Self-similar point vortices and confinement of vorticity

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Abstract

This papers deals with the large time behavior of solutions of the incompressible Euler equations in dimension two. We consider a self-similar configuration of point vortices which grows like the square root of the time. We study the confinement properties of a blob of vorticity initially located around the first point vortex and moving in the velocity field produced by itself and by the other point vortices. We find a sufficient condition on the point vortices such that the vorticity stays confined around the first point vortex at a rate better than the square root of the time. The relevance to the large time behavior of the Euler equations is discussed.

1 Introduction

We consider in this paper the incompressible Euler equations in $\mathbb{R}^2$:

$$\partial_t u + u \cdot \nabla u = -\nabla p, \qquad \operatorname{div} u = 0.$$

An important quantity is the vorticity

$$\omega = \partial_1 u_2 - \partial_2 u_1$$

which is transported by the flow

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

The velocity can be recovered from the vorticity through the so-called Biot-Savart law

$$u(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{2\pi|x-y|^2} \omega(y)\, dy$$

where $x^\perp = (-x_2, x_1)$.

The large time behavior of a perfect incompressible fluid with planar symmetry is still largely unknown. There are only few examples of smooth solutions giving us insight into this matter. A compactly supported radial vorticity is a stationary solution. Vortex patches with elliptical symmetry and the so-called V-states rotate with constant speed. A very important example of vorticity going to infinity is given by the so-called

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vortex pairs. A vortex pair is a couple of two symmetric vortices with opposite sign which travel with constant speed along the axis of symmetry. They are travelling wave solutions and their existence was known for a long time, see for example [14] and [2].

In absence of sufficient examples of smooth solutions, one can look at the point vortex system which is a discrete version of the Euler equations. In the point vortex model it is assumed that the vorticity is a sum of Dirac masses

$$\omega(x,t) = \sum_{j=1}^{n} m_j \delta_{z_j(t)}$$

and the positions $z_j(t) \in \mathbb{R}^2$ must satisfy

$$z_j'(t) = \sum_{k=1;k\neq j}^{n} m_k \frac{(z_j - z_k)^\perp}{2\pi |z_j - z_k|^2}. \tag{1.1}$$

The point vortex system have been introduced in the 1800 century by Helmholtz [7] (see also [13, 27, 12]) and it approximates reasonably well a state with a very large vorticity concentration. Indeed, assume that the initial vorticity is supported in n blobs:

$$\omega_\varepsilon(x,0) = \sum_{i=1}^{n} \omega_{\varepsilon;i}(x,0) \ ,$$

where $\omega_{\varepsilon;i}(x,0)$ is a function with a definite sign supported in a region $\Lambda_{\varepsilon;i}$ such that

$$\Lambda_{\varepsilon;i} = \text{supp } \omega_{\varepsilon;i} \subset D(z_i,\varepsilon) \ ; \ D(z_i,\varepsilon) \cap D(z_j,\varepsilon) = 0 \ \ if \ \ i \neq j \ ,$$

for ε small enough. Here $D(z,r)$ denotes the disk of center z and radius r.

It has been proved that the time evolution of these states via the Euler equation has, for small ε, a similar form:

$$\Lambda_{\varepsilon;i}(t) = \operatorname{supp} \omega_{\varepsilon;i}(t) \subset D(z_i(t), f(\varepsilon,t)) \ ;$$

$$D(z_i(t), f(\varepsilon,t)) \cap D(z_j(t), f(\varepsilon,t)) = 0 \ \text{ if } \ i \neq j \ ,$$

where the z_i are the solutions of the point vortex system and $f(\varepsilon,t)$ is a positive function.

In general the point vortex system has a global solution, but in some cases collapses can happen (for a review on this point see [1]). Moreover there are initial situations in which the point vortices go away indefinitely. It can be proved that the intensity of the vortices and the initial data that produce a collapse are exceptional (see [4], [24]) and hence in general there is a minimal distance between the vortices. Until the time of a possible collapse it can be proved that for $\varepsilon \to 0$ we have $f(\varepsilon,t) \to 0$ and therefore the fluid converges to the point vortex system [23, 24, 19, 3]. For the connection between the Euler flow and the point vortices see also [5, 16, 18, 20, 21, 29].

When studying the large time behavior of the point vortex system, several important facts can be observed:

- If the masses m_j are positive then the positions z_j are bounded. This follows from the conservation of the moment of inertia.
- Two point vortices with opposite masses will translate with constant speed (travelling wave solutions).

- There are examples of four point vortices whose diameter spreads linearly in time. These can be viewed as a superposition of two vortex pairs. In this case the total mass is zero.
- There exists the so-called self-similar point vortices. The configuration of point vortices have a spiral motion, evolving by rotation and dilation of order $O(\sqrt{t})$. In this case the total mass is non zero.

It appears from these examples that three cases must be distinguished: (a) single signed masses, (b) total mass zero and (c) total mass non zero, not single signed masses.

Let us now turn to the existing results in the case of smooth vorticity. There are results for the cases (a) and (b) but not so much about case (c). In the case of single-signed vorticity, confinement of the vorticity like $O(t^{\frac{1}{3}})$ was proved by [17], see also [15], and confinement like $O(t\log t)^{\frac{1}{4}}$ was proved in [11, 28]). A smooth example of vorticity whose diameter spreads linearly in time (and vanishing mass) was constructed in [11]. As far we know, the only result pertaining to case (c) is proved in [9, 10] where the authors make the rescaling $x \sim t^a$ and show that for all $a > \frac{1}{2}$

$$t^{2a}\omega(t^a x, t) \rightharpoonup \Big(\int \omega_0\Big)\delta_0 \qquad \text{as } t \to \infty. \tag{1.2}$$

This is a weak confinement result for the *imbalance* between the positive and negative parts of vorticity. One could imagine that a part of the vorticity of mass $\int \omega_0$ stays confined like $O(t^{\frac{1}{2}})$ while the rest may go fast to infinity via the vortex pair mechanism. A review of these results can be found in [8].

Our initial motivation in writing this paper was to see if the convergence stated in (1.2) holds true for $a = \frac{1}{2}$. Because of the example of self-similar point vortices, it does not hold true for the point vortex system. One way to prove that for smooth vorticity it does not hold true either would be to consider smooth vorticity sharply concentrated around self-similar point vortices and show confinement for the evolved vorticity around the point vortices at a rate better than $t^{\frac{1}{2}}$. In doing that, two problems must be dealt with. The first one is the instability of the self-similar configurations. A tiny modification of the initial configuration makes it non self-similar and the spreading of the point vortices like $t^{\frac{1}{2}}$ is not guaranteed anymore. The second problem is be to prove the confinement itself.

In this paper we consider only the confinement problem and do not deal with the instability issue. More precisely, we consider a toy model where the instability is removed. This is an intermediate model between the point vortex model and the Euler flow with concentrated vorticity. We have n point vortices obeying Eq. (1.1); we consider a blob of vorticity initially posed around a point vortex (for instance the first one) and moving in the velocity field produced by the other ones. Vice versa its motion does not perturb the motion of the other point vortices (hence the instability is removed) that remain governed by the point vortex system. The main result of this article is that we find a condition on the point vortices such that confinement better than $t^{\frac{1}{2}}$ occurs for this toy model, see Theorem 3.1 below.

Even though the initial motivation was to find a counter-example for (1.2) when $a = \frac{1}{2}$, our result may be interesting for other reasons. For instance, long time confinement around point vortices is interesting in itself. Because for positive vorticity we don't know how to prove confinement better than $t^{\frac{1}{4}}$, it seems that we need to have spreading of the point vortices at least like $t^{\frac{1}{4}}$ in order to be able to show long-time confinement.

But the only examples of point vortices that spread at least like $t^{\frac{1}{4}}$ that we know of are the self-similar point vortices.

The plan of the article is the following. In the next section we recall some facts about self-similar point vortices. In Section 3 we introduce our toy model and we prove our main result, Theorem 3.1 below. Finally, in Section 4 we discuss the condition we require on the point vortices for confinement to occur.

2 Self-similar point vortices

A configuration of point vortices is called self-similar if

$$z_j(t) - X = f(t)\big(z_j(0) - X\big) \qquad \forall j \in \{1, \dots, n\} \tag{2.1}$$

where X is independent of the time (the center of mass), and if it verifies the point vortex system:

$$z_j'(t) = \sum_{\substack{k\in\{1,\dots,n\}\\ k\neq j}} m_k \frac{(z_j - z_k)^\perp}{2\pi|z_j - z_k|^2} \qquad \forall j \in \{1, \dots, n\}. \tag{2.2}$$

Above, f is a complex-valued function and the multiplication $f(t)z_j(0)$ must be understood as complex numbers.

Such self-similar configurations are known to exist. For $n = 3$ see [6] (see also [1]), for $n = 4, 5$ see [25] and for $n \geqslant 6$ there is a discussion in [26].

Let us replace the formulae for z_j given in (2.1) into one of the equations of (2.2), say in the first one and use complex notation:

$$f'(t)\big(z_1(0) - X\big) = \sum_{j=2}^{n} m_j \frac{(z_1 - z_j)^\perp}{2\pi|z_1 - z_j|^2} = \sum_{j=2}^{n} \frac{\mathrm{i}m_j}{2\pi(\overline{z}_1 - \overline{z}_j)} = \frac{1}{\overline{f}} \sum_{j=2}^{n} \frac{\mathrm{i}m_j}{2\pi(\overline{z}_1(0) - \overline{z}_j(0))}.$$

We write the above equation under the form

$$f'(t) = \frac{1}{\overline{f}}\Big(\frac{a}{2} + \mathrm{i}b\Big) \qquad \text{where} \qquad \frac{a}{2} + \mathrm{i}b = \frac{1}{z_1(0) - X} \sum_{j=2}^{n} \frac{\mathrm{i}m_j}{2\pi(\overline{z}_1(0) - \overline{z}_j(0))}.$$

Using polar coordinates we write $f(t) = r(t)e^{\mathrm{i}\theta(t)}$ and we deduce from the above relation that

$$r'(t) = \frac{a}{2r} \qquad \text{and} \qquad \theta'(t) = \frac{b}{r^2}.$$

Integrating this system of ODE shows that we have that

$$f(t) = \sqrt{at + 1}\, e^{\mathrm{i}\frac{b}{a}\ln(at+1)}$$

When $a < 0$ we have that the point vortices collapse into the center of mass X at time $t = -1/a$. If $a = 0$ then the point vortices just rotate with constant speed. If $a > 0$ then the point vortices spread like $\sqrt{t}$. This is the case we are interested in, so we will assume in the sequel that $a > 0$.

Let

$$F(x) = \sum_{j=2}^{n} m_j \frac{(x - z_j)^\perp}{2\pi|x - z_j|^2}. \tag{2.3}$$

be the velocity induced by the $z_2, \dots, z_n$ point vortices. Using complex notation, we observe that we can write

$$F(z) = \sum_{j=2}^{n} \frac{\mathrm{i} m_j}{2\pi(\overline{z} - \overline{z}_j)} \tag{2.4}$$

where i is the imaginary unit. Clearly $\overline{F}$ is a holomorphic function so we can derive it with respect to z as a holomorphic function

$$\frac{\partial \overline{F}}{\partial z}(z_1) = \sum_{j=2}^{n} \frac{\mathrm{i} m_j}{2\pi(z_1 - z_j)^2} = \frac{1}{f^2(t)} \sum_{j=2}^{n} \frac{\mathrm{i} m_j}{2\pi(z_1(0) - z_j(0))^2} = \frac{e^{-2\mathrm{i}\frac{b}{a}\ln(at+1)}}{1+at} v \tag{2.5}$$

where

$$v = \sum_{j=2}^{n} \frac{\mathrm{i} m_j}{2\pi(z_1(0) - z_j(0))^2} \tag{2.6}$$

does not depend on the time. We define

$$\alpha = \frac{|v|}{a}$$

and observe that

$$\Big| \frac{\partial \overline{F}}{\partial z}(z_1) \Big| = \frac{|v|}{1+at}$$

so that

$$\alpha = \lim_{t \to \infty} t \Big| \frac{\partial \overline{F}}{\partial z}(z_1) \Big|.$$

3 The toy model and the confinement result

We introduce now our toy model. Consider a system of self-similar point vortices of intensities m_i posed in $z_i(t)$. We consider a blob of vorticity initially supported on a small region around the point $z_1(0)$ and total vorticity m_1 moving via the Euler Equation in the external velocity field $F(x,t)$ produced by the other point vortices. The vortices move according the point vortex law (1.1).

We have that

$$\omega \geqslant 0, \quad \int_{\mathbb{R}^2} \omega = m_1, \quad \text{supp } \omega_0 \subset D(z_1(0), \varepsilon)$$

where $\varepsilon \leqslant 1$. The PDE verified by ω is the following

$$\partial_t \omega + \operatorname{div}((u+F)\omega) = 0$$

where

$$u(x) = \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{2\pi|x-y|^2} \omega(y)\, dy$$

and F is given in (2.3).

Remark 3.1. *The toy model described above is not the vortex-wave model which describes the evolution of a mixture of smooth vorticity and discrete vorticity, see [22]. In the vortex-wave system, the point vortices are perturbed by the smooth vorticity ω. More precisely, the equation for ω does not change but the equations of the point-vortices are modified by replacing the terms*

$$m_1 \frac{(z_j - z_1)^\perp}{2\pi |z_j - z_1|^2}$$

from (2.2) *by*

$$\int_{\mathbb{R}^2} \frac{(z_j - y)^\perp}{2\pi |z_j - y|^2} \omega(y)\, dy.$$

We will prove the following theorem:

Theorem 3.1. *Assume that $\alpha < \frac{1}{2}$. If ε is small enough and $\|\omega_0\|_{L^\infty}$ not too large, then the vorticity stays localized close to z_1 for future times. More precisely, for any $\beta \in \left(\frac{\alpha+1}{3}, \frac{1}{2}\right)$ and for any $N_0 \in \mathbb{N}$ there exists some $M > 0$ and $\varepsilon_0 > 0$ such that if $\|\omega_0\|_{L^\infty} \leqslant \varepsilon^{-N_0}$ and $\varepsilon \leqslant \varepsilon_0$ then*

$$\text{supp } \omega(\cdot, t) \subset D(z_1(t), M(1+t)^\beta) \quad \forall t \geqslant 0.$$

We prove now this result. We will denote by C a generic constant which does not depend on ε, t and n. Changing if necessary the time scale, we can assume that $a = 1$ so that

$$\left| \frac{\partial \overline{F}}{\partial z}(z_1) \right| = \frac{\alpha}{1+t} \tag{3.1}$$

We denote by κ_0 the minimum distance between the point vortices at the initial time so that

$$|z_j(t) - z_k(t)| \geqslant \kappa_0 (1+t)^{\frac{1}{2}} \qquad \forall j \neq k,\ t \geqslant 0.$$

We choose some $M \in (0, \frac{\kappa_0}{2})$.

We will show that the following implication holds true for all times T: if

$$\text{supp } \omega(\cdot, t) \subset D(z_1(t), M(1+t)^\beta) \quad \forall t \in [0, T] \tag{3.2}$$

then

$$\text{supp } \omega(\cdot, t) \subset D(z_1(t), \frac{M}{2}(1+t)^\beta) \quad \forall t \in [0, T] \tag{3.3}$$

By time continuity and assuming $M > \varepsilon$, it follows that (3.2) must hold true for all times.

In what follows we assume (3.2) and we show (3.3). We prove first a bound on the moment of inertia of the vorticity with respect to z_1:

$$I(t) = \frac{1}{m_1} \int_{\mathbb{R}^2} |x - z_1|^2 \omega(x, t)\, dx$$

Lemma 3.1. *We have that*

$$I(t) \leqslant \varepsilon^2 (1+t)^{2\alpha} \exp\Big(\frac{4M \sum_{j \geqslant 2} |m_j|}{\pi \kappa_0^3 (1 - 2\beta)} \Big).$$

Proof. We differentiate I:

$$\begin{aligned} I'(t) &= \frac{1}{m_1}\int_{\mathbb{R}^2} |x-z_1|^2 \partial_t\omega(x,t)\,dx + \frac{2}{m_1}\int_{\mathbb{R}^2} z_1'(t)\cdot(z_1-x)\omega(x,t)\,dx \\ &= -\frac{1}{m_1}\int_{\mathbb{R}^2} |x-z_1|^2 \operatorname{div}((u+F)\omega)\,dx + \frac{2}{m_1}\int_{\mathbb{R}^2} z_1'(t)\cdot(z_1-x)\omega(x,t)\,dx \\ &= \frac{1}{m_1}\int_{\mathbb{R}^2} \nabla(|x-z_1|^2)\cdot(u+F)\omega\,dx + \frac{2}{m_1}\int_{\mathbb{R}^2} z_1'(t)\cdot(z_1-x)\omega(x,t)\,dx \\ &= \frac{2}{m_1}\int_{\mathbb{R}^2} (x-z_1)\cdot(u+F)\omega\,dx + \frac{2}{m_1}\int_{\mathbb{R}^2} z_1'(t)\cdot(z_1-x)\omega(x,t)\,dx. \end{aligned}$$

The usual cancellation properties imply that

$$\int_{\mathbb{R}^2} (x-z_1)\cdot u\;\omega\,dx = 0.$$

Recalling that $z_1' = F(z_1)$ and relation (3.2) we infer that

$$\begin{aligned} I'(t) &= \frac{2}{m_1}\int_{\mathbb{R}^2} (F(z_1)-F(x))\cdot(z_1-x)\omega(x,t)\,dx \\ &\leqslant 2I \sup_{|x-z_1|\leqslant M(1+t)^\beta} \frac{|F(x)-F(z_1)|}{|x-z_1|}. \end{aligned} \tag{3.4}$$

From (2.4), (2.5) and (3.1) and using complex notation (viewing everything as complex numbers) we observe that

$$\begin{aligned} \frac{|F(x)-F(z_1)|}{|x-z_1|} &= \Big|\frac{\overline{F}(x)-\overline{F}(z_1)}{x-z_1}\Big| \\ &= \Big|\sum_{j\geqslant 2} \frac{m_j}{2\pi(x-z_j)(z_1-z_j)}\Big| \\ &= \Big|\sum_{j\geqslant 2}\Big(\frac{m_j}{2\pi(z_1-z_j)^2} - \frac{m_j(x-z_1)}{2\pi(x-z_j)(z_1-z_j)^2}\Big)\Big| \\ &\leqslant \Big|\sum_{j\geqslant 2} \frac{m_j}{2\pi(z_1-z_j)^2}\Big| + \sum_{j\geqslant 2}\Big|\frac{m_j(x-z_1)}{2\pi(x-z_j)(z_1-z_j)^2}\Big| \\ &\leqslant \Big|\frac{\partial\overline{F}}{\partial z}(z_1)\Big| + \sum_{j\geqslant 2}\frac{|m_j||x-z_1|}{2\pi|x-z_j||z_1-z_j|^2} \\ &= \frac{\alpha}{1+t} + \sum_{j\geqslant 2}\frac{|m_j||x-z_1|}{2\pi|x-z_j||z_1-z_j|^2}. \end{aligned}$$

Recall that

$$|z_1-z_j| \geqslant \kappa_0(1+t)^{\frac12} \quad \forall j\geqslant 2.$$

Since $M\leqslant\frac{\kappa_0}{2}$, if $|x-z_1|\leqslant M(1+t)^\beta$ we have that

$$|x-z_1| \leqslant M(1+t)^\beta \leqslant \frac{\kappa_0}{2}(1+t)^{\frac12} \leqslant \frac{|z_1-z_j|}{2}$$

so

$$|x-z_j| \geqslant |z_1-z_j| - |x-z_1| \geqslant \frac{|z_1-z_j|}{2} \geqslant \frac{\kappa_0}{2}(1+t)^{\frac12}$$

We infer from the previous bounds that

$$\sup_{|x-z_1|\leqslant M(1+t)^\beta} \frac{|F(x)-F(z_1)|}{|x-z_1|} \leqslant \frac{\alpha}{1+t} + \frac{M\sum_{j\geqslant 2}|m_j|}{\pi\kappa_0^3}(1+t)^{\beta-\frac{3}{2}}. \tag{3.5}$$

Relation (3.4) implies that

$$I'(t) \leqslant 2I\Big(\frac{\alpha}{1+t} + \frac{M\sum_{j\geqslant 2}|m_j|}{\pi\kappa_0^3}(1+t)^{\beta-\frac{3}{2}}\Big).$$

The Gronwall lemma gives

$$I(t) \leqslant I(0)(1+t)^{2\alpha} e^{\frac{4M\sum_{j\geqslant 2}|m_j|}{\pi\kappa_0^3(1-2\beta)}}$$

Clearly

$$I(0) \leqslant \varepsilon^2$$

and the conclusion follows. □

Remark 3.2. *To prove this lemma we used the hypothesis* (3.2). *The important thing to note is that the bound found in Lemma 3.1 is better than the trivial bound obtained by using directly* (3.2) *in the formula of the moment of inertia.*

We continue with the proof of the theorem. Let Y be a point of $\operatorname{supp}\omega(\cdot,t)$ which is the farthest from z_1. The velocity of Y is $u(x,t)+F(Y)$. To estimate the evolution of

$$R(t) = \max_{x\in\operatorname{supp}\omega(t)} |x-z_1| = |Y-z_1|$$

we need to bound

$$\partial_t|Y-z_1| = \frac{Y-z_1}{|Y-z_1|}\cdot(\partial_t Y - z_1') = \frac{Y-z_1}{|Y-z_1|}\cdot(u(x,t)+F(Y)-F(z_1)).$$

From (3.5) we deduce that

$$\frac{Y-z_1}{|Y-z_1|}\cdot(F(Y)-F(z_1)) \leqslant |Y-z_1|\Big(\frac{\alpha}{1+t} + \frac{M\sum_{j\geqslant 2}|m_j|}{\pi\kappa_0^3}(1+t)^{\beta-\frac{3}{2}}\Big).$$

Next,

$$\begin{aligned}
\frac{Y-z_1}{|Y-z_1|}\cdot u(x,t) &= \frac{Y-z_1}{|Y-z_1|}\cdot\int_{\mathbb{R}^2}\frac{(Y-y)^\perp}{2\pi|Y-y|^2}\omega(y,t)\,dy \\
&= \frac{Y-z_1}{|Y-z_1|}\cdot\int_{|y-z_1|<R/2}\frac{(z_1-y)^\perp}{2\pi|Y-y|^2}\omega(y,t)\,dy \\
&\qquad + \frac{Y-z_1}{|Y-z_1|}\cdot\int_{|y-z_1|>R/2}\frac{(Y-y)^\perp}{2\pi|Y-y|^2}\omega(y,t)\,dy \\
&\leqslant \int_{|y-z_1|<R/2}\frac{|z_1-y|}{2\pi|Y-y|^2}\omega(y,t)\,dy + \int_{|y-z_1|>R/2}\frac{1}{2\pi|Y-y|}\omega(y,t)\,dy \\
&\leqslant \frac{2}{\pi R^2}\int_{|y-z_1|<R/2}|z_1-y|\omega(y,t)\,dy + C\|\omega\|_{L^\infty}^{\frac{1}{2}}\|\omega(y,t)\|_{L^1(|y-z_1|>R/2)}^{\frac{1}{2}}.
\end{aligned}$$

From the estimate of the moment of inertia given in the Lemma we can bound

$$\int_{\mathbb{R}^2} |z_1 - y|\omega(y,t)\,dy \leqslant \Big(\int_{\mathbb{R}^2} \omega\Big)^{\frac12}\Big(\int_{\mathbb{R}^2} |z_1 - y|^2\omega\Big)^{\frac12} \leqslant C\varepsilon(1+t)^{\alpha}.$$

We conclude from the previous relations that

$$\partial_t R \leqslant R\Big(\frac{\alpha}{1+t} + C(1+t)^{\beta-\frac32}\Big) + \frac{C\varepsilon}{R^2}(1+t)^{\alpha} + C\|\omega_0\|_{L^\infty}^{\frac12}\|\omega(y,t)\|_{L^1(|y-z_1|>R/2)}^{\frac12}. \tag{3.6}$$

It remains to bound $\|\omega(y,t)\|_{L^1(|y-z_1|>R/2)}$. This will be done by estimating the higher momenta of the vorticity. More precisely, let

$$I_n(t) = \int_{\mathbb{R}^2} |x - z_1|^n \omega(x,t)\,dx$$

We have the following proposition.

Proposition 3.1. *There exists a constant C such that*

$$I_n(t) \leqslant (Cn\varepsilon)^{\frac n3}(1+t)^{(\alpha+1)\frac n3}.$$

Proof. We derive I_n:

$$\begin{aligned}
I_n'(t) &= \int_{\mathbb{R}^2} |x-z_1|^n \partial_t\omega(x,t)\,dx + n\int_{\mathbb{R}^2} z_1'\cdot(z_1 - x)|x-z_1|^{n-2}\omega(x,t)\,dx\\
&= -\int_{\mathbb{R}^2} |x-z_1|^n \operatorname{div}((u+F)\omega)\,dx + n\int_{\mathbb{R}^2} F(z_1)\cdot(z_1-x)|x-z_1|^{n-2}\omega(x,t)\,dx\\
&= n\int_{\mathbb{R}^2} |x-z_1|^{n-2}(x-z_1)\cdot(u+F)\omega\,dx - n\int_{\mathbb{R}^2} F(z_1)\cdot(x-z_1)|z_1-x|^{n-2}\omega(x,t)\,dx\\
&= n\int_{\mathbb{R}^2} |x-z_1|^{n-2}(x-z_1)\cdot u\omega\,dx + n\int_{\mathbb{R}^2} (F(x)-F(z_1))\cdot(x-z_1)|x-z_1|^{n-2}\omega(x,t)\,dx
\end{aligned}$$

The last integral above can be bounded using relation (3.5):

$$n\int_{\mathbb{R}^2} (F(x)-F(z_1))\cdot(x-z_1)|x-z_1|^{n-2}\omega(x,t)\,dx \leqslant n\Big(\frac{\alpha}{1+t} + C(1+t)^{\beta-\frac32}\Big)I_n$$

We consider now the other integral:

$$\begin{aligned}
n\int_{\mathbb{R}^2} |x-z_1|^{n-2}(x-z_1)\cdot u\omega\,dx &= \frac{n}{2\pi}\iint \frac{(x-z_1)\cdot(x-y)^\perp}{|x-y|^2}|x-z_1|^{n-2}\omega(x,t)\omega(y,t)\,dx\,dy\\
&= \frac{n}{2\pi}\iint \frac{x\cdot(x-y)^\perp}{|x-y|^2}|x|^{n-2}f(x)f(y)\,dx\,dy
\end{aligned}$$

where we used the notation

$$f(x) = \omega(x, t+z_1).$$

Using the skew-symmetry of $\frac{x\cdot(x-y)^\perp}{|x-y|^2}$ when exchanging x and y we further deduce that

$$\begin{aligned}
n\int_{\mathbb{R}^2} |x-z_1|^{n-2}(x-z_1)\cdot u\omega\,dx &= \frac{n}{4\pi}\iint \frac{x\cdot(x-y)^\perp}{|x-y|^2}(|x|^{n-2}-|y|^{n-2})f(x)f(y)\,dx\,dy\\
&\leqslant Cn^2\iint \frac{|x\cdot(x-y)^\perp|(|x|^{n-3}+|y|^{n-3})}{|x-y|}f(x)f(y)\,dx\,dy\\
&\leqslant Cn^2\int |x|^{n-3}f(x)\,dx\int |y|f(y)\,dy\\
&= Cn^2\int |x-z_1|^{n-3}\omega(x,t)\,dx\int |y-z_1|\omega(y,t)\,dy
\end{aligned}$$

By the Hölder inequality and by Lemma 3.1 we have that

$$\int |x-z_1|^{n-3}\omega(x,t) \leqslant \Big(\int \omega(x,t)\Big)^{\frac{3}{n}}\Big(\int |x-z_1|^n\omega(x,t)\Big)^{1-\frac{3}{n}}$$

and

$$\int |z_1-y|\omega(y,t)\,dy \leqslant \Big(\int \omega(y,t)\,dy\Big)^{\frac12}\Big(\int |z_1-y|^2\omega(y,t)\,dy\Big)^{\frac12} \leqslant C\varepsilon(1+t)^{\alpha}.$$

Putting together the above estimates yields the following differential inequality for I_n:

$$I_n'(t) \leqslant \big(\frac{\alpha n}{1+t}+Cn(1+t)^{\beta-\frac32}\big)I_n + Cn^2\varepsilon(1+t)^{\alpha}I_n(t)^{1-\frac{3}{n}}.$$

This is a Bernoulli differential inequality. The function

$$g_n(t)=I_n(t)^{\frac{3}{n}}$$

verifies the following linear differential inequality:

$$g_n' \leqslant g_n\big(\frac{3\alpha}{1+t}+C(1+t)^{\beta-\frac32}\big)+Cn\varepsilon(1+t)^{\alpha}.$$

Observing that $g_n(0)\leqslant C\varepsilon^3$, the Gronwall inequality gives

$$g_n(t)\leqslant Cn\varepsilon(1+t)^{\alpha+1}+C\varepsilon^3(1+t)^{3\alpha}\leqslant Cn\varepsilon(1+t)^{\alpha+1}$$

where we used that $\alpha<\frac12$ and $\varepsilon\leqslant 1$. Since $I_n=g_n^{\frac{n}{3}}$, this completes the proof of the proposition. □

We can now estimate $\|\omega(x,t)\|_{L^1(|x-z_1|>R/2)}$. Starting from this point, the constant C may depend on n but remains independent of ε and t. We use Proposition 3.1 for $2n$ to bound

$$\|\omega(x,t)\|_{L^1(|x-z_1|>R/2)}\leqslant\Big(\frac{2}{R}\Big)^{2n}I_{2n}(t)\leqslant\frac{C\varepsilon^{\frac{2n}{3}}}{R^{2n}}(1+t)^{\frac{2n(\alpha+1)}{3}}.$$

We plug this bound in relation (3.6) to obtain

$$\partial_t R\leqslant R\Big(\frac{\alpha}{1+t}+C(1+t)^{\beta-\frac32}\Big)+\frac{C\varepsilon}{R^2}(1+t)^{\alpha}+\frac{C\varepsilon^{\frac{n}{3}-\frac{N_0}{2}}}{R^n}(1+t)^{\frac{n(\alpha+1)}{3}}.$$

where we used the hypothesis $\|\omega_0\|_{L^\infty}\leqslant\varepsilon^{-N_0}$. The function

$$g(t)=R(t)e^{-\int_0^t(\frac{\alpha}{1+s}+C(1+s)^{\beta-\frac32})ds}=R(t)(1+t)^{-\alpha}e^{\frac{2C}{1-2\beta}((1+t)^{\beta-\frac12}-1)}$$

verifies

$$g'(t)\leqslant\Big[\frac{C\varepsilon}{R^2}(1+t)^{\alpha}+\frac{C\varepsilon^{\frac{n}{3}-\frac{N_0}{2}}}{R^n}(1+t)^{\frac{n(\alpha+1)}{3}}\Big](1+t)^{-\alpha}e^{\frac{2C}{1-2\beta}((1+t)^{\beta-\frac12}-1)}. \tag{3.7}$$

Clearly

$$1\geqslant e^{\frac{2C}{1-2\beta}((1+t)^{\beta-\frac12}-1)}\geqslant e^{-\frac{2C}{1-2\beta}}$$

so the quantity $e^{\frac{2C}{1-2\beta}((1+t)^{\beta-\frac{1}{2}}-1)}$ is of the order of a constant for all times t and can effectively be neglected. So $g(t)$ and $R(t)(1+t)^{-\alpha}$ are of the same order:

$$g(t) \leqslant R(t)(1+t)^{-\alpha} \leqslant C_1 g(t) \tag{3.8}$$

where $C_1 = e^{\frac{2C}{1-2\beta}}$. We further deduce from (3.7) that

$$g' \leqslant \frac{C_2\varepsilon}{g^2}(1+t)^{-2\alpha} + \frac{C_2\varepsilon^{\frac{n}{3}-\frac{N_0}{2}}}{g^n}(1+t)^{\frac{n(1-2\alpha)}{3}-\alpha} \tag{3.9}$$

for some constant C_2. We show that this differential inequality implies the following bound for g:

$$g(t) < K_\varepsilon(1+t)^{\frac{1}{n+1}(n\frac{1-2\alpha}{3}+1)} \tag{3.10}$$

for some constant K_ε to be determined later.

We first impose that (3.10) holds true at the initial time. In view of (3.8) and recalling that $R(0) \leqslant \varepsilon$, the condition

$$K_\varepsilon > \varepsilon \tag{3.11}$$

ensures that (3.10) holds true at time $t = 0$. We assume this condition on K. If (3.10) does not globally hold true, let T^* be the first time when it breaks down. We show that, up to time T^*, we have that

$$\frac{\varepsilon}{g^2}(1+t)^{-2\alpha} \leqslant L_\varepsilon \frac{(1+t)^{\frac{n(1-2\alpha)}{3}}}{g^n} \tag{3.12}$$

and

$$\frac{\varepsilon^{\frac{n}{3}-\frac{N_0}{2}}}{g^n}(1+t)^{\frac{n(1-2\alpha)}{3}-\alpha} \leqslant L_\varepsilon \frac{(1+t)^{\frac{n(1-2\alpha)}{3}}}{g^n} \tag{3.13}$$

for some constant L_ε to be determined later.

Clearly (3.12) is equivalent to

$$g \leqslant \Big(\frac{L_\varepsilon}{\varepsilon}\Big)^{\frac{1}{n-2}}(1+t)^{\frac{1}{n-2}(\frac{n(1-2\alpha)}{3}+2\alpha)}$$

which follows from (3.10) if we assume that

$$K_\varepsilon \leqslant \Big(\frac{L_\varepsilon}{\varepsilon}\Big)^{\frac{1}{n-2}}. \tag{3.14}$$

Indeed, one can readily check that

$$\frac{1}{n+1}\Big(n\frac{1-2\alpha}{3}+1\Big) < \frac{1}{n-2}\Big(\frac{n(1-2\alpha)}{3}+2\alpha\Big).$$

Relation (3.13) is obvious if

$$L_\varepsilon > \varepsilon^{\frac{n}{3}-\frac{N_0}{2}}. \tag{3.15}$$

We deduce from (3.9), (3.12) and (3.13) that

$$g' \leqslant 2C_2L_\varepsilon\frac{(1+t)^{\frac{n(1-2\alpha)}{3}}}{g^n} \qquad \forall t \in [0, T^*].$$

Upon integration

$$
\begin{aligned}
g(t)^{n+1} &\leqslant g(0)^{n+1} + 2C_2L_\varepsilon \frac{n+1}{\frac{n(1-2\alpha)}{3}+1}(1+t)^{\frac{n(1-2\alpha)}{3}+1} \\
&\leqslant \varepsilon^{n+1} + \frac{6C_2L_\varepsilon}{1-2\alpha}(1+t)^{\frac{n(1-2\alpha)}{3}+1} \\
&\leqslant \Big(\varepsilon^{n+1} + \frac{6C_2L_\varepsilon}{1-2\alpha}\Big)(1+t)^{\frac{n(1-2\alpha)}{3}+1}
\end{aligned}
$$

so

$$
g(t) \leqslant \Big(\varepsilon^{n+1} + \frac{6C_2L_\varepsilon}{1-2\alpha}\Big)^{\frac{1}{n+1}}(1+t)^{\frac{1}{n+1}(\frac{n(1-2\alpha)}{3}+1)} \qquad \forall t \in [0, T^*].
$$

We infer that if we further assume that

$$
\Big(\varepsilon^{n+1} + \frac{6C_2L_\varepsilon}{1-2\alpha}\Big)^{\frac{1}{n+1}} < K_\varepsilon \tag{3.16}
$$

then relation (3.10) holds true at time T^*. This proves that the time T^* can't be finite.

We conclude that if the conditions (3.11), (3.14), (3.15) and (3.16) hold true, then (3.10) holds true globally. One can readily check that if ε is sufficiently small, then there exist some K_ε and L_ε verifying (3.11), (3.14), (3.15) and (3.16). Indeed, one can choose for instance

$$
L_\varepsilon = \varepsilon^{1+\frac{n}{4}}, \qquad K_\varepsilon = \varepsilon^{\frac{n}{4(n-2)}}, \qquad n > 12 + 6N_0 \tag{3.17}
$$

and, once n is fixed, an ε small enough such that

$$
\Big(\varepsilon^{n+1} + \frac{6C_2}{1-2\alpha}\varepsilon^{1+\frac{n}{4}}\Big)^{\frac{1}{n+1}} < \varepsilon^{\frac{n}{4(n-2)}}
$$

which is possible because

$$
(1+\frac{n}{4})\frac{1}{n+1} > \frac{n}{4(n-2)}.
$$

For this choice of ε, n, K_ε and L_ε relation (3.10) holds true. We deduce then from (3.8) that

$$
R(t) \leqslant C_1 g(t)(1+t)^\alpha \leqslant C_1 K_\varepsilon (1+t)^{\frac{1}{n+1}(n\frac{1-2\alpha}{3}+1)+\alpha}.
$$

Because

$$
\lim_{n\to\infty} \frac{1}{n+1}(n\frac{1-2\alpha}{3}+1)+\alpha = \frac{1+\alpha}{3}
$$

there exists some n such that

$$
\frac{1}{n+1}(n\frac{1-2\alpha}{3}+1)+\alpha \leqslant \beta
$$

(recall that $\beta > \frac{1+\alpha}{3}$). For such an n we have that

$$
R(t) \leqslant C_1 K_\varepsilon (1+t)^\beta
$$

and we proved (3.3) provided that

$$
C_1 K_\varepsilon \leqslant \frac{M}{2}.
$$

To conclude the proof of Theorem 3.1 it suffices to observe that, if ε is sufficiently small, then the value of K_ε given in (3.17) satisfies the above condition.

4 Comments about the condition imposed on the point vortices

A legitimate question is if there is a system of self-similar point vortices such that $\alpha < 1/2$ (otherwise our result is empty). We discuss how to realize this condition.

We know that if

$$m_1 m_2 + m_1 m_3 + m_2 m_3 = 0 \tag{4.1}$$

and

$$m_1 m_2 |z_1 - z_2|^2 + m_1 m_3 |z_1 - z_3|^2 + m_2 m_3 |z_2 - z_3|^2 = 0 \tag{4.2}$$

then $\omega = m_1 \delta_{z_1} + m_2 \delta_{z_2} + m_3 \delta_{z_3}$ defines a self-similar system of point vortices. We have

$$\frac{d}{dt}|z_2 - z_3|^2 = \frac{2Am_1}{\pi}\Big(\frac{1}{|z_1 - z_3|^2} - \frac{1}{|z_1 - z_2|^2}\Big) \tag{4.3}$$

where A is the area of the triangle produced by the three vortices with orientation, i.e., reckoned positive if (z_1, z_2, z_3) appear counterclockwise and negative if (z_1, z_2, z_3) appear clockwise.

We choose the following system of point vortices:

$$z_1(0) = 0,\ m_1 = 1, \qquad z_2(0) = \lambda - \mathrm{i},\ m_2 = -\frac{1}{\lambda^2}, \qquad z_3(0) = \lambda,\ m_3 = \frac{1}{\lambda^2 - 1}$$

where λ is a large positive number. One can check that conditions (4.1) and (4.2) are verified, so this is a self-similar system of point vortices.

With the notations from Section 2, we have that

$$|z_2(t) - z_3(t)|^2 = |f(t)|^2 |z_2(0) - z_3(0)|^2 = 1 + at$$

so

$$\frac{d}{dt}|z_2 - z_3|^2 = a$$

We deduce from (4.3) that

$$a = \frac{2A(0)m_1}{\pi}\Big(\frac{1}{|z_1(0) - z_3(0)|^2} - \frac{1}{|z_1(0) - z_2(0)|^2}\Big) = \frac{1}{\pi\lambda(\lambda^2 + 1)}.$$

Next, we use the formula for v given in (2.6) to deduce that

$$v = \frac{\mathrm{i}m_2}{2\pi(z_1(0)^2 - z_2(0)^2)} + \frac{\mathrm{i}m_3}{2\pi(z_1(0)^2 - z_3(0)^2)} = -\frac{1}{\pi\lambda(\lambda - \mathrm{i})^2(\lambda^2 - 1)}$$

We conclude that

$$\alpha = \frac{|v|}{a} = \frac{1}{\lambda^2 - 1} \to 0 \qquad \text{as } \lambda \to +\infty.$$

Therefore α can be made as small as we want.

Now that we convinced ourselves that the sufficient condition $\alpha < \frac{1}{2}$ for confinement can be satisfied, we could ask ourselves if this condition is also necessary. We do not have an answer to this question but the following calculations could give us some indications.

Assume that $\alpha \geqslant \frac{1}{2}$ and that confinement occurs, *i.e.* $\operatorname{supp}\omega(\cdot,t) \subset D(z_1(t), M(1+t)^\beta)$ for some $\beta < \frac{1}{2}$. Let

$$X_1(t) = \frac{1}{m_1}\int x\omega(x,t)\,dx$$

be the center of mass of the vorticity ω and

$$h(t) = z_1 - X_1.$$

Then $h(t) = O\big((1+t)^\beta\big)$. Let us differentiate h:

$$\begin{aligned}
h'(t) &= z_1' - \frac{1}{m_1}\int x\partial_t\omega(x,t)\,dx \\
&= F(z_1) + \frac{1}{m_1}\int x\operatorname{div}\big[(u(x,t)+F(x))\omega(x,t)\big]\,dx \\
&= F(z_1) - \frac{1}{m_1}\int (u(x,t)+F(x))\omega(x,t)\big]\,dx \\
&= \frac{1}{m_1}\int (F(z_1)-F(x))\omega(x,t)\,dx
\end{aligned}$$

Recalling that $|x-z_1| \leqslant M(1+t)^\beta$ for $x\in\operatorname{supp}\omega$ we deduce as in the proof of Lemma 3.1 that

$$\overline{F}(z_1) - \overline{F}(x) = \frac{\partial\overline{F}}{\partial z}(z_1)(z_1-x) + O\big((1+t)^{\beta-\frac{3}{2}}\big)$$

so

$$\overline{h}'(t) = h\Big(\frac{\partial\overline{F}}{\partial z}(z_1) + O\big((1+t)^{\beta-\frac{3}{2}}\big)\Big) \equiv hu. \tag{4.4}$$

If $u = x+\mathrm{i}y$, then the equation for h can be written under the form of a system of 2 linear ODEs:

$$h' = Ah \qquad \text{where} \qquad A = \begin{pmatrix} x & -y \\ -y & -x\end{pmatrix}$$

When we compute the eigenvalues of A we find $\pm|u|$. Because

$$\Big|\frac{\partial\overline{F}}{\partial z}(z_1)\Big| \sim \frac{\alpha}{t}$$

as $t\to\infty$ and since $\beta < \frac{1}{2}$ we deduce that

$$|u| \sim \frac{\alpha}{t} \qquad \text{as } t\to\infty.$$

We infer that the largest eigenvalue of A is equivalent to $\frac{\alpha}{t}$, so one might expect the solution of $h' = Ah$ to behave, for at least some initial conditions, like t^α as $t\to\infty$ (which would contradict our assumptions). But this is probably not true. In fact, if we neglect the small remainder $O\big((1+t)^{\beta-\frac{3}{2}}\big)$ in (4.4), then one can find the exact asymptotic behavior of the solution. More precisely, making the change of variables $s = \ln(1+at)$ and using polar coordinates one can prove that the solution of

$$\overline{\ell}'(t) = \ell\,\frac{\partial\overline{F}}{\partial z}(z_1) \tag{4.5}$$

exhibits growth like $t^{\alpha\sqrt{1-b^2/|v|^2}}$ if $|b| < |v|$, logarithmic growth if $|b| = |v|$ and stays bounded if $|b| > |v|$.

The calculations given above does not show that the hypothesis $\alpha < \frac{1}{2}$ is optimal for confinement to occur. But they show that our method cannot yield a better result. Indeed, it is hard to see how to take into account the fine oscillations that occur in the solutions of system (4.5) and their interaction with the "small" remainder $O\big((1+t)^{\beta-\frac{3}{2}}\big)$, so we must contend ourselves with the upper bounds obtained through the eigenvalues of the matrix A.

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