(u^\nu) \cdot u^\nu \leq 0$. The relation (10) must be modified accordingly in an inequality instead of an equality and this explains why there is an inequality in (9) instead of an equality. In short, the rigorous derivation of (9) goes like that: add the energy inequality (6) to the equation of v multiplied by v and subtract the equation of u^ν multiplied by v and the equation of v multiplied by u^ν.

We now go back to (9). We write

$$\begin{aligned}
&2\nu\alpha \int_0^t \int_{\partial\Omega} u^\nu \cdot w + 2\nu \int_0^t \int_\Omega D(u^\nu) \cdot D(w) \\
&\quad = 2\nu\alpha \int_0^t \int_{\partial\Omega} \big|w + \frac{v}{2}\big|^2 + 2\nu \int_0^t \int_\Omega \big|D\big(w + \frac{v}{2}\big)\big|^2 - \frac{\nu\alpha}{2} \int_0^t \int_{\partial\Omega} |v|^2 - \frac{\nu}{2} \int_0^t \int_\Omega |D(v)|^2.
\end{aligned}$$

On the other hand, $\Big|\int_\Omega w \cdot \nabla v \cdot w\Big| \leq \|w\|_{L^2}^2 \|\nabla v\|_{L^\infty}$. We deduce from the above relations plugged in (9) that

$$\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 + 2\int_0^t \|w\|_{L^2}^2 \|\nabla v\|_{L^\infty} + \nu\alpha \int_0^t \int_{\partial\Omega} |v|^2 + \nu \int_0^t \int_\Omega |D(v)|^2.$$

Since $\|v\|_{L^2(\partial\Omega)} \leq C\|v\|_{H^1(\Omega)}$ and $w = u^\nu - v$, the Gronwall inequality implies that

$$\|u^\nu(t) - v(t)\|_{L^2}^2 \leq \Big[\|u^\nu(0) - v(0)\|_{L^2}^2 + \nu C(\Omega,\alpha) \int_0^t \|v\|_{H^1}^2\Big] \exp\Big(2\int_0^t \|\nabla v\|_{L^\infty}\Big).$$

As $v \in L^\infty_{loc}\big([0,T); H^s\big)$ and $H^s(\Omega) \hookrightarrow Lip(\Omega)$, we deduce the desired conclusion:

$$\sup_{[0,t]} \|u^\nu - v\|_{L^2} \xrightarrow{\nu \to 0} 0 \qquad \text{for all } t \in [0,T).$$

This completes the proof of Theorem 1.

Part II
The anisotropic inviscid limit

II.1 Notations and preliminary estimates

For a vector field $u = (u_1, u_2, u_3)$ we denote the horizontal component by $u_h = (u_1, u_2)$. We also define the horizontal variable $x_h = (x_1, x_2)$, we denote by $\nabla_h = (\partial_1, \partial_2)$ the horizontal gradient, by $\Delta_h = \partial_1^2 + \partial_2^2$ the horizontal Laplacian and by D_3 the operator $D_3 = (Id, \partial_3)$. All norms with respect to x are supposed to be taken in $\mathbb{H}$ unless otherwise specified. We will use the following anisotropic Lebesgue spaces:

$$L^{p,q} = \Big\{ f \text{ measurable } ; \ \|f\|_{L^{p,q}} = \Big\| \|f\|_{L^p(dx_h)} \Big\|_{L^q(dx_3)} < \infty \Big\}.$$

The following remark shows that the order of integrations is important.

Remark 5 *Let $(X_1;\mu_1), (X_2;\mu_2)$ be two measure spaces, $1 \le p \le q \le \infty$ and $f : X_1 \times X_2 \to \mathbb{R}$. Then*

$$\Big\| \|f\|_{L^p(X_1,\mu_1)} \Big\|_{L^q(X_2,\mu_2)} \le \Big\| \|f\|_{L^q(X_2,\mu_2)} \Big\|_{L^p(X_1,\mu_1)}.$$

For a proof see [7]. We continue with a version of the Gagliardo-Nirenberg inequality for the anisotropic Lebesgue spaces.

Lemma 6 *Suppose that $p \in [2, +\infty)$, $q \in [2, +\infty]$ and choose a number a in the interval $a \in \Big[\max\big(\frac{1}{p}, \frac{1}{q} + \frac{2}{p} - \frac{1}{2}\big), \min\big(\frac{2}{p}, \frac{1}{p} + \frac{1}{q}\big) \Big]$. There exists a constant C such that*

$$\|f\|_{L^{p,q}} \le C \|f\|_{L^2}^a \|\nabla_h f\|_{L^2}^{\frac{1}{2}+\frac{1}{q}-a} \|D_3 f\|_{L^2}^{\frac{2}{p}-a} \|\nabla_h D_3 f\|_{L^2}^{\frac{1}{2}-\frac{2}{p}-\frac{1}{q}+a}.$$

Proof. We consider first the case $p > 2$. We use the well-known Gagliardo-Nirenberg inequality in $\mathbb{R}^2$ to write, for fixed x_3,

$$\|f(x_h, x_3)\|_{L^p(dx_h)} \le C \|f(x_h, x_3)\|_{L^2(dx_h)}^{\lambda} \|\nabla_h f(x_h, x_3)\|_{L^2(dx_h)}^{1-\lambda}, \qquad \lambda = \frac{2}{p}.$$

We next apply the Hölder inequality in the vertical direction and use Remark 5,

$$\begin{aligned} \|f\|_{L^{p,q}} &\le C \Big\| \|f\|_{L^2(dx_h)}^{\lambda} \|\nabla_h f\|_{L^2(dx_h)}^{1-\lambda} \Big\|_{L^q(dx_3)} \\ &\le C \Big\| \|f\|_{L^2(dx_h)} \Big\|_{L^{\lambda q_1}(dx_3)}^{\lambda} \Big\| \|\nabla_h f\|_{L^2(dx_h)} \Big\|_{L^{(1-\lambda) q_2}(dx_3)}^{1-\lambda} \\ &\le C \Big\| \|f\|_{L^{\lambda q_1}(dx_3)} \Big\|_{L^2(dx_h)}^{\lambda} \Big\| \|\nabla_h f\|_{L^{(1-\lambda) q_2}(dx_3)} \Big\|_{L^2(dx_h)}^{1-\lambda}, \end{aligned} \tag{11}$$

where $q_1, q_2 \in [1, \infty]$ are such that

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad 2 \le \lambda q_1 \le +\infty, \quad 2 \le (1-\lambda) q_2 \le +\infty.$$

We now recall the Gagliardo-Nirenberg inequality in the vertical direction

$$\|f\|_{L^{\lambda q_1}(dx_3)} \le C \|f\|_{L^2(dx_3)}^{\beta_1} \|D_3 f\|_{L^2(dx_3)}^{1-\beta_1}, \qquad \beta_1 = \frac{1}{2} + \frac{1}{\lambda q_1},$$

so that, after applying the Hölder inequality,

$$\Big\| \|f\|_{L^{\lambda q_1}(dx_3)} \Big\|_{L^2(dx_h)}^{\lambda} \le C \Big\| \|f\|_{L^2(dx_3)}^{\beta_1} \|D_3 f\|_{L^2(dx_3)}^{1-\beta_1} \Big\|_{L^2(dx_h)}^{\lambda} \le C \|f\|_{L^2}^{\lambda \beta_1} \|D_3 f\|_{L^2}^{\lambda(1-\beta_1)}.$$

We obtain in a similar manner that

$$\big\|\|\nabla_h f\|_{L^{(1-\lambda)q_2}(dx_3)}\big\|^{1-\lambda}_{L^2(dx_h)} \leq C\|\nabla_h f\|^{(1-\lambda)\beta_2}_{L^2}\|\nabla_h D_3 f\|^{(1-\lambda)(1-\beta_2)}_{L^2}, \qquad \beta_2 = \frac{1}{2} + \frac{1}{(1-\lambda)q_2}.$$

For a in the given range, we choose q_1 and q_2 such that $a = \frac{1}{p} + \frac{1}{q_1}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. We observe that for this choice of constants, the various restrictions that appear in the proof are satisfied. The conclusion in the case $p > 2$ now follows by plugging the last two equations into (11). The case $p = 2$ is entirely similar except that we choose $q_1 = q$ and $q_2 = \infty$. This implies at the end the desired conclusion for $a = \frac{1}{2} + \frac{1}{q}$. We finally observe that the choice $a = \frac{1}{2} + \frac{1}{q}$ is the only one allowed by the hypothesis in the case $p = 2$. This completes the proof. □

As an immediat consequence of the previous lemma and of the inequality $\|f\|_{L^2} \leq \|D_3 f\|_{L^2} = (\|f\|^2_{L^2} + \|\partial_3 f\|_{L^2})^{\frac{1}{2}}$ we deduce the following corollary.

Corollary 7 *There exists a constant C such that*

$$\|f\|_{L^4} \leq C\|f\|^{\frac{1}{2}}_{L^2}\|\nabla_h f\|^{\frac{1}{4}}_{L^2}\|\nabla_h D_3 f\|^{\frac{1}{4}}_{L^2},$$
$$\|f\|_{L^{4,\infty}} \leq \|f\|^{\frac{1}{4}}_{L^2}\|\nabla_h f\|^{\frac{1}{4}}_{L^2}\|D_3 f\|^{\frac{1}{4}}_{L^2}\|\nabla_h D_3 f\|^{\frac{1}{4}}_{L^2} \leq C\|D_3 f\|^{\frac{1}{2}}_{L^2}\|\nabla_h D_3 f\|^{\frac{1}{2}}_{L^2},$$
$$\|f\|_{L^{4,2}} \leq C\|f\|^{\frac{1}{2}}_{L^2}\|\nabla_h f\|^{\frac{1}{2}}_{L^2},$$
$$\|f\|_{L^{2,\infty}} \leq C\|f\|^{\frac{1}{2}}_{L^2}\|D_3 f\|^{\frac{1}{2}}_{L^2} \leq C\|D_3 f\|_{L^2}.$$

As a consequence of Lemma 6 combined with the anisotropic Hölder inequality we have the following lemma.

Lemma 8 *There exists a constant C such that*

$$\Big|\int_{\mathbb{H}} fgh\,dx\Big| \leq C\|f\|^{\frac{1}{2}}_{L^2}\|\nabla_h f\|^{\frac{1}{2}}_{L^2}\|h\|^{\frac{1}{2}}_{L^2}\|\nabla_h h\|^{\frac{1}{2}}_{L^2}\|g\|^{\frac{1}{2}}_{L^2}\|D_3 g\|^{\frac{1}{2}}_{L^2}.$$

Proof. Simply write

$$\Big|\int_{\mathbb{H}} fgh\,dx\Big| \leq \|f\|_{L^{4,2}}\|g\|_{L^{2,\infty}}\|h\|_{L^{4,2}} \leq C\|f\|^{\frac{1}{2}}_{L^2}\|\nabla_h f\|^{\frac{1}{2}}_{L^2}\|h\|^{\frac{1}{2}}_{L^2}\|\nabla_h h\|^{\frac{1}{2}}_{L^2}\|g\|^{\frac{1}{2}}_{L^2}\|D_3 g\|^{\frac{1}{2}}_{L^2}.$$

□

We end this section with a result on the anisotropic Stokes problem that will be used in a later proof.

Proposition 9 *Let $0 < \varepsilon \leq 1$ and $u_0 \in L^2(\mathbb{H})$ be a divergence free vector field, tangent to the boundary and such that $\partial_3 u_0 \in L^2(\mathbb{H})$. Let v be the solution of the Stokes problem with initial data u_0:*

$$\partial_t v - \nu(\partial_1^2 + \partial_2^2)v - \varepsilon\partial_3^2 v = -\nabla q, \quad \operatorname{div} v = 0, \quad v\Big|_{t=0} = u_0,$$

supplemented with the Navier boundary conditions,

$$v_3 = 0, \quad \partial_3 v_1 = \alpha' v_1, \quad \partial_3 v_2 = \alpha' v_2 \quad \text{on} \quad \partial\mathbb{H},$$

where $\alpha' \geq 0$ *and* $\nu > 0$. *For any* η*, there exists a time* T_η *independent of* ε *such that*

$$\int_0^{T_\eta} \|\nabla_h v(\tau)\|_{L^2}^2 d\tau \leq \eta.$$

Moreover, the following inequality holds true:

$$\begin{aligned} \|D_3 v(t)\|_{L^2}^2 &+ \int_0^t \|\nabla_h D_3 v(\tau)\|_{L^2}^2 d\tau \\ &\leq \max\Big(1, \frac{1}{2\nu}\Big) \Big[\|\partial_3 u_0\|_{L^2}^2 + (1+\alpha'^2)\|u_0\|_{L^2}^2 + \alpha' \|u_h(0)\|_{L^2(\partial\mathbb{H})}^2\Big]. \end{aligned} \tag{12}$$

Proof. The existence of T_η for fixed ε is trivial. We will prove that it is independent of ε. Let w be the solution of the following system

$$\partial_t w - \nu(\partial_1^2 + \partial_2^2) w = 0 \quad \text{on } \mathbb{H}, \quad w\big|_{t=0} = u_0.$$

Obviously, w is given explicitly as a convolution integral in terms of the fundamental solution of the Laplacian in $\mathbb{R}^2$:

$$w(t,x) = \int_{\mathbb{R}^2} \frac{1}{4\pi t\nu} e^{-\frac{|y|^2}{4t\nu}} u_0(x_h - y, x_3)\, dy.$$

From this explicit formula, it is clear that $w(t)$ is divergence free and tangent to the boundary for all times t. The difference $\psi = v - w$ verifies then the following equation

$$\partial_t \psi - \nu(\partial_1^2 + \partial_2^2)\psi - \varepsilon \partial_3^2 v = -\nabla q, \quad \operatorname{div} \psi = 0, \quad \psi\big|_{t=0} = 0,$$

and ψ is tangent to the boundary. We now take the L^2 scalar product with ψ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{L^2}^2 + \nu\|\nabla_h \psi\|_{L^2}^2 - \varepsilon \int_{\mathbb{H}} \partial_3^2 v \cdot \psi\, dx = 0. \tag{13}$$

Integrating by parts the last integral yields

$$-\varepsilon \int_{\mathbb{H}} \partial_3^2 v \cdot \psi\, dx = \varepsilon \int_{\mathbb{H}} \partial_3 v \cdot \partial_3 \psi\, dx + \varepsilon \int_{\partial\mathbb{H}} \partial_3 v \cdot \psi\, ds.$$

We use that $\psi_3 = 0$ and v verifies the Navier boundary conditions on the boundary to replace the term $\partial_3 v \cdot \psi$ with $\alpha' v_h \cdot \psi_h$. We further replace v by $w + \psi$ and observe that we can write

$$-\varepsilon \int_{\mathbb{H}} \partial_3^2 v \cdot \psi\, dx = \varepsilon \|\partial_3 \psi + \frac{1}{2}\partial_3 w\|_{L^2}^2 - \frac{\varepsilon}{4}\|\partial_3 w\|_{L^2}^2 + \varepsilon\alpha' \|\psi_h + \frac{1}{2} w_h\|_{L^2(\partial\mathbb{H})}^2 - \frac{\varepsilon\alpha'}{4}\|w_h\|_{L^2(\partial\mathbb{H})}^2.$$

Using this relation in (13) and integrating in time we get

$$\begin{aligned}\int_0^t \|\nabla_h\psi(\tau)\|_{L^2}^2 d\tau &\leq \frac{\varepsilon}{4\nu}\int_0^t \|\partial_3 w(\tau)\|_{L^2}^2 d\tau + \frac{\varepsilon\alpha'}{4\nu}\int_0^t \|w_h(\tau)\|_{L^2(\partial\mathbb{H})}^2 d\tau\\ &\leq \frac{t\varepsilon}{4\nu}\|\partial_3 u_0\|_{L^2}^2 + \frac{t\varepsilon\alpha'}{4\nu}\|u_h(0)\|_{L^2(\partial\mathbb{H})}^2,\end{aligned}$$

where we used the explicit formula for w to deduce that $\|\partial_3 w\|_{L^2} \leq \|\partial_3 u_0\|_{L^2}$ and that $\|w_h\|_{L^2(\partial\mathbb{H})} \leq \|u_h(0)\|_{L^2(\partial\mathbb{H})}$. Observe that the hypothesis together with standard trace results implies that the trace of the initial data on the boundary is square-integrable. We finally obtain that

$$\int_0^t \|\nabla_h v(\tau)\|_{L^2}^2 d\tau \leq 2\int_0^t \|\nabla_h w(\tau)\|_{L^2}^2 d\tau + \frac{t}{2\nu}\|\partial_3 u_0\|_{L^2}^2 + \frac{t\alpha'}{2\nu}\|u_h(0)\|_{L^2(\partial\mathbb{H})}^2.$$

The first part of the proposition follows immediately since w is independent of ε and it clearly verifies that $\lim\limits_{t\to 0}\int_0^t \|\nabla_h w(\tau)\|_{L^2}^2 d\tau = 0$.

To prove (12), we multiply the equation for v by $-\partial_3^2 v$ and integrate in space to obtain,

$$-\int_{\mathbb{H}} \partial_t v\cdot\partial_3^2 v\,dx + \nu\int_{\mathbb{H}} \triangle_h v\cdot\partial_3^2 v\,dx + \varepsilon\int_{\mathbb{H}} |\partial_3^2 v|^2\,dx = \int_{\mathbb{H}} \partial_3^2 v\cdot\nabla q\,dx. \tag{14}$$

Using the Navier boundary conditions and integrating by parts we get

$$\begin{aligned}-\int_{\mathbb{H}} \partial_t v\cdot\partial_3^2 v\,dx &= \int_{\mathbb{H}} \partial_t\partial_3 v\cdot\partial_3 v\,dx + \int_{\partial\mathbb{H}} \partial_t v\cdot\partial_3 v\,ds\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{H}} |\partial_3 v|^2\,dx + \int_{\partial\mathbb{H}} \partial_t v_h\cdot\partial_3 v_h\,ds\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\big(\|\partial_3 v\|_{L^2}^2 + \alpha'\|v_h\|_{L^2(\partial\mathbb{H})}^2\big).\end{aligned} \tag{15}$$

The second term on the left hand side of (14) can be written as

$$\int_{\mathbb{H}} \triangle_h v\cdot\partial_3^2 v\,dx = -\int_{\mathbb{H}} \nabla_h v\cdot\nabla_h\partial_3^2 v\,dx = \int_{\mathbb{H}} |\nabla_h\partial_3 v|^2\,dx + \int_{\partial\mathbb{H}} \nabla_h v\cdot\partial_3\nabla_h v\,ds.$$

From the Navier boundary conditions we have $\nabla_h v_3 = 0$ and $\nabla_h\partial_3 v_h = \alpha'\nabla_h v_h$ on $\partial\mathbb{H}$, so we deduce that $\nabla_h v\cdot\nabla_h\partial_3 v = \nabla_h v_h\cdot\nabla_h\partial_3 v_h = \alpha'|\nabla_h v_h|^2$ on $\partial\mathbb{H}$. Therefore,

$$\int_{\mathbb{H}} \triangle_h v\cdot\partial_3^2 v\,dx = \|\nabla_h\partial_3 v\|_{L^2}^2 + \alpha'\|\nabla_h v_h\|_{L^2(\partial\mathbb{H})}^2. \tag{16}$$

Integrating by parts and using that $\operatorname{div} v = 0$, the pressure term can be written as

$$\int_{\mathbb{H}} \partial_3^2 v\cdot\nabla q\,dx = -\int_{\partial\mathbb{H}} \partial_3^2 v_3 q\,ds = -\alpha'\int_{\partial\mathbb{H}} \partial_3 v_3 q\,ds.$$

Therefore, from (14),(15) and (16) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\partial_3 v\|_{L^2}^2+\alpha'\|v_h\|_{L^2(\partial\mathbb{H})}^2\big)+\nu\|\nabla_h\partial_3 v\|_{L^2}^2+\alpha'\nu\|\nabla_h v_h\|_{L^2(\partial\mathbb{H})}^2+\varepsilon\|\partial_3^2 v\|_{L^2}^2$$
$$=-\alpha'\int_{\partial\mathbb{H}}\partial_3 v_3 q\,ds. \quad (17)$$

To estimate the pressure term, observe that q verifies the system of equations

$$\begin{aligned}&\triangle q=0 \quad \text{in } \mathbb{H}\\ &\partial_3 q=\varepsilon\partial_3^2 v_3=\varepsilon\alpha'\partial_3 v_3 \quad \text{on } \partial\mathbb{H}.\end{aligned}$$

On one hand

$$-\alpha'\int_{\partial\mathbb{H}}\partial_3 v_3\, q\,ds=-\frac{1}{\varepsilon}\int_{\partial\mathbb{H}}q\,\partial_3 q\,ds=\frac{1}{\varepsilon}\|\nabla q\|_{L^2}^2$$

and on the other hand

$$\|\nabla q\|_{L^2}^2=-\alpha'\varepsilon\int_{\partial\mathbb{H}}\partial_3 v_3\, q\,ds=\alpha'\varepsilon\int_{\mathbb{H}}\partial_3 v\cdot\nabla q\,dx\le\alpha'\varepsilon\|\nabla q\|_{L^2}\|\partial_3 v\|_{L^2}.$$

We deduce first that $\|\nabla q\|_{L^2}\le\alpha'\varepsilon\|\partial_3 v\|_{L^2}$ and secondly that

$$-\alpha'\int_{\partial\mathbb{H}}\partial_3 v_3 q\,ds\le{\alpha'}^2\varepsilon\|\partial_3 v\|_{L^2}^2.$$

Plugging this into (17) and integrating in time yields

$$\|\partial_3 v(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla_h\partial_3 v(\tau)\|_{L^2}^2 d\tau\le\|\partial_3 u_0\|_{L^2}^2+\alpha'\|u_h(0)\|_{L^2(\partial\mathbb{H})}^2$$
$$+2{\alpha'}^2\varepsilon\int_0^t\|\partial_3 v(\tau)\|_{L^2}^2 d\tau. \quad (18)$$

On the other hand, the standard L^2 energy estimate for v (obtained by multiplying the equation for v by v) implies that

$$\|v(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla_h v(\tau)\|_{L^2}^2\,d\tau+2\varepsilon\int_0^t\|\partial_3 v(\tau)\|_{L^2}^2\,d\tau\le\|u_0\|_{L^2}^2. \quad (19)$$

Using first this relation to bound the last term in (18) and adding the resulting inequality to (19) yields equation (12). This completes the proof. □

Remark 10 *It is not difficult to see by a density argument that the hypothesis $\partial_3 u_0\in L^2$ is in fact not necessary to prove the existence of T_η.*

II.2 *A priori* estimates

We will now deduce some *a priori* estimate for the velocity. For the sake of conciseness, we write u for u^ε. In the following, C, C_1, C_2, C_3 will denote some constants independent of ε that may change from one relation to another. Multiplying the equation of u by u, integrating in space and performing some straightforward integrations by parts we get the following L^2 energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \nu\|\nabla_h u\|_{L^2}^2 + \varepsilon\|\partial_3 u\|_{L^2}^2 + \varepsilon\alpha'\|u_h\|_{L^2(\partial\mathbb{H})}^2 = 0. \tag{20}$$

Integrating in time implies that

$$\|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla_h u(\tau)\|_{L^2}^2\, d\tau + 2\varepsilon\int_0^t \|\partial_3 u(\tau)\|_{L^2}^2\, d\tau \le \|u_0\|_{L^2}^2. \tag{21}$$

Now we multiply (3) by $-\partial_3^2 u$ and integrate in space to obtain

$$-\int_{\mathbb{H}} \partial_t u\cdot\partial_3^2 u\, dx + \nu\int_{\mathbb{H}} \triangle_h u\cdot\partial_3^2 u\, dx + \varepsilon\int_{\mathbb{H}} |\partial_3^2 u|^2\, dx = \int_{\mathbb{H}} u\cdot\nabla u\cdot\partial_3^2 u\, dx + \int_{\mathbb{H}} \partial_3^2 u\cdot\nabla p\, dx. \tag{22}$$

The first two terms can be treated as in the proof of Proposition 9:

$$-\int_{\mathbb{H}} \partial_t u\cdot\partial_3^2 u\, dx = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\partial_3 u\|_{L^2}^2 + \alpha'\|u_h\|_{L^2(\partial\mathbb{H})}^2\big), \tag{23}$$

and

$$\int_{\mathbb{H}} \triangle_h u\cdot\partial_3^2 u\, dx = \|\nabla_h\partial_3 u\|_{L^2}^2 + \alpha'\|\nabla_h u_h\|_{L^2(\partial\mathbb{H})}^2. \tag{24}$$

We now go to the nonlinear term. We integrate by parts and write

$$\int_{\mathbb{H}} u\cdot\nabla u\cdot\partial_3^2 u\, dx = -\int_{\mathbb{H}} \partial_3(u\cdot\nabla u)\cdot\partial_3 u\, dx - \int_{\partial\mathbb{H}} u\cdot\nabla u\cdot\partial_3 u\, ds. \tag{25}$$

Since u is divergence free and tangent to the boundary, one has that

$$\int_{\mathbb{H}} u\cdot\nabla\partial_3 u\cdot\partial_3 u\, dx = \frac{1}{2}\int_{\mathbb{H}} u\cdot\nabla\big(|\partial_3 u|^2\big)\, dx = -\frac{1}{2}\int_{\mathbb{H}} \operatorname{div} u|\partial_3 u|^2\, dx = 0.$$

Next we observe that

$$\begin{aligned}-\int_{\mathbb{H}} \partial_3(u\cdot\nabla u)\cdot\partial_3 u\, dx &= -\int_{\mathbb{H}} \partial_3 u\cdot\nabla u\cdot\partial_3 u\, dx\\ &= -\int_{\mathbb{H}} \partial_3 u_h\cdot\nabla_h u\cdot\partial_3 u\, dx - \int_{\mathbb{H}} \partial_3 u_3|\partial_3 u|^2\, dx\\ &= -\int_{\mathbb{H}} \partial_3 u_h\cdot\nabla_h u\cdot\partial_3 u\, dx + \int_{\mathbb{H}} \operatorname{div}_h u_h\, |\partial_3 u|^2\, dx.\end{aligned}$$

Since the right hand side is a sum of terms of the form $\displaystyle\int_{\mathbb{H}} \partial_3 u\, \partial_3 u\, \nabla_h u\ dx$, we deduce in view of Lemma 8 that

$$-\int_{\mathbb{H}} \partial_3(u \cdot \nabla u) \cdot \partial_3 u\, dx \leq C \|\partial_3 u\|_{L^2} \|\nabla_h D_3 u\|_{L^2}^{\frac{3}{2}} \|\nabla_h u\|_{L^2}^{\frac{1}{2}}. \tag{26}$$

We now go to the boundary term in (25). Since $u_3 = 0$ on $\partial\mathbb{H}$, we see that $u \cdot \nabla u_3 = 0$ on the boundary. Therefore

$$-\int_{\partial\mathbb{H}} u\cdot\nabla u\cdot\partial_3 u\, ds = -\int_{\partial\mathbb{H}} u\cdot\nabla u_h\cdot\partial_3 u_h\, ds = -\alpha' \int_{\partial\mathbb{H}} u\cdot\nabla u_h\cdot u_h\, ds = \frac{\alpha'}{2} \int_{\partial\mathbb{H}} \mathrm{div}_h u_h\ |u_h|^2\, ds,$$

where we used the Navier boundary conditions. We now return to an integral on $\mathbb{H}$

$$\begin{aligned}
-\int_{\partial\mathbb{H}} u \cdot \nabla u \cdot \partial_3 u\, ds &= -\frac{\alpha'}{2} \int_{\mathbb{H}} \partial_3(\mathrm{div}_h u_h\ |u_h|^2)\, dx \\
&= -\frac{\alpha'}{2} \int_{\mathbb{H}} \partial_3 \mathrm{div}_h u_h\ |u_h|^2\, dx - \alpha' \int_{\mathbb{H}} \mathrm{div}_h u_h\ u_h \cdot \partial_3 u_h\, dx \\
&= \alpha' \int_{\mathbb{H}} \partial_3 u_h \cdot \nabla_h u_h \cdot u_h\, dx - \alpha' \int_{\mathbb{H}} \mathrm{div}_h u_h\ u_h \cdot \partial_3 u_h\, dx \\
&\leq C \|u\|_{L^2}^{\frac{1}{2}} \|\nabla_h u\|_{L^2} \|\partial_3 u_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h D_3 u\|_{L^2},
\end{aligned} \tag{27}$$

where we used Lemma 8.

It remains to estimate the pressure term. Taking the restriction to the boundary of the third component of the equation for u (3), we first get that

$$\partial_3 p = \varepsilon \partial_3^2 u_3 = -\varepsilon(\partial_1\partial_3 u_1 + \partial_2 \partial_3 u_2) = -\varepsilon\alpha'(\partial_1 u_1 + \partial_2 u_2) = \varepsilon\alpha' \partial_3 u_3 \quad \text{on } \partial\mathbb{H}.$$

Next, taking the divergence of (3), we deduce that

$$\triangle p = -\sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i \quad \text{in } \mathbb{H}.$$

We observe that the pressure can be written as $p = p_1 + p_2$, where p_1 verifies the Neumann problem

$$\begin{aligned}
&\triangle p_1 = -\sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i \quad \text{in } \mathbb{H}, \\
&\partial_3 p_1 = 0 \quad \text{on } \partial\mathbb{H}.
\end{aligned}$$

and p_2 verifies

$$\triangle p_2 = 0 \quad \text{in } \mathbb{H}, \tag{28}$$

$$\partial_3 p_2 = \varepsilon\alpha' \partial_3 u_3 \quad \text{on } \partial\mathbb{H}. \tag{29}$$

Let us denote by S, respectively A, the extension operator by even, respectively odd, reflection with respect to the plane $\{x_3 = 0\}$, *i.e.* for $f : \mathbb{H} \to \mathbb{R}$, we define $S(f), A(f) : \mathbb{R}^3 \to \mathbb{R}$ by

$$S(f)(x) = \begin{cases} f(x), & x_3 \geq 0 \\ f(x_h, -x_3), & x_3 < 0 \end{cases} \quad \text{and} \quad A(f)(x) = \begin{cases} f(x), & x_3 \geq 0 \\ -f(x_h, -x_3), & x_3 < 0. \end{cases}$$

The estimates for the Neumann problems for the Laplacian are well-known. However, we require precise estimates, namely we want to use homogeneous Sobolev norms. For this reason we briefly show how to obtain such estimates. We introduce the following extensions of p_1 and u defined by $\widetilde{p}_1 = S(p_1)$ and $\widetilde{u} = \big(S(u_1), S(u_2), A(u_3)\big)$. It is a well-known trick to observe that $\widetilde{p}_1$ verifies

$$\triangle \widetilde{p}_1 = -S\Big(\sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i\Big) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3).$$

Since $\widetilde{u}$ is continuous across the boundary one has that the first order derivatives of $\widetilde{u}$ contain no Dirac mass concentrated on the interface $\{x_3 = 0\}$ and are computed by differentiating piecewise $\widetilde{u}$ for $x_3 < 0$ and $x_3 > 0$. This implies that $\operatorname{div} \widetilde{u} = 0$ in the sense of distributions of $\mathbb{R}^3$. Next we observe that the function $\partial_i \widetilde{u}_j \partial_j \widetilde{u}_i$ is even with respect to x_3 for all $i, j \in \{1, 2, 3\}$, so that

$$S\Big(\sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i\Big) = \sum_{i,j=1}^{3} \partial_i \widetilde{u}_j \partial_j \widetilde{u}_i.$$

Therefore, using also that $\operatorname{div} \widetilde{u} = 0$, we deduce that

$$\triangle \widetilde{p}_1 = -\sum_{i,j=1}^{3} \partial_i \widetilde{u}_j \partial_j \widetilde{u}_i = -\sum_{i,j=1}^{3} \partial_i \partial_j (\widetilde{u}_i \widetilde{u}_j),$$

that is,

$$\widetilde{p}_1 = -\sum_{i,j=1}^{3} \partial_i \partial_j \triangle^{-1} (\widetilde{u}_i \widetilde{u}_j).$$

Since the operator $\partial_i \partial_j \triangle^{-1}$ is bounded in $L^2(\mathbb{R}^3)$, we conclude that

$$\|p_1\|_{L^2} \leq \|\widetilde{p}_1\|_{L^2(\mathbb{R}^3)} \leq C \|\widetilde{u}\|^2_{L^4(\mathbb{R}^3)} = C\sqrt{2} \|u\|^2_{L^4}. \tag{30}$$

We proceed similarly to deduce

$$\begin{aligned} \|\partial_3 p_1\|_{L^2} &\leq \|\partial_3 \widetilde{p}_1\|_{L^2(\mathbb{R}^3)} \\ &\leq C \|\partial_3 (\widetilde{u} \otimes \widetilde{u})\|_{L^2(\mathbb{R}^3)} \\ &\leq C \|\partial_3 u\|_{L^{4,2}} \|u\|_{L^{4,\infty}} \\ &\leq C \|\partial_3 u\|^{\frac{1}{2}}_{L^2} \|\nabla_h \partial_3 u\|^{\frac{1}{2}}_{L^2} \|D_3 u\|^{\frac{1}{2}}_{L^2} \|\nabla_h D_3 u\|^{\frac{1}{2}}_{L^2}, \end{aligned} \tag{31}$$

where we used Corollary 7.

Next, we estimate p_2 as we did with q at the end of the proof of Proposition 9. Hence, we conclude that

$$-\alpha' \int_{\partial\mathbb{H}} p_2 \partial_3 u_3 \, ds \leq {\alpha'}^2 \varepsilon \|\partial_3 u\|_{L^2}^2. \tag{32}$$

We can now estimate the pressure term from (22). Integrating by parts and using that $\operatorname{div} u = 0$, one has that

$$\begin{aligned}
\int_{\mathbb{H}} \partial_3^2 u \cdot \nabla p \, dx &= -\int_{\partial\mathbb{H}} p \, \partial_3^2 u_3 \, ds = -\alpha' \int_{\partial\mathbb{H}} p \, \partial_3 u_3 \, ds \\
&= \alpha' \int_{\mathbb{H}} \partial_3 (p_1 \, \partial_3 u_3) \, dx - \alpha' \int_{\partial\mathbb{H}} p_2 \, \partial_3 u_3 \, ds \\
&= -\alpha' \int_{\mathbb{H}} \partial_3 p_1 \operatorname{div}_h u_h \; dx - \alpha' \int_{\mathbb{H}} p_1 \, \partial_3 \operatorname{div}_h u_h \; dx - \alpha' \int_{\partial\mathbb{H}} p_2 \, \partial_3 u_3 \, ds.
\end{aligned} \tag{33}$$

In view of (30), (31) and of Corollary 7 we infer that

$$-\alpha' \int_{\mathbb{H}} p_1 \, \partial_3 \operatorname{div}_h u_h \; dx \leq \alpha' \|p_1\|_{L^2} \|\partial_3 \operatorname{div}_h u_h \; \|_{L^2} \leq C \|u\|_{L^2} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h D_3 u\|_{L^2}^{\frac{3}{2}} \tag{34}$$

and

$$-\alpha' \int_{\mathbb{H}} \partial_3 p_1 \operatorname{div}_h u_h \; dx \leq C \|\nabla_h u_h\|_{L^2} \|D_3 u\|_{L^2} \|\nabla_h D_3 u\|_{L^2}. \tag{35}$$

We deduce from (32), (33), (34) and (35) the following bound for the pressure term

$$\begin{aligned}
&\int_{\mathbb{H}} \partial_3^2 u \cdot \nabla p \, dx \\
&\quad \leq C \|u\|_{L^2} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h D_3 u\|_{L^2}^{\frac{3}{2}} + C \|\nabla_h u_h\|_{L^2} \|D_3 u\|_{L^2} \|\nabla_h D_3 u\|_{L^2} + \varepsilon {\alpha'}^2 \|\partial_3 u\|_{L^2}^2.
\end{aligned} \tag{36}$$

Finally, by adding (20) to (22) and using relations (23), (24), (25), (26), (27) and (36), we obtain

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big(\|D_3 u\|_{L^2}^2 + \alpha' \|u_h\|_{L^2(\partial\mathbb{H})}^2 \big) + \nu \|\nabla_h D_3 u\|_{L^2}^2 + \alpha' \nu \|\nabla_h u_h\|_{L^2(\partial\mathbb{H})}^2 \\
+ \varepsilon \|\partial_3 u\|_{L^2}^2 + \varepsilon \alpha' \|u_h\|_{L^2(\partial\mathbb{H})}^2 + \varepsilon \|\partial_3^2 u\|_{L^2}^2 \\
\leq C \|D_3 u\|_{L^2} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h D_3 u\|_{L^2}^{\frac{3}{2}} + \varepsilon {\alpha'}^2 \|\partial_3 u\|_{L^2}^2,
\end{aligned} \tag{37}$$

where the constant C is independent of ε and ν.

II.2.1 Global estimates for small data

We now prove that there exists a constant K such that if $\|u_0\|_{L^2}^{\frac{1}{2}}\|D_3u_0\|_{L^2}^{\frac{1}{2}} \leq K\nu$, then one has *a priori* estimates that show that, for $\varepsilon \geq 0$ the quantity D_3u is bounded in time with values in L^2 and $\nabla_h D_3 u$ is square integrable in time and space. For the sake of conciseness, we introduce the notation $G^2(t) = \|D_3u(t)\|_{L^2}^2 + \alpha'\|u_h(t)\|_{L^2(\partial\mathbb{H})}^2$. From standard trace theorems, we know that $\|u_h\|_{L^2(\partial\mathbb{H})} \leq C\|D_3u\|_{L^2}$, so the norm G is equivalent to $\|D_3u\|_{L^2}$. Using that

$$C\|D_3u\|_{L^2}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h D_3u\|_{L^2}^{\frac{3}{2}} \leq \frac{\nu}{2}\|\nabla_h D_3u\|_{L^2}^2 + \frac{C_1}{2\nu^3}\|D_3u\|_{L^2}^4\|\nabla_h u\|_{L^2}^2$$

in (37), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}G^2 + \nu\|\nabla_h D_3u\|_{L^2}^2 \leq \frac{C_1}{\nu^3}G^4\|\nabla_h u\|_{L^2}^2 + 2\varepsilon{\alpha'}^2\|\partial_3 u\|_{L^2}^2. \tag{38}$$

Gronwall's lemma together with relation (21) implies that

$$\begin{aligned}
G^2(t) &+ \nu\int_0^t \|\nabla_h D_3u(\tau)\|_{L^2}^2\,d\tau \\
&\leq \Big(G^2(0) + 2\varepsilon{\alpha'}^2\int_0^t \|\partial_3 u(\tau)\|_{L^2}^2\,d\tau\Big)\exp\Big(\frac{C_1}{\nu^3}\int_0^t G^2(\tau)\|\nabla_h u(\tau)\|_{L^2}^2\,d\tau\Big) \\
&\leq (1+{\alpha'}^2)G^2(0)\exp\Big(\frac{C_1}{2\nu^4}\|u_0\|_{L^2}^2\sup_{[0,t]}G^2\Big).
\end{aligned}$$

We argue by contradiction assuming that

$$T = \sup\{t \geq 0;\ G(\tau) \leq 2\sqrt{1+{\alpha'}^2}G(0)\ \forall \tau \in [0,t]\}$$

is finite. Then from the above relation we get that

$$G^2(T) \leq (1+{\alpha'}^2)G^2(0)\exp\Big(\frac{2C_1(1+{\alpha'}^2)}{\nu^4}\|u_0\|_{L^2}^2 G^2(0)\Big).$$

Therefore, if we choose K_1 small enough such that $\exp(2C_1(1+{\alpha'}^2)K_1^4) < 4$ and if we assume that $\|u_0\|_{L^2}^{\frac{1}{2}}G^{\frac{1}{2}} \leq K_1\nu$, then we deduce that $G(T) < 2\sqrt{1+{\alpha'}^2}G(0)$ and this contradicts the maximality of T. We conclude that, for such a choice of K_1, one has that

$$G^2(t) + \nu\int_0^t \|\nabla_h D_3u(\tau)\|_{L^2}^2\,d\tau \leq 4(1+{\alpha'}^2)G^2(0) \qquad \forall t \geq 0.$$

Since G is equivalent to $\|D_3u\|_{L^2}$, these are the desired *a priori* estimates under the desired smallness assumption.

II.2.2 Local estimates for large data

To obtain local estimates for large data we proceed as in the previous subsection and define

$$T = \sup\{t \geq 0;\ G(\tau) \leq 2\sqrt{1+\alpha'^2}G(0)\ \forall \tau \in [0,t]\}. \tag{39}$$

We prove that, for fixed initial data u_0, there exists a time T_0 independent of ε such that $T \geq T_0$. The local time existence T_0 will be constructed in (43). We assume in the sequel that $t \in [0,T]$.

Let us go back to (38) and use that $G \leq 2\sqrt{1+\alpha'^2}G(0)$ to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}G^2 \leq 16\frac{C_1}{\nu^3}(1+\alpha'^2)^2 G^4(0)\|\nabla_h u\|_{L^2}^2 + 2\varepsilon\alpha'^2\|\partial_3 u\|_{L^2}^2.$$

We now integrate in time and use estimate (21) to deduce that

$$G^2(t) \leq (1+\alpha'^2)G^2(0) + 16\frac{C_1}{\nu^3}(1+\alpha'^2)^2 G^4(0)\int_0^t \|\nabla_h u(\tau)\|_{L^2}^2\,d\tau \tag{40}$$

for all $t \in [0,T]$. We now estimate $\displaystyle\int_0^t \|\nabla_h u(\tau)\|_{L^2}^2\,d\tau$ and prove that it can be made as small as we want independently of ε. To do that, we compare it with the solution of the Stokes equation and use Proposition 9.

Let v be the solution of the anisotropic Stokes equation

$$\partial_t v - \nu(\partial_1^2+\partial_2^2)v - \varepsilon\partial_3^2 v = -\nabla q, \quad \operatorname{div} v = 0, \quad v\big|_{t=0} = u_0,$$

supplemented with the Navier boundary conditions. Then $w = u - v$ satisfies the equation

$$\partial_t w - \nu(\partial_1^2+\partial_2^2)w - \varepsilon\partial_3^2 w + (v+w)\cdot\nabla(v+w) = -\nabla(p-q), \quad \operatorname{div} w = 0, \quad w\big|_{t=0} = 0,$$

and also the Navier boundary conditions. We multiply this equation by w and integrate by parts to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \nu\|\nabla_h w\|_{L^2}^2 + \varepsilon\|\partial_3 w\|_{L^2}^2 + \varepsilon\alpha'\|w_h\|_{L^2(\partial\mathbb{H})}^2 = -\int_{\mathbb{H}} v\cdot\nabla v\cdot w\,dx - \int_{\mathbb{H}} w\cdot\nabla v\cdot w\,dx.$$

We can bound the above nonlinear terms in the following manner: we split the first integral in two parts

$$-\int_{\mathbb{H}} v\cdot\nabla v\cdot w\,dx = -\int_{\mathbb{H}} v_3\partial_3 v\cdot w\,dx - \int_{\mathbb{H}} v_h\cdot\nabla_h v\cdot w\,dx.$$

Using the anisotropic Hölder inequality and Corollary 7 we get

$$-\int_{\mathbb{H}} v_3\partial_3 v\cdot w\,dx \leq \|v\|_{L^{4,\infty}}\|\partial_3 v\|_{L^{4,2}}\|w\|_{L^2} \leq C\|D_3 v\|_{L^2}\|\nabla_h D_3 v\|_{L^2}\|w\|_{L^2},$$

and

$$
\begin{aligned}
-\int_{\mathbb{H}} v_h \cdot \nabla_h v \cdot w \, dx &\leq \|v\|_{L^{4,\infty}} \|\nabla_h v\|_{L^2} \|w\|_{L^{4,2}} \\
&\leq C \|D_3 v\|_{L^2}^{\frac{1}{2}} \|\nabla_h D_3 v\|_{L^2}^{\frac{1}{2}} \|\nabla_h v\|_{L^2} \|w\|_{L^2}^{\frac{1}{2}} \|\nabla_h w\|_{L^2}^{\frac{1}{2}} \\
&\leq \frac{\nu}{6} \|\nabla_h w\|_{L^2}^2 + \frac{C}{\nu^{\frac{1}{3}}} \|D_3 v\|_{L^2}^{\frac{2}{3}} \|\nabla_h D_3 v\|_{L^2}^{\frac{2}{3}} \|\nabla_h v\|_{L^2}^{\frac{4}{3}} \|w\|_{L^2}^{\frac{2}{3}}.
\end{aligned}
$$

Similarly for the second integral,

$$
\begin{aligned}
-\int_{\mathbb{H}} w_3 \partial_3 v \cdot w \, dx &\leq C \|D_3 v\|_{L^{4,2}} \|w\|_{L^{2,\infty}} \|w\|_{L^{4,2}} \\
&\leq C \|D_3 v\|_{L^2}^{\frac{1}{2}} \|\nabla_h D_3 v\|_{L^2}^{\frac{1}{2}} \|D_3 w\|_{L^2} \|w\|_{L^2}^{\frac{1}{2}} \|\nabla_h w\|_{L^2}^{\frac{1}{2}} \\
&\leq \frac{\nu}{6} \|\nabla_h w\|_{L^2}^2 + \frac{C}{\nu^{\frac{1}{3}}} \|D_3 v\|_{L^2}^{\frac{2}{3}} \|\nabla_h D_3 v\|_{L^2}^{\frac{2}{3}} \|D_3 w\|_{L^2}^{\frac{4}{3}} \|w\|_{L^2}^{\frac{2}{3}},
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_{\mathbb{H}} w_h \cdot \nabla_h v \cdot w \, dx &\leq C \|\nabla_h v\|_{L^{2,\infty}} \|w\|_{L^{4,2}}^2 \\
&\leq C \|\nabla_h D_3 v\|_{L^2} \|w\|_{L^2} \|\nabla_h w\|_{L^2} \\
&\leq \frac{\nu}{6} \|\nabla_h w\|_{L^2}^2 + \frac{C}{\nu} \|\nabla_h D_3 v\|_{L^2}^2 \|w\|_{L^2}^2.
\end{aligned}
$$

We deduce from the above relations that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|w\|_{L^2}^2 + \nu \|\nabla_h w\|_{L^2}^2 \leq{}& C_2 \|D_3 v\|_{L^2} \|\nabla_h D_3 v\|_{L^2} \|w\|_{L^2} + \frac{C_2}{\nu} \|\nabla_h D_3 v\|_{L^2}^2 \|w\|_{L^2}^2 \\
&+ \frac{C_2}{\nu^{\frac{1}{3}}} \|D_3 v\|_{L^2}^{\frac{2}{3}} \|\nabla_h D_3 v\|_{L^2}^{\frac{2}{3}} \|\nabla_h v\|_{L^2}^{\frac{4}{3}} \|w\|_{L^2}^{\frac{2}{3}} \\
&+ \frac{C_2}{\nu^{\frac{1}{3}}} \|D_3 v\|_{L^2}^{\frac{2}{3}} \|\nabla_h D_3 v\|_{L^2}^{\frac{2}{3}} \|D_3 w\|_{L^2}^{\frac{4}{3}} \|w\|_{L^2}^{\frac{2}{3}},
\end{aligned}
\tag{41}
$$

for some constant C_2 independent of ε. Next, since $t \in [0,T]$ one has that $\|D_3 u\|_{L^2} \leq 2\sqrt{1+\alpha'^2} G(0)$ so

$$
\|D_3 w\|_{L^2}^2 \leq 8(1+\alpha'^2) G^2(0) + 2\|D_3 v\|_{L^2}^2.
$$

Set now

$$
K_0^2 = 2\max\Big(1, \frac{1}{2\nu}\Big) \left[\|\partial_3 u_0\|_{L^2}^2 + (1+\alpha'^2)\|u_0\|_{L^2}^2 + \alpha' \|u_h(0)\|_{L^2(\partial\mathbb{H})}^2 \right] + 8(1+\alpha'^2) G^2(0).
$$

In view of (12), we have that

$$
\|D_3 v(t)\|_{L^2} \leq K_0, \quad \|D_3 w(t)\|_{L^2} \leq K_0, \quad \int_0^t \|\nabla_h D_3 v(\tau)\|_{L^2}^2 d\tau \leq K_0^2,
$$

for all $t \in [0,T]$. We therefore deduce from (41) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \nu\|\nabla_h w\|_{L^2}^2 \leq C_3 K_0^2 \|\nabla_h D_3 v\|_{L^2} + C_3\|\nabla_h D_3 v\|_{L^2}^2 \|w\|_{L^2}^2 \\ + C_3 K_0^{\frac{4}{3}} \|\nabla_h D_3 v\|_{L^2}^{\frac{2}{3}} \|\nabla_h v\|_{L^2}^{\frac{4}{3}} + C_3 K_0^{\frac{8}{3}} \|\nabla_h D_3 v\|_{L^2}^{\frac{2}{3}}.$$

Gronwall's lemma now implies that

$$\nu \int_0^t \|\nabla_h w(\tau)\|_{L^2}^2 d\tau \leq C_3 \exp\Big(C_3 \int_0^t \|\nabla_h D_3 v(\tau)\|_{L^2}^2 d\tau\Big) \Big[K_0^2 \int_0^t \|\nabla_h D_3 v(\tau)\|_{L^2} d\tau \\ + K_0^{\frac{4}{3}} \int_0^t \|\nabla_h D_3 v(\tau)\|_{L^2}^{\frac{2}{3}} \|\nabla_h v(\tau)\|_{L^2}^{\frac{4}{3}} d\tau + K_0^{\frac{8}{3}} \int_0^t \|\nabla_h D_3 v(\tau)\|_{L^2}^{\frac{2}{3}} d\tau\Big].$$

By the Hölder inequality, we have that

$$\int_0^t \|\nabla_h D_3 v(\tau)\|_{L^2} d\tau \leq K_0 t^{\frac{1}{2}}, \qquad \int_0^t \|\nabla_h D_3 v(\tau)\|_{L^2}^{\frac{2}{3}} d\tau \leq K_0^{\frac{2}{3}} t^{\frac{2}{3}},$$

and that

$$\int_0^t \|\nabla_h D_3 v(\tau)\|_{L^2}^{\frac{2}{3}} \|\nabla_h v(\tau)\|_{L^2}^{\frac{4}{3}} d\tau \leq K_0^{\frac{2}{3}} \Big(\int_0^t \|\nabla_h v(\tau)\|_{L^2}^2 d\tau\Big)^{\frac{2}{3}}.$$

We finally get the following bound

$$\nu \int_0^t \|\nabla_h w(\tau)\|_{L^2}^2 d\tau \leq C_3 \exp\Big(C_3 K_0^2\Big) \Big[K_0^3 t^{\frac{1}{2}} + K_0^{\frac{10}{3}} t^{\frac{2}{3}} + K_0^2 \Big(\int_0^t \|\nabla_h v(\tau)\|_{L^2}^2 d\tau\Big)^{\frac{2}{3}}\Big].$$

Consequently,

$$\begin{aligned} \int_0^t \|\nabla_h u(\tau)\|_{L^2}^2 d\tau &\leq 2\int_0^t \|\nabla_h v(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \|\nabla_h w(\tau)\|_{L^2}^2 d\tau \\ &\leq 2\int_0^t \|\nabla_h v(\tau)\|_{L^2}^2 d\tau \\ &\quad + 2\frac{C_3}{\nu} \exp\Big(C_3 K_0^2\Big) \Big[K_0^3 t^{\frac{1}{2}} + K_0^{\frac{10}{3}} t^{\frac{2}{3}} + K_0^2 \Big(\int_0^t \|\nabla_h v(\tau)\|_{L^2}^2 d\tau\Big)^{\frac{2}{3}}\Big] \\ &\equiv A(t). \end{aligned} \tag{42}$$

Proposition 9 tells us that $\lim\limits_{t\to 0} \displaystyle\int_0^t \|\nabla_h v(\tau)\|_{L^2}^2 d\tau = 0$ uniformly with respect to ε. From the formula for $A(t)$, we see that

$$\lim_{t\to 0} A(t) = 0 \quad \text{uniformly with respect to } \varepsilon.$$

We also observe that even though the strong solution may exist only locally in time, the expression $A(t)$ is globally defined since v is globally defined. We conclude that there exists a time T_0 independent of ε such that

$$16\frac{C_1}{\nu^3}(1+\alpha'^2) G^2(0) A(T_0) < 1. \tag{43}$$

We prove now that $T \geq T_0$. Assume by contradiction that $T < T_0$. From (40), (42) and (43) we get

$$\begin{aligned} G^2(T) &\leq (1+\alpha'^2)G^2(0) + 16\frac{C_1}{\nu^3}(1+\alpha'^2)^2 G^4(0) \int_0^{T_0} \|\nabla_h u(\tau)\|_{L^2}^2 \, d\tau \\ &\leq (1+\alpha'^2)G^2(0) + 16\frac{C_1}{\nu^3}(1+\alpha'^2)^2 G^4(0) A(T_0) \\ &< 2(1+\alpha'^2)G^2(0). \end{aligned}$$

This contradicts the maximality of T given in (39) and we deduce that we must necessarily have that $T \geq T_0$.

Therefore we have the desired *a priori* local estimates.

II.3 Passing to the limit

Once the *a priori* estimates completed, the existence of solutions to system (3)-(4) with non-zero vertical viscosity follows with standard arguments, see for instance [11, 5, 3].

Now, thanks to the *a priori* estimates proved in Section II.2, we have at our disposal a sequence of solutions u^ε of system (3)-(4) such that $D_3 u^\varepsilon$ is bounded in $L^\infty(0,T;L^2)$ and $\nabla_h D_3 u^\varepsilon$ is bounded in $L^2([0,T]\times\mathbb{H})$ independently of ε. In particular, u^ε is bounded in $L^2_{loc}\big([0,T);H^1\big)$. Here, T may be finite or not, depending if we consider the case of large or small data. We will show that, under this assumption, u^ε converges to a solution of the limit system and then Theorem 2 will be proved.

Let $\mathbb{P}$ denote the Leray projector. From equation (3) written under the form

$$\partial_t u^\varepsilon = \nu\mathbb{P}(\partial_1^2+\partial_2^2)u + \varepsilon\mathbb{P}\partial_3^2 u - \mathbb{P}(u\cdot\nabla u)$$

we obtain that the sequence $\partial_t u^\varepsilon$ is bounded in $L^2(0,T;H^{-2}_{loc})$ independently of ε. We deduce that the sequence u^ε is bounded and equicontinuous in time with values in H^{-2}_{loc}. By the Arzela-Ascoli theorem, it is precompact in $L^\infty_{loc}\big([0,T);H^{-3}_{loc}\big)$. Since u^ε is bounded in $L^2_{loc}\big([0,T);H^1\big)$, standard interpolation arguments imply that it is possible to extract a subsequence (which we still denote by u^ε) converging strongly to some u in $L^2_{loc}\big((0,T)\times\mathbb{H}\big)$. Without loss of generality, this subsequence also converges weakly in $L^2_{loc}\big([0,T);H^1\big)$ to u. With this information, it is trivial to pass to the limit in the first equation of (3) to obtain that the limit velocity u verifies the first equation of (5) in the sense of distributions $\mathcal{D}'\big((0,T)\times\mathbb{H}\big)$. We observe that since u^ε has a limit in the sense of distributions, so does $\partial_3^2 u^\varepsilon$; hence $\varepsilon\partial_3^2 u^\varepsilon \to 0$ in the sense of distributions $\mathcal{D}'\big((0,T)\times\mathbb{H}\big)$. The second equation of (5) is trivially verified as the divergence free condition commutes with the limit in the sense of distributions. Next, since u^ε converges uniformly in time with values in H^{-3}_{loc} we obtain on one hand that $u \in C^0\big([0,T);H^{-3}_{loc}\big)$, and on the other hand that the initial data of the limit velocity is the limit of $u^\varepsilon(0)$. Thus, the third condition in (5) is also verified. Finally, the boundary condition in (5) follows from the weak convergence of u_3^ε to u_3 in $L^2_{loc}\big([0,T);H^1\big)$.

II.4 A final remark

A legitimate question is if there are higher order estimates valid independently of ε up to the boundary. We claim that it is impossible to obtain *a priori* estimates independent of ε in spaces of functions for which the Navier boundary conditions make sense. For example, H^2 estimates independent of ε are not possible to obtain. Indeed, such a priori estimates would imply that the limit velocity belongs to those spaces and therefore verifies the Navier boundary conditions. This is a contradiction since, as we show below, in general the limit equation (5) does not preserve the Navier boundary conditions. More precisely, we show that there exists a $C_0^\infty(\mathbb{H})$ divergence free vector field u_0 and a sequence of times $t_n \to 0$ such that if u denotes the solution of the limit system (5), then $u(t_n)$ does not verify the Navier boundary conditions. We observe next that the (local) strong solution u is very smooth. Indeed, the extension of the initial data by reflection as discussed in the Introduction still belongs to $C_0^\infty(\mathbb{R}^3)$. Furthermore, in the system (5) posed in $\mathbb{R}^3$ one can ignore the Laplacian term when making energy estimates, so we see that the same theory of existence of solutions as for the Euler equation can be deduced for equation (5). Therefore, the local solution u belongs to $C^\infty\big([0,T); \bigcap_{s\in\mathbb{R}} H^s(\overline{\mathbb{H}})\big)$. Note that if such a solution verifies the Navier boundary conditions almost everywhere in time, by time continuity it actually verifies it everywhere.

Assume by contradiction that for all $C_0^\infty(\mathbb{H})$ divergence free vector fields u_0, there exists T_0 such that $u(t)$ verifies the Navier boundary conditions for all $t \in [0, T_0]$.

The pressure p verifies the following Neumann problem for the Laplacian:

$$\begin{aligned} &\triangle p = -\operatorname{div}(u\cdot\nabla u) \quad \text{in } \mathbb{H}\\ &\partial_3 p = 0 \quad \text{on } \partial\mathbb{H}. \end{aligned}$$

Using the well-known formula for the Green's function associated to the Neumann problem of the Laplacian in the half-space, we deduce that

$$p(x) = \frac{1}{4\pi}\int_{\mathbb{H}}\Big(\frac{1}{|x-y|} + \frac{1}{|x-\overline{y}|}\Big)\operatorname{div}(u\cdot\nabla u)(y)\,dy, \tag{44}$$

where $\overline{y} = (y_1, y_2, -y_3)$.

We now differentiate with respect to x_3 the equation for u_h:

$$\partial_t\partial_3 u_h - \nu(\partial_1^2+\partial_2^2)\partial_3 u_h + \partial_3(u\cdot\nabla u_h) = -\nabla_h\partial_3 p.$$

Taking the restriction to the boundary and using that $\partial_3 u_h = \alpha' u_h$ on the boundary, we deduce that

$$\alpha'\partial_t u_h - \alpha'\nu(\partial_1^2+\partial_2^2)u_h + \partial_3(u\cdot\nabla u_h) = -\nabla_h\partial_3 p = 0 \qquad \text{on } \partial\mathbb{H}.$$

On the other hand, taking the restriction to the boundary of the equation of u_h and multiplying by α' gives

$$\alpha'\partial_t u_h - \alpha'\nu(\partial_1^2+\partial_2^2)u_h + \alpha' u\cdot\nabla u_h = -\alpha'\nabla_h p \qquad \text{on } \partial\mathbb{H}.$$

We infer that we must necessarily have that

$$\partial_3(u \cdot \nabla u_h) - \alpha' u \cdot \nabla u_h = \alpha' \nabla_h p \qquad \text{on } \partial\mathbb{H}$$

for all $t \in [0, T_0]$. Using the formula for p given in (44), letting $t \to 0$ and using that u_0 vanishes in a neighborhood of the boundary we deduce that the initial data must verify the following relation

$$\nabla_{x_h} \int_{\mathbb{H}} \frac{1}{|x-y|} \operatorname{div}(u_0 \cdot \nabla u_0)(y)\, dy = 0 \qquad \text{for all } x \in \partial\mathbb{H}. \tag{45}$$

Therefore, the initial data must verify the above relation if we want the solution to verify the Navier boundary conditions. It is not difficult to see that there exists initial data for which (45) is not verified. Indeed, integrating twice by parts in (45) implies

$$\sum_{i,j=1}^{3} \int_{\mathbb{H}} F_x^{ij}(y) u_{0,i}(y) u_{0,j}(y)\, dy = 0 \qquad \text{for all } x \in \partial\mathbb{H}, \tag{46}$$

where

$$F_x^{ij}(y) = \nabla_{x_h} \Big[\frac{\partial^2}{\partial y_i\, \partial y_j} \Big(\frac{1}{|x-y|} \Big) \Big]. \tag{47}$$

Let g be a fixed non-trivial $C_0^\infty(\mathbb{H})$ divergence free vector field and $z \in \mathbb{H}$ be arbitrary. Replacing u_0 by the function $y \mapsto g\big(\frac{y-z}{\varepsilon}\big)$ in (46) and changing variables we get

$$\sum_{i,j=1}^{3} \int_{\mathbb{H}} F_x^{ij}(z + \varepsilon\eta) g_i(\eta) g_j(\eta)\, d\eta = 0 \qquad \text{for all } x \in \partial\mathbb{H}.$$

Letting $\varepsilon \to 0$ we finally get that

$$\sum_{i,j=1}^{3} \gamma_{ij} F_x^{ij}(z) = 0 \qquad \text{for all } x \in \partial\mathbb{H} \text{ and } z \in \mathbb{H},$$

where $\gamma_{ij} = \displaystyle\int_{\mathbb{H}} g_i(\eta) g_j(\eta) d\eta$. From the explicit formula for F given in (47), we infer that for every $z \in \mathbb{H}$, the function

$$x \longmapsto \sum_{i,j=1}^{3} \gamma_{ij} \frac{\partial^2}{\partial z_i\, \partial z_j} \Big(\frac{1}{|x-z|} \Big)$$

must be constant on the boundary $\partial\Omega$. Since this function can be bounded by $C|x-z|^{-3}$, it vanishes when $|x| \to \infty$ and z is fixed. Since it is constant on the boundary, it must actually vanish on the boundary. Writing that it vanishes at $x = 0$, we obtain that

$$\operatorname{tr}(\Gamma)|z|^2 = 3\Gamma z \cdot z \qquad \text{for all } z \in \mathbb{H},$$

where Γ is the matrix $\Gamma = (\gamma_{ij})$. Obviously, the above relation can hold true only if Γ is a multiple of the identity. Clearly, there exists g such that Γ is not a multiple of the identity.

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