INCOMPRESSIBLE FLOW AROUND A SMALL OBSTACLE AND THE VANISHING VISCOSITY LIMIT

DRAGOŞ IFTIMIE, MILTON C. LOPES FILHO,
AND HELENA J. NUSSENZVEIG LOPES

ABSTRACT. In this article we consider viscous flow in the exterior of an obstacle satisfying the standard no-slip boundary condition at the surface of the obstacle. We seek conditions under which solutions of the Navier-Stokes system in the exterior domain converge to solutions of the Euler system in the full space when both viscosity and the size of the obstacle vanish. We prove that this convergence is true assuming two hypotheses: first, that the initial exterior domain velocity converges strongly in L^2 to the full-space initial velocity and second, that the diameter of the obstacle is smaller than a suitable constant times viscosity, or, in other words, that the obstacle is sufficiently small. The convergence holds as long as the solution to the limit problem is known to exist and stays sufficiently smooth. This work complements the study of incompressible flow around small obstacles, which has been carried out in [4, 5, 6]

1. INTRODUCTION

The purpose of the present work is to study the asymptotic behavior of families of solutions of the incompressible Navier-Stokes equations, in two and three space dimensions, in the exterior of a single smooth obstacle, when both viscosity and the size of the obstacle become small. More precisely, let Ω be a smooth and bounded domain in $\mathbb{R}^n$, $n = 2, 3$, such that Ω is connected and simply connected if $n = 2$ and $\mathbb{R}^3 \setminus \Omega$ is connected and simply connected if $n = 3$. Let $\varepsilon > 0$ and set $\Pi_\varepsilon = \mathbb{R}^n \setminus \varepsilon\overline{\Omega}$. Let u_0 be a smooth, divergence-free vector field in $\mathbb{R}^n$, which gives rise to a smooth solution u of the Euler equations, defined on an interval $[0, T]$. Let $u^{\nu,\varepsilon} \in L^\infty((0, T); L^2(\Pi_\varepsilon)) \cap C^0_w([0, T]; L^2(\Pi_\varepsilon)) \cap L^2((0, T); H^1_0(\Pi_\varepsilon))$ be a weak Leray solution of the incompressible Navier-Stokes equations, with viscosity ν, in Π_ε, satisfying the no-slip boundary condition at $\partial\Pi_\varepsilon$. We prove that there exists a constant $C = C(u_0, \Omega, T) > 0$ such that if the following hypothesis holds:

[H] We have that

$$\sup_{\varepsilon \leq C\nu} \|u^{\nu,\varepsilon}(\cdot, 0) - u_0\|_{L^2(\Pi_\varepsilon)} \to 0,$$

as $\nu \to 0$, then we have that $\sup_{\varepsilon \leq C\nu} \|u^{\nu,\varepsilon} - u\|_{L^\infty((0,T);L^2(\Pi_\varepsilon))} \to 0$, as $\nu \to 0$. Furthermore, if we assume that [H] occurs at a certain rate in ν we can obtain an explicit error estimate in L^2.

In addition, we prove that if we fix an initial vorticity ω_0 in $\mathbb{R}^2$, smooth and compactly supported in $\mathbb{R}^2 \setminus \{0\}$ and consider $u^{\nu,\varepsilon}(\cdot, 0) = K^\varepsilon[\omega_0] + mH^\varepsilon$, where K^ε denotes the Biot-Savart operator in Π_ε, while H^ε is the normalized generator of the harmonic vector fields in Π_ε, and $m = \int \omega_0$, then hypothesis [H] is satisfied. In the case of the dimension three, if we fix an initial vorticity ω_0 in $\mathbb{R}^3$, smooth, divergence-free and compactly supported in $\mathbb{R}^3 \setminus \{0\}$ and consider $u^{\nu,\varepsilon}(\cdot, 0) = K^\varepsilon[\omega_0]$, where K^ε again denotes the Biot-Savart operator in Π_ε, then it is proved in [4] that hypothesis [H] is satisfied. In both cases we have rates for the convergence of the initial data in such a way that $\|u^{\nu,\varepsilon} - u\|_{L^2(\Pi_\varepsilon)} = \mathcal{O}(\sqrt{\nu})$ when $\nu \to 0$, uniformly in time.

A central theme in incompressible hydrodynamics is the vanishing viscosity limit, something naturally associated with the physical phenomena of turbulence and of boundary layers. In particular, a natural question to ask is whether the limiting flow associated with the limit of vanishing viscosity satisfies the incompressible Euler equations. This is known to be true in the absence of material boundaries, see [1, 18] for the two dimensional case and [9, 24] for the three dimensional case. Also, if the boundary conditions are of Navier type, see [2, 8, 17, 28], noncharacteristic, see [26] or for certain symmetric 2D flows, see [15, 16, 19], convergence to an Euler solution remains valid. The most relevant case from the physical point of view corresponds to no slip boundary conditions. In this case, we have results on criteria for convergence to solutions of the Euler system, see [10, 11, 25, 27], and also results when the data for the boundary layer equations are analytic, see [13, 21, 22, 23], but the general problem remains wide open. To be a bit more precise, let us assume that u^0 is a solution of the Euler equations in the exterior domain Π and that u^ν is a solution of the Navier-Stokes equations with viscosity ν, with no-slip boundary condition in Π. Suppose further that u^ν and u^0 have the same initial velocity v_0 and that both u^0 and the family $\{u^\nu\}$ are smooth, defined on a fixed time

interval $[0, T]$. It is easy to see that

$$\delta E(\nu, t) \equiv \|u^0(\cdot, t) - u^\nu(\cdot, t)\|_{L^2(\Pi)}$$

is uniformly bounded in ν and $t \in [0, T]$ (by $2\|v_0\|_{L^2}$ if v_0 has finite energy), but it is not known whether $\delta E \to 0$ when $\nu \to 0$. In fact, given the experimentally and numerically observed behavior of high Reynolds number flows in the presence of boundaries, it is reasonable to conjecture that δE does not, in general, vanish as $\nu \to 0$; see, for instance, Section 15.6 of [20] for illustrations of high Reynolds number flow past a cylinder. Of course, this leaves open the possibility that u^ν might approach another solution of the Euler equations, different from u^0.

This article contains an answer to the following question: can we make δE small, by making both the viscosity and the obstacle small? This problem was one of the main motivations underlying the authors' research on incompressible flow around small obstacles. Our previous results include the small obstacle limit for the 2D inviscid equations, see [5, 14] and for the viscous equations, see [4, 6]. The work we present here is a natural outgrowth of this research effort.

The remainder of this article is divided into three sections. In Section 2, we state and prove our main result, namely the convergence in the small viscosity and small obstacle limit, assuming convergence of the initial data. In Section 3 we study the problem of convergence of the initial data: for two space dimensions, we adapt techniques developed in our previous work, while in the three dimensional case we refer the reader to recent work by D. Iftimie and J. Kelliher, see [4]. In Section 4, we interpret the smallness condition on the obstacle as the condition that the *local* Reynolds number stays below a certain (small) constant. In addition, still in Section 4, we obtain an enstrophy estimate for the wake generated by the small obstacle and we list some open problems.

2. Main theorem

We use the notation from the introduction to state and prove our main result. We consider the initial-value problem for the Navier-Stokes equations

in Π_ε, with no-slip boundary condition, given by:

$$(1) \quad \begin{cases} \partial_t u^{\nu,\varepsilon} + u^{\nu,\varepsilon} \cdot \nabla u^{\nu,\varepsilon} = -\nabla p^{\nu,\varepsilon} + \nu \Delta u^{\nu,\varepsilon}, & \text{in } \Pi_\varepsilon \times (0,\infty) \\ \text{div } u^{\nu,\varepsilon} = 0 & \text{in } \Pi_\varepsilon \times [0,\infty) \\ u^{\nu,\varepsilon}(x,t) = 0 & \text{for } x \in \partial\Pi_\varepsilon, \quad t > 0 \\ u^{\nu,\varepsilon}(t=0) = \vartheta^\varepsilon(x) & \text{for } x \in \Pi_\varepsilon, \quad t = 0. \end{cases}$$

We assume that the initial velocity $\vartheta^\varepsilon \in L^2_{\text{loc}}(\Pi_\varepsilon)$ is divergence-free and tangent to $\partial\Pi_\varepsilon$, but we do not assume that it satisfies the no-slip boundary condition. In three dimensions we assume further that $\vartheta^\varepsilon \in L^2(\Pi_\varepsilon)$. Under these hypotheses it was shown by H. Kozono and M. Yamazaki, see [12], that, in two dimensions, there is a unique global strong solution to (1) with initial velocity ϑ^ε, while, in three dimensions, there is a global Leray weak solution $u^{\nu,\varepsilon}$ of (1), see [3]. More precisely, in three dimensions there exists

$$u^{\nu,\varepsilon} \in L^\infty([0,\infty); L^2(\Pi_\varepsilon)) \cap C^0_w([0,\infty); L^2(\Pi_\varepsilon)) \cap L^2_{\text{loc}}([0,\infty); H^1_0(\Pi_\varepsilon))$$

such that $u^{\nu,\varepsilon}$ is a distributional solution of (1) and the following energy inequality holds true:

$$(2) \quad \|u^{\nu,\varepsilon}(t)\|_{L^2(\Pi_\varepsilon)} + 2\nu \int_0^t \|\nabla u^{\nu,\varepsilon}(s)\|_{L^2(\Pi_\varepsilon)}\, ds \leq \|u_0^\varepsilon\|_{L^2(\Pi_\varepsilon)} \qquad \forall t \geq 0.$$

Both ϑ^ε and $u^{\nu,\varepsilon}(\cdot, t)$ are defined only in Π_ε, but we will consider them as defined on the whole space by extending them to be identically zero inside $\varepsilon\Omega$.

Let u_0 be a smooth, divergence-free vector field defined in all $\mathbb{R}^n$, and let u be the corresponding smooth solution of the Euler equations; in two dimensions u is globally defined while in three dimensions it is defined, at least, on an interval $[0,T]$.

We are now ready to state our main result.

Theorem 1. *Assume that*

$$\|\vartheta^\varepsilon - u_0\|_{L^2(\mathbb{R}^n)} \to 0 \ \text{as } \varepsilon \to 0.$$

Fix $T > 0$, arbitrary if $n = 2$, and smaller than the time of existence of the smooth Euler solution if $n = 3$. Then there exists a constant $C_1 = C_1(\Omega, u_0, T)$ such that, if $\varepsilon \leq C_1\nu$, then

$$\|u^{\nu,\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0 \ \text{as } \nu \to 0.$$

Moreover, if we assume that $\|\vartheta^\varepsilon - u_0\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\sqrt{\nu})$, then there exists also $C_2 = C_2(T, u_0, \Omega)$ such that

$$\|u^{\nu,\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq C_2\sqrt{\nu},$$

for all $0 < \varepsilon < C_1\nu$ and all $0 \leq t \leq T$.

Before we proceed with the proof, we require two technical lemmas. To state the first lemma we must introduce some notation. As in the statement of the theorem, u denotes the smooth Euler solution in $\mathbb{R}^n$. In dimension two, we denote by $\psi = \psi(x, t)$ the stream function for the velocity field u, chosen so that $\psi(0, t) = 0$. In dimension three, ψ denotes the unique divergence free vector field vanishing for $x = 0$ and whose curl is u. In other words, we set

$$\psi(x, t) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp \cdot u(y, t)}{2\pi |x - y|^2} dy + \int_{\mathbb{R}^2} \frac{y^\perp \cdot u(y, t)}{2\pi |y|^2} dy$$

in dimension two so that $u = \nabla^\perp \psi$ and

$$\psi(x, t) = -\int_{\mathbb{R}^3} \frac{x - y}{4\pi |x - y|^3} \times u(y, t) dy - \int_{\mathbb{R}^3} \frac{y}{4\pi |y|^3} \times u(y, t) dy$$

in dimension three so that $u = \operatorname{curl} \psi$. In both two and three dimensions one has that ψ and $\nabla \psi$ are uniformly bounded on the time interval $[0, T]$.

Let $R > 0$ be such that the ball of radius R, centered at the origin, contains Ω. Let $\varphi = \varphi(r)$ be a smooth function on $\mathbb{R}_+$ such that $\varphi(r) \equiv 0$ if $0 \leq r \leq R + 1$, $\varphi \geq 0$ and $\varphi(r) \equiv 1$ if $r \geq R + 2$. Set $\varphi^\varepsilon = \varphi^\varepsilon(x) = \varphi(|x|/\varepsilon)$ and

$$u^\varepsilon = \nabla^\perp(\varphi^\varepsilon \psi)$$

in dimension two and

$$u^\varepsilon = \operatorname{curl}(\varphi^\varepsilon \psi)$$

in dimension three. In both dimensions two and three, the vector field u^ε is divergence free and vanishes in a neighborhood of the boundary.

We also re-define the pressure $p = p(x, t)$ from the Euler equation in $\mathbb{R}^n$ with data u_0 so that $p(0, t) = 0$.

Lemma 2. *Fix $T > 0$. There exist constants $K_i > 0$, $i = 1, \ldots, 5$ such that, for any $0 < \varepsilon < \varepsilon_0$ and any $0 \leq t < T$ we have:*

(1) $\|\nabla u^\varepsilon\|_{L^2}^2 \leq K_1$,

(2) $\|u^\varepsilon\|_{L^\infty} \leq K_2$,

(3) $\|u^\varepsilon - u\|_{L^2} + \|u^\varepsilon - \varphi^\varepsilon u\|_{L^2} \leq K_3\varepsilon$,

(4) $\|\nabla\psi\nabla\varphi^\varepsilon\|_{L^\infty} + \|\psi\nabla^2\varphi^\varepsilon\|_{L^\infty} \le K_4/\varepsilon$,

(5) $\|p\nabla\varphi^\varepsilon\|_{L^2} + \|\partial_t\psi\nabla\varphi^\varepsilon\|_{L^2} \le K_5\varepsilon$.

Above, we used the notation $\|\nabla\psi\nabla\varphi^\varepsilon\|_{L^\infty} = \sum_{i,j}\|\partial_i\psi\partial_j\varphi^\varepsilon\|_{L^\infty}$ in dimension two and $\|\nabla\psi\nabla\varphi^\varepsilon\|_{L^\infty} = \sum_{i,j,k}\|\partial_i\psi_k\partial_j\varphi^\varepsilon\|_{L^\infty}$ in dimension three. Similar notations were used for the other terms.

Proof. Some of the inequalities above can be improved in dimension three. However, it turns out that these improvements have no effect on the statement of Theorem 1. Therefore, to avoid giving separate proofs in dimension three we chose to state these weaker estimates.

Recall that both u and ∇u are uniformly bounded. First we write

$$\partial_i u^\varepsilon = \partial_i \nabla^\perp(\varphi^\varepsilon\psi) = u\partial_i\varphi^\varepsilon + \partial_i\psi\nabla^\perp\varphi^\varepsilon + \psi\partial_i\nabla^\perp\varphi^\varepsilon + \varphi^\varepsilon\partial_i u$$

in dimension two and

$$\partial_i u^\varepsilon = \partial_i \operatorname{curl}(\varphi^\varepsilon\psi) = u\partial_i\varphi^\varepsilon + \nabla\varphi^\varepsilon \times \partial_i\psi + \partial_i\nabla\varphi^\varepsilon \times \psi + \varphi^\varepsilon\partial_i u$$

in dimension three. The supports of the first three terms of the right-hand sides of the relations above are contained in the annulus $\varepsilon(R+1) < |x| < \varepsilon(R+2)$, whose Lebesgue measure is $\mathcal{O}(\varepsilon^2)$. Furthermore, $|\nabla\varphi^\varepsilon| = \mathcal{O}(1/\varepsilon)$, $|\nabla^2\varphi^\varepsilon| = \mathcal{O}(1/\varepsilon^2)$ and $|\psi(x,t)| = \mathcal{O}(\varepsilon)$ for $|x| < \varepsilon(R+2)$, since $\psi(0,t) = 0$. Taking L^2 norms in the expressions above gives the first estimate.

Next we observe that $u^\varepsilon = \varphi^\varepsilon u + \psi\nabla^\perp\varphi^\varepsilon$ or $u^\varepsilon = \varphi^\varepsilon u + \nabla\varphi^\varepsilon \times \psi$. Clearly $\varphi^\varepsilon u$ is bounded and to bound the second term, we use again that $\psi(0,t) = 0$, which proves the second estimate. For the third estimate, observe that $u^\varepsilon - u$ and $u^\varepsilon - \varphi^\varepsilon u$ are bounded, as we have just proved, and have support in the ball $|x| < \varepsilon(R+2)$. For the fourth estimate, we use again that $\psi(0,t) = 0$. The last estimate follows from two facts: that the functions whose L^2-norm we are estimating have support on the ball $|x| < \varepsilon(R+2)$ and that they are both bounded, since $p(0,t) = 0$ and $\psi_t(0,t) = 0$. □

We also require a modified Poincaré inequality, stated below.

Lemma 3. *Let Ω be the obstacle under consideration and let R be such that $\Omega \subset B_R$. Consider the scaled obstacles $\varepsilon\Omega$ and the exterior domains Π_ε. Then, if $W \in H_0^1(\Pi_\varepsilon)$ we have*

$$\|W\|_{L^2(\Pi_\varepsilon \cap B_{(R+2)\varepsilon})} \le K_6\,\varepsilon\,\|\nabla W\|_{L^2(\Pi_\varepsilon \cap B_{(R+2)\varepsilon})}.$$

Proof. The case $\varepsilon = 1$ is standard. The remainder of the proof requires a scaling argument. Let $W \in H_0^1(\Pi_\varepsilon)$ and set $Y = Y(x) = W(\varepsilon x)$. Then $Y \in H_0^1(\Pi_1)$. Using the case $\varepsilon = 1$ there exists a constant K_6 such that $\|Y\|_{L^2(\Pi_1 \cap B_{R+2})} \leq K_6 \|\nabla Y\|_{L^2(\Pi_1 \cap B_{R+2})}$. Undoing the scaling we find:

$$\|Y\|_{L^2(\Pi_1 \cap B_{R+2})}^2 = \int_{\Pi_1 \cap B_{R+2}} |W(\varepsilon x)|^2 \, dx = \frac{\|W\|_{L^2(\Pi_\varepsilon \cap B_{(R+2)\varepsilon})}^2}{\varepsilon^n};$$

$$\|\nabla Y\|_{L^2(\Pi_1 \cap B_{R+2})}^2 = \int_{\Pi_1 \cap B_{R+2}} \varepsilon^2 |\nabla W(\varepsilon x)|^2 \, dx = \varepsilon^{2-n} \|\nabla W\|_{L^2(\Pi_\varepsilon \cap B_{(R+2)\varepsilon})}^2.$$

The desired result follows immediately.

$\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1: We begin by noting that, since u is a smooth solution of the Euler equations in $\mathbb{R}^n \times [0, T]$, it follows that

$$(3) \qquad \|u^{\nu,\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^2(\varepsilon\Omega)} \equiv \|u(\cdot, t)\|_{L^2(\varepsilon\Omega)} \leq C\varepsilon \leq C\sqrt{\nu},$$

if $\varepsilon < C\nu$. Hence it remains only to estimate the L^2-norm of the difference in Π_ε, which we do below.

We first give the proof in two dimensions and then we indicate how the proof should be adapted to three dimensions.

2.1. **Case** $n = 2$. The vector field u^ε is divergence free and satisfies the equation

$$u_t^\varepsilon = -\varphi^\varepsilon u \cdot \nabla u - \varphi^\varepsilon \nabla p + \partial_t \psi \nabla^\perp \varphi^\varepsilon.$$

We set $W^{\nu,\varepsilon} \equiv u^{\nu,\varepsilon} - u^\varepsilon$. The vector field $W^{\nu,\varepsilon}$ is divergence free, vanishes on the boundary and satisfies:

$$\partial_t W^{\nu,\varepsilon} - \nu \Delta W^{\nu,\varepsilon} = -u^{\nu,\varepsilon} \cdot \nabla u^{\nu,\varepsilon} - \nabla p^{\nu,\varepsilon} + \nu \Delta u^\varepsilon + \varphi^\varepsilon u \cdot \nabla u + \varphi^\varepsilon \nabla p - \partial_t \psi \nabla^\perp \varphi^\varepsilon.$$

We perform an energy estimate, multiplying this equation by $W^{\nu,\varepsilon}$ and integrating over Π_ε. We obtain

$$(4) \quad \frac{1}{2} \frac{d}{dt} \|W^{\nu,\varepsilon}\|_{L^2}^2 + \nu \|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 = -\nu \int_{\Pi_\varepsilon} \nabla W^{\nu,\varepsilon} \cdot \nabla u^\varepsilon \, dx$$

$$- \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(u^{\nu,\varepsilon} \cdot \nabla) u^{\nu,\varepsilon}] \, dx + \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(\varphi^\varepsilon u \cdot \nabla) u] \, dx$$

$$+ \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot \varphi^\varepsilon \nabla p \, dx - \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot \partial_t \psi \nabla^\perp \varphi^\varepsilon \, dx.$$

We will examine each one of the five terms on the right-hand-side of identity (4). We look at the first term. We use Cauchy-Schwarz and Young's inequalities followed by Lemma 2, item (1), to obtain

$$(5) \qquad \left| \nu \int_{\Pi_\varepsilon} \nabla W^{\nu,\varepsilon} \cdot \nabla u^\varepsilon \, dx \right| \le \frac{\nu}{2} \Big(\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 + K_1 \Big).$$

Next we look at the second and third terms together. We write

$$
\begin{aligned}
|\mathcal{I}| \equiv & \left| - \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(u^{\nu,\varepsilon} \cdot \nabla) u^{\nu,\varepsilon}] \, dx + \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(\varphi^\varepsilon u \cdot \nabla) u] \, dx \right| \\
= & \left| - \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [((W^{\nu,\varepsilon} + u^\varepsilon) \cdot \nabla)(W^{\nu,\varepsilon} + u^\varepsilon)] \, dx + \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(\varphi^\varepsilon u \cdot \nabla) u] \, dx \right| \\
= & \left| - \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(W^{\nu,\varepsilon} \cdot \nabla) u^\varepsilon] \, dx - \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(u^\varepsilon \cdot \nabla) u^\varepsilon] \, dx \right. \\
& \left. + \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(\varphi^\varepsilon u \cdot \nabla) u] \, dx \right|,
\end{aligned}
$$

where we used the fact that $\int W^{\nu,\varepsilon} \cdot [((W^{\nu,\varepsilon} + u^\varepsilon) \cdot \nabla) W^{\nu,\varepsilon}] = 0$. Finally, we add and subtract $\int W^{\nu,\varepsilon} \cdot [(u^\varepsilon \cdot \nabla) u]$ to obtain

$$
\begin{aligned}
|\mathcal{I}| = & \left| \int_{\Pi_\varepsilon} \left\{ -W^{\nu,\varepsilon} \cdot (W^{\nu,\varepsilon} \cdot \nabla u^\varepsilon) + W^{\nu,\varepsilon} \cdot [u^\varepsilon \cdot \nabla(u - u^\varepsilon)] \right. \right. \\
& \left. \left. + W^{\nu,\varepsilon} \cdot [((\varphi^\varepsilon u - u^\varepsilon) \cdot \nabla) u] \right\} \, dx \right| \\
\le & \left| \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot (W^{\nu,\varepsilon} \cdot \nabla u^\varepsilon) \, dx \right| + \left| \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [u^\varepsilon \cdot \nabla(u - u^\varepsilon)] \, dx \right| \\
& + \left| \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [((\varphi^\varepsilon u - u^\varepsilon) \cdot \nabla) u] \, dx \right| \\
= & \left| \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot (W^{\nu,\varepsilon} \cdot \nabla u^\varepsilon) \, dx \right| + \left| \int_{\Pi_\varepsilon} (u - u^\varepsilon) \cdot [(u^\varepsilon \cdot \nabla) W^{\nu,\varepsilon}] \, dx \right| \\
& + \left| \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [((\varphi^\varepsilon u - u^\varepsilon) \cdot \nabla) u] \, dx \right| \\
\le & \left| \int_{\Pi_\varepsilon} (W^{\nu,\varepsilon} \cdot \nabla u^\varepsilon) \cdot W^{\nu,\varepsilon} \, dx \right| + \|u - u^\varepsilon\|_{L^2} \|u^\varepsilon\|_{L^\infty} \|\nabla W^{\nu,\varepsilon}\|_{L^2} \\
& + \|W^{\nu,\varepsilon}\|_{L^2} \|\varphi^\varepsilon u - u^\varepsilon\|_{L^2} \|\nabla u\|_{L^\infty}.
\end{aligned}
$$

For each $i = 1, 2$ we have that

$$(6) \qquad \partial_i u^\varepsilon = \partial_i \psi \nabla^\perp \varphi^\varepsilon + \psi \partial_i \nabla^\perp \varphi^\varepsilon + \partial_i \varphi^\varepsilon u + \varphi^\varepsilon \partial_i u.$$

Therefore,

$$(7) \quad |\mathcal{I}| \leq (\|\nabla\psi\nabla\varphi^\varepsilon\|_{L^\infty} + \|\psi\nabla^2\varphi^\varepsilon\|_{L^\infty})\|W^{\nu,\varepsilon}\|^2_{L^2(A_\varepsilon)} + \|\varphi^\varepsilon\nabla u\|_{L^\infty}\|W^{\nu,\varepsilon}\|^2_{L^2}$$

$$+ \|u - u^\varepsilon\|_{L^2}\|u^\varepsilon\|_{L^\infty}\|\nabla W^{\nu,\varepsilon}\|_{L^2} + \|W^{\nu,\varepsilon}\|_{L^2}\|\varphi^\varepsilon u - u^\varepsilon\|_{L^2}\|\nabla u\|_{L^\infty},$$

where A_ε is the set $\Pi_\varepsilon \cap B_{(R+2)\varepsilon}$, which contains the support of $\nabla\varphi^\varepsilon$.

Hence, using Lemma 2, items (2), (3) and (4), together with Lemma 3, in the inequality (7), we find

$$(8) \quad |\mathcal{I}| \leq \frac{K_4}{\varepsilon}\varepsilon^2 K_6^2\|\nabla W^{\nu,\varepsilon}\|^2_{L^2} + \|\varphi^\varepsilon\nabla u\|_{L^\infty}\|W^{\nu,\varepsilon}\|^2_{L^2}$$

$$+ K_2 K_3 \varepsilon\|\nabla W^{\nu,\varepsilon}\|^2_{L^2} + K_3\varepsilon\|\nabla u\|_{L^\infty}\|W^{\nu,\varepsilon}\|_{L^2}.$$

Next we look at the fourth and fifth terms in (4). Recall that we chose the pressure p in such a way that $p(0,t) = 0$. We find

$$|\mathcal{J}| \equiv \left|\int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot \varphi^\varepsilon\nabla p\, dx - \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot \partial_t\psi\nabla^\perp\varphi^\varepsilon\, dx\right|$$

$$\leq \left|\int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot p\nabla\varphi^\varepsilon\, dx\right| + \left|\int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot \partial_t\psi\nabla^\perp\varphi^\varepsilon\, dx\right|.$$

We estimate each term above to obtain, using Lemma 2 item (5),

$$(9) \quad |\mathcal{J}| \leq (\|p\nabla\varphi^\varepsilon\|_{L^2} + \|\partial_t\psi\nabla^\perp\varphi^\varepsilon\|_{L^2})\|W^{\nu,\varepsilon}\|_{L^2} \leq K_5\varepsilon\|W^{\nu,\varepsilon}\|_{L^2}.$$

We use estimates (5), (8) and (9) in the energy identity (4) to deduce that

$$(10) \quad \frac{1}{2}\frac{d}{dt}\|W^{\nu,\varepsilon}\|^2_{L^2} + \nu\|\nabla W^{\nu,\varepsilon}\|^2_{L^2} \leq \frac{\nu}{2}\|\nabla W^{\nu,\varepsilon}\|^2_{L^2} + \frac{\nu}{2}K_1 + K_2 K_3\varepsilon\|\nabla W^{\nu,\varepsilon}\|_{L^2}$$

$$+ K_4 K_6^2\varepsilon\|\nabla W^{\nu,\varepsilon}\|^2_{L^2} + \|\varphi^\varepsilon\nabla u\|_{L^\infty}\|W^{\nu,\varepsilon}\|^2_{L^2} + \varepsilon(K_5 + K_3\|\nabla u\|_{L^\infty})\|W^{\nu,\varepsilon}\|_{L^2}$$

$$\leq \frac{\nu}{2}\|\nabla W^{\nu,\varepsilon}\|^2_{L^2} + \frac{\nu}{2}K_1 + \frac{\nu}{4}\|\nabla W^{\nu,\varepsilon}\|^2_{L^2} + K_2^2 K_3^2\frac{\varepsilon^2}{\nu} + K_4 K_6^2\varepsilon\|\nabla W^{\nu,\varepsilon}\|^2_{L^2}$$

$$+ K_0\|W^{\nu,\varepsilon}\|^2_{L^2} + \frac{\|W^{\nu,\varepsilon}\|^2_{L^2}}{2} + \frac{\widetilde{K}_5^2\varepsilon^2}{2}.$$

Above we have used the notation $K_0 = \sup_\varepsilon \|\varphi^\varepsilon\nabla u\|_{L^\infty}$ and $\widetilde{K}_5 = K_5 + K_3\|\nabla u\|_{L^\infty}$.

At this point we choose ε so that

$$(11) \qquad 0 < \varepsilon < \min\left\{\varepsilon_0\,, \frac{\nu}{8K_4\,K_6^2}\right\}.$$

With this choice, letting $y = y(t) = \|W^{\nu,\varepsilon}\|^2_{L^2}$, we obtain

$$(12) \qquad \frac{dy}{dt} \leq \nu K_1 + 2K_2^2 K_3^2\frac{\varepsilon^2}{\nu} + \widetilde{K}_5^2\varepsilon^2 + (2K_0 + 1)y \leq C_1'\nu + C_2'y.$$

From Gronwall's inequality it follows that

$$(13) \qquad \|u^{\nu,\varepsilon} - u^{\varepsilon}\|_{L^2(\Pi_{\varepsilon})}^2 \leq C(T, u_0, \Omega)\left(\nu + \|\vartheta^{\varepsilon} - u_0^{\varepsilon}\|_{L^2(\Pi_{\varepsilon})}^2\right).$$

If $\|\vartheta^{\varepsilon} - u_0\|_{L^2} \to 0$ then it follows from (13) together with Lemma 2, item (3), and from (3), that $\|u^{\nu,\varepsilon} - u\|_{L^2} \to 0$, as desired, where the constant C_1 can be chosen to be $(8K_4 K_6^2)^{-1}$. If we assume further that $\|\vartheta^{\varepsilon} - u_0\|_{L^2} = \mathcal{O}(\sqrt{\nu})$ then the second part of the statement of Theorem 1 easily follows. This concludes the proof in the two dimensional case.

2.2. **Case $n = 3$.** The proof in dimension three is similar to the previous one. There are two differences: notation and the justification that we can multiply the equation of $W^{\nu,\varepsilon}$ by $W^{\nu,\varepsilon}$.

First, about notation. One has to replace everywhere the term $\partial_t \psi \nabla^{\perp} \varphi^{\varepsilon}$ by $\nabla \varphi^{\varepsilon} \times \partial_t \psi$ and also relation (6) becomes

$$\partial_i u^{\varepsilon} = \nabla \varphi^{\varepsilon} \times \partial_i \psi + \partial_i \nabla \varphi^{\varepsilon} \times \psi + \partial_i \varphi^{\varepsilon} u + \varphi^{\varepsilon} \partial_i u.$$

These two modifications are just changes of notations. These new terms are of the same type as the old ones, so the estimates that follow are not affected.

Second, we multiplied the equation of $W^{\nu,\varepsilon}$ by $W^{\nu,\varepsilon}$. The solution $u^{\nu,\varepsilon}$, and therefore $W^{\nu,\varepsilon}$ too, is not better than $L^{\infty}(0, T; L^2(\Pi_{\varepsilon})) \cap L^2(0, T; H^1(\Pi_{\varepsilon}))$. But it is well-known that some of the trilinear terms that appear when multiplying the equation of $W^{\nu,\varepsilon}$ by $W^{\nu,\varepsilon}$ are not well defined in dimension three with this regularity only. In other words, one cannot multiply directly the equation of $W^{\nu,\varepsilon}$ by $W^{\nu,\varepsilon}$. Nevertheless, there is a classical trick that allows us to perform this multiplication if the weak solution $u^{\nu,\varepsilon}$ verifies the energy inequality. What we are trying to do, is to subtract the equation of u^{ε} from the equation of $u^{\nu,\varepsilon}$ and to multiply the result by $u^{\nu,\varepsilon} - u^{\varepsilon}$. This is the same as multiplying the equation of $u^{\nu,\varepsilon}$ by $u^{\nu,\varepsilon}$, adding the equation of u^{ε} times u^{ε} and subtracting the equation of $u^{\nu,\varepsilon}$ times u^{ε} and the equation of u^{ε} times $u^{\nu,\varepsilon}$. Since u^{ε} is smooth, all these operations are legitimate except for the multiplication of the equation of $u^{\nu,\varepsilon}$ by $u^{\nu,\varepsilon}$. Formally, multiplying the equation of $u^{\nu,\varepsilon}$ by $u^{\nu,\varepsilon}$ and integrating in space and time from 0 to t yields the energy equality, *i.e.* relation (2) where the sign $\leq$ is replaced by $=$. Since we assumed that the energy inequality holds true, the above operations are justified provided that the relation we get at the end is an inequality instead of an equality. But an inequality is, of course, sufficient for our purpose. Finally, to be completely

rigorous, one has to integrate in time from the begining. That is, we would obtain at the end relation (12) integrated in time. Clearly, the result of the application of the Gronwall lemma in (12) is the same as in (12) integrated in time. This completes the proof in dimension three.

Remark: The proof above is closely related to the proof of Kato's criterion for the vanishing viscosity limit in bounded domains, see [10]. Both results are based on estimating the difference between Navier-Stokes solutions and Euler solutions by means of energy methods. In Kato's argument, the difference is estimated in terms of the Navier-Stokes solution, on which Kato's criterion was imposed. In contrast, our proof estimates the difference in terms of the full-space Euler solution, which is smooth in the context of interest.

3. COMPACTLY SUPPORTED INITIAL VORTICITY

Now that we are in possession of Theorem 1 we will examine two asymptotic problems for which we can prove the convergence condition on the initial velocity. We focus on flows with compactly supported vorticity, and the diameter of the support of vorticity becomes the order one length scale, relative to which the obstacle is small.

Let us begin with the three dimensional case. We consider an initial vorticity ω_0 which is assumed to be smooth, compactly supported in $\mathbb{R}^3 \setminus \{0\}$, and divergence-free. Let $\varepsilon > 0$ be sufficiently small so that the support of ω_0 is contained in Π_ε. The domain Π_ε is assumed simply connected so that there exists a unique divergence-free vector field, tangent to $\partial\Pi_\varepsilon$, in $L^2(\Pi_\varepsilon)$, whose curl is ω_0, see, for example, [4]. We take ϑ^ε to be this unique vector field. We take u_0 to be the unique divergence-free vector field in $L^2(\mathbb{R}^3)$ whose curl is ω_0, given by the full space Biot-Savart law.

In [4], D. Iftimie and J. Kelliher studied the small obstacle asymptotics for viscous flow in Π_ε, for fixed viscosity, in three dimensions. They proved that the small obstacle limit converges to the appropriate Leray solution of the Navier-Stokes equations in the full space. One important ingredient in their proof was precisely to verify strong convergence of the initial data; in our notation, Iftimie and Kelliher proved that

$$\|\vartheta^\varepsilon - u_0\|_{L^2(\mathbb{R}^3)} = O(\varepsilon^{\frac{3}{2}}),$$

as $\varepsilon \to 0$.

We may hence apply Theorem 1 to obtain the following corollary.

Corollary 4. *Let $\omega_0 \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ and consider u_0 and ϑ^ε defined as above. Fix $T > 0$ and assume that the solution $u = u(x,t)$ of the incompressible Euler equations in $\mathbb{R}^3$, with initial velocity u_0, exists up to time T. Let $u^{\nu,\varepsilon}$ be a Leray solution of* (1) *with initial velocity ϑ^ε. Then there exist constants $C_1 = C_1(\Omega, \omega_0, T) > 0$ and $C_2 = C_2(\Omega, \omega_0, T) > 0$ such that*

$$\|u^{\nu,\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^2(\mathbb{R}^3)} \leq C_2 \sqrt{\nu},$$

for all $0 < \varepsilon < C_1 \nu$ and all $0 \leq t \leq T$.

Next we discuss at length the case $n = 2$. In dimension two the exterior domain is no longer simply connected. This means that the vorticity formulation of the Euler equations is incomplete, and we must specify the harmonic part of the initial velocity as well as the initial vorticity, see [5] for a thorough discussion of this issue. To specify the asymptotic problem we wish to consider, we must choose the initial data for (1).

Let ω_0 be smooth and compactly supported in $\mathbb{R}^2 \setminus \{0\}$. Let K denote the operator associated with the Biot-Savart law in the full plane and set $H = x^\perp/(2\pi|x|^2)$, to be its kernel. Let K^ε be the operator associated with the Biot-Savart in Π_ε, i.e., $K^\varepsilon = \nabla^\perp \Delta_{\Pi_\varepsilon}^{-1}$, where Δ_{Π_ε} is the Dirichlet Laplacian in Π_ε. Let H^ε be the generator of the harmonic vector fields in Π_ε, normalized so that its circulation around $\partial \Pi_\varepsilon$ is one. The divergence-free vector fields in Π^ε with curl equal to ω_0 are of the form $K^\varepsilon[\omega_0] + \alpha H^\varepsilon$, with $\alpha \in \mathbb{R}$, see [5]. In [6] the authors studied the asymptotic behavior, as $\varepsilon \to 0$, of solutions of (1) with ν fixed and initial velocity $K^\varepsilon[\omega_0] + \alpha H^\varepsilon$. It was shown in [6] that $u^{\nu,\varepsilon}$ converges to a solution of the Navier-Stokes equations in the full plane with initial data $K[\omega_0] + (\alpha - m)H$, where $m = \int \omega_0$, as long as $|\alpha - m|$ is sufficiently small.

For the vanishing viscosity limit, we must consider only the case $\alpha = m$. There are two reasons for this. First, $K^\varepsilon[\omega_0] + \alpha H^\varepsilon$ converges weakly to $K[\omega_0] + (\alpha - m)H$ in distributions, see Lemma 10 in [6], but, as we shall see, this convergence is not strong in L^2 (see Remark 1 following the proof of the next lemma). Second, one cannot expect solutions of the Euler equations in the full plane with initial velocity $K[\omega_0] + (\alpha - m)H$ to be smooth (even existence is not clear) unless $\alpha = m$.

In view of this discussion, set $u_0 = K[\omega_0]$ and $\vartheta^\varepsilon = K^\varepsilon[\omega_0] + mH^\varepsilon$. With this notation, we can prove strong convergence in L^2 of the initial data, as follows.

Lemma 5. *Fix ε_0 such that the support of ω_0 does not intersect Ω_ε for any $\varepsilon < \varepsilon_0$. There exists a constant $C > 0$, depending on Ω and ω_0 such that*

$$\|\vartheta^\varepsilon - u_0\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon.$$

Proof. We begin the proof with a construction whose details can be found in [5]. In Section 2 of [5], an explicit formula for both K^ε and H^ε can be found in terms of a conformal map T, which takes Π into the exterior of the unit disk centered at zero. The construction of T and its behavior near infinity are contained in Lemma 2.1 of [5]. Using identities (3.5) and (3.6) in [5], we have that the vector field H^ε can be written explictly as

$$H^\varepsilon = H^\varepsilon(x) = \frac{1}{2\pi\varepsilon}DT^t(x/\varepsilon)\frac{(T(x/\varepsilon))^\perp}{|T(x/\varepsilon)|^2}.$$

and the operator K^ε can be written as an integral operator with kernel $\mathcal{K}^\varepsilon$, given by

$$\mathcal{K}^\varepsilon = \frac{1}{2\pi\varepsilon}DT^t(x/\varepsilon)\Big(\frac{(T(x/\varepsilon) - T(y/\varepsilon))^\perp}{|T(x/\varepsilon) - T(y/\varepsilon)|^2} - \frac{(T(x/\varepsilon) - (T(y/\varepsilon))^*)^\perp}{|T(x/\varepsilon) - (T(y/\varepsilon))^*|^2}\Big),$$

where $x^* = x/|x|^2$ denotes the inversion with respect to the unit circle. Furthermore, we recall Theorem 4.1 of [5], from which we obtain

$$\|\vartheta^\varepsilon\|_{L^\infty(\Pi_\varepsilon)} \leq C\|\omega_0\|_{L^\infty}^{1/2}\|\omega_0\|_{L^1}^{1/2},$$

for some constant $C > 0$.

To understand the behavior for ε small in the expressions above, we need to understand the behavior of $T(x)$ for large x. We use Lemma 1 in [7], which is a more detailed version of Lemma 2.1 in [5], to find that there exists a constant $\beta > 0$ such that

(14) $$T(x/\varepsilon) = \beta x\varepsilon^{-1} + h(x/\varepsilon),$$

with $h = h(x)$ a bounded, holomorphic function on Π_1 satisfying $|Dh(x)| \leq C/|x|^2$. Therefore,

(15) $$|DT(x/\varepsilon) - \beta\mathbb{I}| \leq C\frac{\varepsilon^2}{|x|^2}.$$

We will need a further estimate on the bounded holomorphic function $h = h(z) = T(z) - \beta z$, namely that

$$(16) \qquad |h(z_1) - h(z_2)| \le C \frac{|z_1 - z_2|}{|z_1||z_2|},$$

for some constant $C > 0$ independent of z_1, z_2. This estimate holds since, by construction (see Lemma 2.1 in [5]), we have that $h(z) = g(1/z)$ with g a holomorphic function on $(\Pi_1)^*$, whose derivatives are bounded in the closure of $(\Pi_1)^*$. Here, $(\Pi_1)^*$ denotes the image of Π_1 through the mapping $x \mapsto x^* = x/|x|^2$ to which we add $\{0\}$. Here, we are using the following fact: If D is a bounded domain with ∂D a C^1 Jordan curve then any bounded function $f \in C^1(D)$ with bounded derivatives is globally Lipschitz in D. This fact is a nice exercise in basic analysis, which we leave to the reader.

Therefore we have

$$(17) \qquad |h(z_1) - h(z_2)| = \left| g\Big(\frac{1}{z_1}\Big) - g\Big(\frac{1}{z_2}\Big) \right| \le C \left| \frac{1}{z_1} - \frac{1}{z_2} \right| = C \frac{|z_1 - z_2|}{|z_1||z_2|}.$$

In order to estimate $\|\vartheta^\varepsilon - u_0\|_{L^2(\Pi_\varepsilon)}$ we use the fact that the support of ω_0 is contained in Π_ε for ε sufficiently small ε to write

$$2\pi[\vartheta^\varepsilon(x) - u_0(x)] = \int_{\Pi_\varepsilon} \Big(\frac{1}{\varepsilon} DT^t(x/\varepsilon) \frac{(T(x/\varepsilon) - T(y/\varepsilon))^\perp}{|T(x/\varepsilon) - T(y/\varepsilon)|^2} - \frac{(x-y)^\perp}{|x-y|^2} \Big) \omega_0(y)\, dy$$

$$+ \int_{\Pi_\varepsilon} \frac{1}{\varepsilon} DT^t(x/\varepsilon) \Big(\frac{(T(x/\varepsilon))^\perp}{|T(x/\varepsilon)|^2} - \frac{(T(x/\varepsilon) - (T(y/\varepsilon))^*)^\perp}{|T(x/\varepsilon) - (T(y/\varepsilon))^*|^2} \Big) \omega_0(y)\, dy \equiv \mathfrak{A}^\varepsilon + \mathfrak{B}^\varepsilon.$$

Let us begin by estimating $\mathfrak{B}^\varepsilon$. We make the change of variables $\eta = \varepsilon T(y/\varepsilon)$, whose Jacobian is $J = |\det(DT^{-1})(\eta/\varepsilon)|$, a bounded function. Additionally, we set $z = \varepsilon T(x/\varepsilon)$. With this we find:

$$\mathfrak{B}^\varepsilon = DT^t(x/\varepsilon) \int_{\{|\eta|>\varepsilon\}} \Big(\frac{z^\perp}{|z|^2} - \frac{(z - \varepsilon^2 \eta^*)^\perp}{|z - \varepsilon^2 \eta^*|^2} \Big) \omega_0(\varepsilon T^{-1}(\eta/\varepsilon))\, Jd\eta.$$

We observe now that there exists ρ independently of ε such that the support of $\omega_0(\varepsilon T^{-1}(\eta/\varepsilon))$ is contained in the set $\{|\eta| > \rho\}$. Moreover, one can bound $|z - \varepsilon^2 \eta^*| \ge |z| - \varepsilon^2|\eta^*| \ge |z| - \varepsilon^2/\rho \ge |z|/2$ provided that $\varepsilon^2 \le \rho/2$. Therefore we can write

$$|\mathfrak{B}^\varepsilon| \le C \int_{\{|\eta|>\rho\}} \frac{\varepsilon^2 |\eta^*|}{|z||z - \varepsilon^2 \eta^*|} |\omega_0(\varepsilon T^{-1}(\eta/\varepsilon))|\, Jd\eta \le C \frac{\varepsilon^2}{|z|^2},$$

where C depends on the support of ω_0, on the L^1-norm of ω_0 and on the domain Ω through the bounds on the conformal map T and its derivatives.

Finally, we use this estimate in the integral of the square of $\mathfrak{B}^\varepsilon$:

$$\int_{\Pi_\varepsilon} |\mathfrak{B}^\varepsilon|^2 \, dx \leq C\varepsilon^4 \int_{\{|z|>\varepsilon\}} \frac{1}{|z|^4} \, dz \leq C\varepsilon^2,$$

as desired.

Next we treat $\mathfrak{A}^\varepsilon$. First we re-write $\mathfrak{A}^\varepsilon$ in a more convenient form:

$$\mathfrak{A}^\varepsilon = \int_{\Pi_\varepsilon} \frac{1}{\beta} DT^t(x/\varepsilon) \Big(\frac{\beta}{\varepsilon} \frac{(T(x/\varepsilon) - T(y/\varepsilon))^\perp}{|T(x/\varepsilon) - T(y/\varepsilon)|^2} - \frac{(x-y)^\perp}{|x-y|^2} \Big) \omega_0(y) \, dy$$

$$+ \int_{\Pi_\varepsilon} \Big(\frac{1}{\beta} DT^t(x/\varepsilon) - \mathbb{I} \Big) \frac{(x-y)^\perp}{|x-y|^2} \omega_0(y) \, dy$$

$$\equiv \mathfrak{A}^\varepsilon{}_1 + \mathfrak{A}^\varepsilon{}_2.$$

By (15), the term $\mathfrak{A}^\varepsilon{}_2$ can be easily estimated:

$$|\mathfrak{A}^\varepsilon{}_2| \leq \frac{\varepsilon^2}{|x|^2} \int_{\Pi_\varepsilon} \frac{1}{|x-y|} |\omega_0(y)| \, dy \leq C \frac{\varepsilon^2}{|x|^2},$$

so this reduces to an estimate similar to the one we found for $\mathfrak{B}^\varepsilon$.

Next we examine $\mathfrak{A}^\varepsilon{}_1$. We use the expression for T given in (14) to write

$$\mathfrak{A}^\varepsilon{}_1 = \frac{DT^t(\frac{x}{\varepsilon})}{\beta} \int_{\Pi_\varepsilon} \Big(\frac{(x - y + (\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon}) - h(\frac{y}{\varepsilon})])^\perp}{|x - y + (\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon}) - h(\frac{y}{\varepsilon})]|^2} - \frac{(x-y)^\perp}{|x-y|^2} \Big) \omega_0(y) \, dy.$$

With this we have:

$$|\mathfrak{A}^\varepsilon{}_1| \leq C \int_{\Pi_\varepsilon} \frac{|\frac{\varepsilon}{\beta}[h(\frac{x}{\varepsilon}) - h(\frac{y}{\varepsilon})]|}{|x - y + (\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon}) - h(\frac{y}{\varepsilon})]||x-y|} |\omega_0(y)| \, dy.$$

We will make use several times of the estimate we obtained for h given in (17). First

$$(18) \qquad |\frac{\varepsilon}{\beta}|[h(\frac{x}{\varepsilon}) - h(\frac{y}{\varepsilon})]| \leq C \frac{\varepsilon^2 |x-y|}{|\beta||x||y|}.$$

Using (18) gives

$$|\mathfrak{A}^\varepsilon{}_1| \leq C \int_{\Pi_\varepsilon} \frac{\varepsilon^2}{|x - y + (\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon}) - h(\frac{y}{\varepsilon})]||\beta||x||y|} |\omega_0(y)| \, dy.$$

Let $R, r > 0$ be such that the support of ω_0 is contained in the disk of radius R and outside the disk of radius r. We will estimate $\mathfrak{A}^\varepsilon{}_1$ in two regions: $|x| \geq 2R$ and $|x| < 2R$. Also, recall that the estimate of the L^2-norm of $\mathfrak{A}^\varepsilon{}_1$ is to be

performed in Π_ε so we may assume throughout that $|x| \geq C\varepsilon$. Suppose first that $|x| \geq 2R$. Then we find:

$$|\mathfrak{A}^\varepsilon{}_1| \leq C\frac{\varepsilon^2}{|x|^2}.$$

Above we used that $r < |y| \leq |x|/2$ and hence $|x-y+(\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon})-h(\frac{y}{\varepsilon})]| \geq C|x|$ if ε is sufficiently small, since h is bounded. Finally, in the region $C\varepsilon \leq |x| < 2R$ we use (16) and the fact that $|y|$ is of order 1 to bound

$$|x - y + (\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon}) - h(\frac{y}{\varepsilon})] \geq ||x - y| - \varepsilon^2(|x - y|/|x||y|)| \geq \frac{|x - y|}{2}$$

for ε small enough. Therefore

$$|\mathfrak{A}^\varepsilon{}_1| \leq C\frac{\varepsilon^2}{|x|} \int_{\Pi_\varepsilon} \frac{|\omega_0(y)|}{|x - y|}\, dy \leq C\frac{\varepsilon^2}{|x|} \leq C\varepsilon.$$

Clearly this last portion has L^2-norm in the region $|x| < 2R$ bounded by $C\varepsilon$.

$\square$

Remark 1: Let $\omega_0 \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ and $\alpha \in \mathbb{R}$, $\alpha \neq m$. We observe that $K^\varepsilon[\omega_0] + \alpha H^\varepsilon$ does not converge strongly in L^2 to $K[\omega_0] + (\alpha - m)H$. We argue by contradiction, assuming this convergence holds. In view of the Lemma above, $K^\varepsilon[\omega_0] + mH^\varepsilon$ converges strongly to $K[\omega_0]$ so we must have

$$(\alpha - m)H^\varepsilon \to (\alpha - m)H.$$

This does not hold, as it can be easily seen in the case of the exterior of the disk. In this case, $H^\varepsilon = H$ outside the disk of radius ε, but H^ε vanishes for $|x| < \varepsilon$. Since $\|H\|_{L^2(\{|x|<\varepsilon\})} = \infty$, we have a contradiction.

Remark 2: Note that if we were willing to confine our analysis to the exterior of a small disk, the proof of Lemma 5 would be much simpler. Indeed, let $\Omega_\varepsilon = \{|x| > \varepsilon\}$. Then the conformal map T is the identity, so $\mathfrak{A}^\varepsilon \equiv 0$ and all that is needed is the easier estimate for $\mathfrak{B}^\varepsilon$.

Remark 3: The constant $\alpha - m$ is precisely the circulation of ϑ^ε around the boundary of Π_ε. The condition $\alpha - m = 0$ is physically reasonable, in particular because viscous flows vanish at the boundary, and therefore, so does their circulation. This is the condition for the small obstacle limit of ideal flow to satisfy Euler equations in the full plane, see [5] and also for the small obstacle limit of viscous flows to satisfy the full plane Navier-Stokes equations for all viscosities, see [6]. The argument in [6] required sufficiently

small $\alpha - m$ to obtain the appropriate limit when $\varepsilon \to 0$, and the smallness condition was actually $\alpha - m = \mathcal{O}(\nu)$ as $\nu \to 0$.

We conclude this section with the formal statement of a corollary which encompasses Theorem 1 and Lemma 5.

Corollary 6. *Let $\omega_0 \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ and consider u_0 and ϑ^ε defined as in Lemma 5. Let $u = u(x,t)$ be the global smooth solution of the incompressible Euler equations in $\mathbb{R}^2$, with initial velocity u_0. Let $u^{\nu,\varepsilon}$ be the solution of (1) with initial velocity ϑ^ε. Fix $T > 0$. There exist constants $C_1 = C_1(\Omega, \omega_0, T) > 0$ and $C_2 = C_2(\Omega, \omega_0, T) > 0$ such that*

$$\|u^{\nu,\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C_2 \sqrt{\nu},$$

for all $0 < \varepsilon < C_1 \nu$ and all $0 \leq t \leq T$.

4. Physical interpretation and conclusions

The behavior of incompressible viscous flow past a bluff body, such as a long cylinder or a sphere is a classical problem in fluid dynamics, to the extent of having conference series devoted to it, see *http://www.mae.cornell.edu/bbviv5/*. Let us consider the simplest situation, two-dimensional flow of a viscous fluid with kinematic viscosity μ, filling the whole plane minus a disk of diameter L, with constant driving velocity U at infinity. The disturbance caused by the disk, known as its *wake*, depends only on the *Reynolds number* associated with the flow, given by

$$Re \equiv \frac{LU}{\mu}.$$

The observed behavior of the wake begins, for small Re, as a steady solution of the Navier-Stokes equations, but the wake undergoes a series of bifurcations as Re grows, progressively developing steady recirculation zones ($4 < Re < 40$, periodic recirculation and a Von Karman street ($40 < Re < 200$), nonperiodic vortex shedding ($200 < Re < 400$), leading to turbulence ($Re > 400$). See [20], Section 15.6, for details and illustrations.

In our problem, which involves nearly inviscid flow past a *small* bluff body, the qualitative behavior of the wake of the small obstacle is determined by the *local* Reynolds number, which encodes the way in which an observer at the scale of the obstacle experiences the flow. Basically, by making our obstacle small, we are making the flow more viscous at its scale. Roughly speaking, a change of

variables $x = \varepsilon y$ leaves the obstacle fixed while the Reynolds number behaves like $\dfrac{\varepsilon}{\nu}$. Thus, the assumption $\varepsilon \leq C_1 \nu$ corresponds to bounded Reynolds number. Let us make this argument more precise.

We assume that the Navier-Stokes system under consideration, (1), is dimensional, i.e. has time and space measured in seconds and meters, and mass normalized so that fluid density is one. In these units, the kinematic viscosity for air is $\mu = 14.5 \times 10^{-6} m^2/sec$, and for water it is $\mu = 1.138 \times 10^{-6} m^2/sec$, both at $15^o C$.

Let us restrict our discussion to the two-dimensional case. The smallness condition in Theorem 1, (11), reads $\varepsilon < C_1 \mu$, and the *dimensional* constant C_1, requires closer scrutiny. Actually, the constant C_1 is given by:

$$C_1 = \frac{1}{8 K_4 K_6^2},$$

where K_4 appears in Lemma 2, item (4), and K_6 is from Lemma 3. K_6 is a non-dimensional constant that depends on the shape of the obstacle Ω. The constant K_4 can be chosen as

$$K_4 = \sup_{x \in \mathbb{R}^n, \, t \in [0,T]} \varepsilon (|\nabla \psi \nabla \varphi^\varepsilon|(x,t) + |\psi \nabla^2 \varphi^\varepsilon|(x,t)).$$

The function ψ above is the stream function of the full-plane Euler flow, adjusted so that $\psi(0,t) = 0$. Also, $\varepsilon \nabla \varphi^\varepsilon$ is $\mathcal{O}(1)$, localized near the obstacle and $\varepsilon \nabla^2 \varphi^\varepsilon$ is $\mathcal{O}(1/\varepsilon)$, also localized near the obstacle. Therefore, both terms included in K_4 are associated with first derivatives of the stream function at the obstacle, i.e. with the local velocity $u(0,t)$. Therefore, we can write $K_4 = \widetilde{K}_4 \sup_t |u(0,t)|$ (we can also assume that the limiting Euler flow is stationary, to avoid the time dependence).

From the point of view of the obstacle, the inviscid velocity $u(0,t)$ acts as a constant (in space) forcing velocity imposed at infinity, and therefore, the qualitative behavior of the wake of the obstacle is determined by the local Reynolds number $Re_{\text{loc}} \equiv u(0,t)\varepsilon/\mu$. Clearly, condition (11) can be rewritten as

$$Re_{\text{loc}} < \frac{1}{8 \widetilde{K}_4 K_6^2}.$$

The non-dimensional constant K_6 is related to the constant in the Poincaré inequality in the unit disk. Examining our proof for the case of the disk, we

cannot make the constant $\widetilde{K}_4 K_6^2$ smaller than something of the order of 10. Therefore, our result is restricted to rather viscous wakes.

When it occurs, the turbulence is caused by vorticity shed by the obstacle through boundary layer separation. The main difficulty in studying the vanishing viscosity limit in the presence of boundaries is the fact that, although the Navier-Stokes equations do have a vorticity form, valid in the bulk of the fluid, the vorticity equation does not satisfy a useful boundary condition, so that we cannot control the amount of vorticity added to the flow by the boundary layer. In the proof of Theorem 1, we found a way of controlling the kinetic energy of the wake without making explicit reference to the vorticity. At this point, it is reasonable to ask whether we can control the vorticity content of the wake as well. To answer that, we introduce the *enstrophy* $\Omega^{\nu,\varepsilon}(t)$ of the flow:

$$\Omega^{\nu,\varepsilon}(t) \equiv \frac{1}{2} \int_{\Pi_\varepsilon} |\operatorname{curl} u^{\nu,\varepsilon}|^2 \, dx.$$

Of course, enstrophy measures how much vorticity is in the flow, but its behavior as $\nu \to 0$ is also involved in the statistical structure of a turbulent wake.

Corollary 7. *For any $T > 0$ there exists a constant $C > 0$, independent of ν such that*

$$\int_0^T \Omega^{\nu,\varepsilon}(t) \, dt \le C.$$

Proof. We go back to relations (10) and (12) and include the viscosity term which had been ignored. We find:

$$\frac{dy}{dt} + \frac{\nu}{4} \|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 \le C_1'\nu + C_2'y.$$

We next integrate in time to obtain

$$\|W^{\nu,\varepsilon}(\cdot,T)\|_{L^2}^2 - \|W^{\nu,\varepsilon}(\cdot,0)\|_{L^2}^2 + \frac{\nu}{4}\int_0^T \|\nabla W^{\nu,\varepsilon}(\cdot,t)\|_{L^2}^2 \, dt$$

$$\le C_1'T\nu + C_2'\int_0^T \|W^{\nu,\varepsilon}(\cdot,t)\|_{L^2}^2 \, dt.$$

Now we use Theorem 1 and ignore a term with good sign to obtain

$$\frac{\nu}{4}\int_0^T \|\nabla W^{\nu,\varepsilon}(\cdot,t)\|_{L^2}^2 \, dt \le CT\nu + \|W^{\nu,\varepsilon}(\cdot,0)\|_{L^2}^2 \le C'T\nu,$$

where we used Lemma 5 together with item (3) from Lemma 2 to estimate the
initial data term. From this we conclude that

$$\int_0^T \|\nabla W^{\nu,\varepsilon}(\cdot, t)\|_{L^2}^2 \, dt \le C.$$

Finally, we observe that

$$\Omega^{\nu,\varepsilon} \le C\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 + C\|\nabla u^\varepsilon\|_{L^2}^2 \le C\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 + CK_1,$$

by item (1) in Lemma 2. This concludes the proof.

$$\square$$

Finally, let us consider some open questions naturally associated with the re-
search presented here. First, one would like to weaken, and ultimately remove,
the smallness condition on the size of the obstacle; this is the most physically
interesting follow-up problem. Second, one would also like to consider two
dimensional flows with nonzero initial circulation at the obstacle, in order to
study the interaction of the vanishing viscosity and vanishing obstacle limits
in more detail. This would improve the connection of the present work with
the authors' previous results in [5, 6]. An easier version of this second prob-
lem would be to consider an initial circulation of the form $\gamma = \gamma(\nu)$ and find
out how fast γ has to vanish as $\nu \to 0$ in order to retain our result. A third
problem is to describe more precisely the asymptotic structure of the difference
between the full-space Euler flow and the approximating small viscosity, small
obstacle flows.

References

[1] J.-Y. Chemin. *A remark on the inviscid limit for two-dimensional incompressible fluids.*
Comm. Partial Differential Equations **21** (1996), no. 11-12, 1771–1779.

[2] T. Clopeau, A. Mikelić and R. Robert. *On the vanishing viscosity limit for the* 2D *incom-
pressible Navier-Stokes equations with the friction type boundary conditions.* Nonlinearity
11 (1998), no. 6, 1625–1636.

[3] E. Hopf. *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen.* Math. Nachr. **4**, (1951), 213–231.

[4] D. Iftimie and J. Kelliher, *Remarks on the vanishing obstacle limit for a 3D viscous incompressible fluid.* Preprint available at http://math.univ-lyon1.fr/~iftimie/ARTICLES/viscoushrink3d.pdf.

[5] D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes. *Two dimensional incompressible ideal flow around a small obstacle.* Comm. Partial Differential Equations **28** (2003), no. 1-2, 349–379.

[6] D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes. *Two dimensional incompressible viscous flow around a small obstacle.* Math. Ann. **336** (2006), no. 2, 449–489.

[7] D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes. *Confinement of vorticity in two dimensional ideal incompressible exterior flow.* Quart. Appl. Math. **65** (2007), 499–521.

[8] D. Iftimie and G. Planas. *Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions.* Nonlinearity **19** (2006), no. 4, 899–918.

[9] T. Kato. *Nonstationary flows of viscous and ideal fluids in R^3.* J. Functional Analysis **9** (1972), 296–305.

[10] T. Kato. *Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary.* Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983), 85–98, Math. Sci. Res. Inst. Publ., **2**, Springer, New York, 1984.

[11] J. Kelliher. *On Kato's conditions for vanishing viscosity.* Indiana Univ. Math. J. **56** (2007), no. 4, 1711–1721.

[12] H. Kozono and M. Yamazaki. *Local and global unique solvability of the Navier-Stokes exterior problem with Cauchy data in the space $L^{n,\infty}$.* Houston J. Math. **21** (1995), no. 4, 755–799.

[13] M. Lombardo, M. Cannone and M. Sammartino, *Well-posedness of the boundary layer equations.* SIAM J. Math. Anal. **35** (2003), no. 4, 987–1004.

[14] M. C. Lopes Filho. *Vortex dynamics in a two-dimensional domain with holes and the small obstacle limit.* SIAM J. Math. Anal. **39** (2007), 422–436.

[15] M. C. Lopes Filho, A. Mazzucato and H. J. Nussenzveig Lopes. *Vanishing viscosity limit for incompressible flow inside a rotating circle.* To appear, Phys. D, 2008, doi:10.1016/j.physd.2008.03.009.

[16] M. C. Lopes Filho, A. Mazzucato, H. J. Nussenzveig Lopes and M. Taylor. *Vanishing Viscosity Limits and Boundary Layers for Circularly Symmetric 2D Flows.* To appear, Bull. Braz. Soc. Math., 2008.

[17] M. C. Lopes Filho, H. J. Nussenzveig Lopes and G. Planas. *On the inviscid limit for two-dimensional incompressible flow with Navier friction condition.* SIAM J. Math. Anal. **36** (2005), no. 4, 1130–1141.

[18] A. Majda. *Remarks on weak solutions for vortex sheets with a distinguished sign.* Indiana Univ. Math. J. **42** (1993), no. 3, 921–939.

[19] S. Matsui. *Example of zero viscosity limit for two-dimensional nonstationary Navier-Stokes flows with boundary.* Japan J. Indust. Appl. Math. **11** (1994), no. 1, 155–170.

[20] R. J. Panton, *Incompressible Flow*, John Wiley & Sons, New York, 1984.

[21] M. Sammartino and R. Caflisch, *Zero viscosity limit for analytic solutions of the Navier-Stokes equations.* Proc. of the VIIIth Inter. Conf. on Waves and Stability in Continuous Media, Part II (Palermo, 1995) Rend. Circ. Mat. Palermo (2) Suppl. No. **45** part II (1996) 595–605.

[22] M. Sammartino and R. Caflisch, *Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space I, existence for Euler and Prandtl equations.* Comm. Math. Phys. **192** (1998), no. 2, 433-461.

[23] M. Sammartino and R. Caflisch, *Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space II, construction of the Navier-Stokes solution.* Comm. Math. Phys. **192** (1998), no. 2, 463-491.

[24] H. S. G. Swann. *The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in R_3.* Trans. Amer. Math. Soc. **157** (1971) 373–397.

[25] R. Temam and X. Wang. *On the behavior of the solutions of the Navier-Stokes equations at vanishing viscosity.* Dedicated to Ennio De Giorgi. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1998), no. 3-4, 807–828.

[26] R. Temam and X. Wang. *Boundary layers associated with incompressible Navier-Stokes equations: the noncharacteristic boundary case.* J. Differential Equations **179** (2002), no. 2, 647–686.

[27] X. Wang. *A Kato type theorem on zero viscosity limit of Navier-Stokes flows.* Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000). Indiana Univ. Math. J. **50** (2001), Special Issue, 223–241.

[28] Yuelong Xiao and Zhouping Xin. *On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition.* Comm. Pure and Appl. Math. **60** (2007), no. 7, 1027–1055.

Université de Lyon,, Université Lyon 1, CNRS, UMR 5208 Institut Camille Jordan, Bâtiment du Doyen Jean Braconnier, 43, blvd du 11 novembre 1918, F–69622 Villeurbanne Cedex, France

   E-mail address: dragos.iftimie@univ-lyon1.fr

Departamento de Matemática, IMECC, Caixa Postal 6065, Universidade Estadual de Campinas - UNICAMP, Campinas, SP 13083-970, Brazil

   E-mail address: mlopes@ime.unicamp.br

Departamento de Matemática, IMECC, Caixa Postal 6065, Universidade Estadual de Campinas - UNICAMP, Campinas SP 13083-970, Brazil

   E-mail address: hlopes@ime.unicamp.br