TWO-DIMENSIONAL INCOMPRESSIBLE VISCOUS FLOW AROUND A SMALL OBSTACLE

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Abstract. In this work we study the asymptotic behavior of viscous incompressible 2D flow in the exterior of a small material obstacle. We fix the initial vorticity ω_0 and the circulation γ of the initial flow around the obstacle. We prove that, if γ is sufficiently small, the limit flow satisfies the full-plane Navier-Stokes system, with initial vorticity $\omega_0 + \gamma\delta$, where δ is the standard Dirac measure. The result should be contrasted with the corresponding inviscid result obtained by the authors in [15], where the effect of the small obstacle appears in the coefficients of the PDE and not only in the initial data. The main ingredients of the proof are $L^p - L^q$ estimates for the Stokes operator in an exterior domain, a priori estimates inspired on Kato's fixed point method, energy estimates, renormalization and interpolation.

Contents

1. Introduction

The purpose of this work is to study the influence of a material obstacle on the behavior of two-dimensional incompressible viscous flows when the size of the obstacle is small compared to that of a reference spatial scale. More precisely, we fix both an initial vorticity ω_0, smooth and compactly supported, and the circulation γ of the initial velocity around the boundary of the obstacle, while homothetically contracting the obstacle to a point P outside the support of ω_0. The initial vorticity ω_0 and the circulation γ uniquely determine a family of divergence-free initial velocities u_0^ε with curl $u_0^\varepsilon = \omega_0$ and $u_0^\varepsilon(x) \to 0$ at infinity; here ε denotes the size of the obstacle. The size of the support of the initial vorticity ω_0 can be used as reference spatial scale. Let $u^\varepsilon = u^\varepsilon(x,t)$ be a solution of the Navier-Stokes equations with initial data u_0^ε and no-slip data at the boundary of the small obstacle. Our problem is to determine the asymptotic behavior of u^ε as $\varepsilon \to 0$. We will show that u^ε converges to a solution of the Navier-Stokes equations in the full plane with initial vorticity $\omega_0 + \gamma\delta(x-P)$, as long as γ is sufficiently small. More precisely, we prove the following theorem.

Theorem 1. *Let $\Omega_\varepsilon = \varepsilon\Omega$ be a 2D simply-connected smooth obstacle, ω_0 a smooth function compactly supported in $\mathbb{R}^2 \setminus \{0\}$, independent of ε and γ a real number independent of ε. Consider the Navier-Stokes equations in the exterior of Ω_ε with homogeneous Dirichlet boundary conditions and assume that the initial velocity has vorticity ω_0 and circulation around the obstacle equal to γ. Let u^ε denote the corresponding global solution. There exists a constant $\gamma_0 > 0$ such that if $|\gamma| \leq \gamma_0$, then u^ε converges to the solution of the Navier-Stokes equations in $\mathbb{R}^2$ with initial vorticity given by $\omega_0 + \gamma\delta_0$.*

There is a sharp contrast between the behavior of ideal and viscous flows around a small obstacle. In [15], the authors studied the vanishing obstacle problem for incompressible, ideal, two-dimensional flow. The ideal flow assumption is physically incorrect in the presence of material boundaries, and part of the motivation for the present work (and of [15]) is to explore more precisely this incorrectness from a mathematical standpoint. The main result in [15] is that the limit vorticity in the ideal case satisfies a modified vorticity equation of the form $\omega_t + u \cdot \nabla\omega = 0$, with div $u = 0$ and curl $u = \omega + \gamma\delta(x-P)$. In other words, for ideal flow the correction due to the vanished obstacle appears as time-independent additional convection centered at P, whereas in the viscous case, the correction appears on the initial data and gets convected and diffused as it evolves.

The small obstacle limit is an instance of the general problem of PDE on singularly perturbed domains. There is a large literature on such problems, specially in the elliptic case, see [23] for a broad overview. Asymptotic behavior of fluid flow on singularly perturbed domains is a natural subject for analytical investigation which is virtually unexplored. The present work, together with [15], may be regarded as a first attempt to address this class of problems.

There is a natural connection between the approximation problem as we have formulated it and the issue of uniqueness for the limit problem. In fact, from a technical point of view, our work is closely related to the classical uniqueness result due to Y. Giga, T. Miyakawa and H. Osada, on solutions of the incompressible 2D Navier-Stokes equations with measures as initial data, see [14]. Some of the more striking similarities are: the difficulties with locally infinite kinetic energy, the use of L^p estimates for the linearized problem and the use of Kato-type norms to estimate the nonlinearity. The smallness condition on the mass of the point vortices in the initial data, required in the uniqueness result, is closely related to our smallness condition on the circulation.

The remainder of this work is organized in eleven sections. In Section 2 we summarize L^p estimates for the time-dependent Stokes problem on exterior domains. In Section 3 we formulate precisely the problem we wish to discuss and write uniform estimates for the initial data. In Section 4 we study the asymptotic behavior of the initial data. In Section 5 we discuss physical motivation for our problem and we establish the small obstacle asymptotics for circularly symmetric flows, a linear version of our problem. In Section 6 we derive a priori estimates in the initial layer for the nonlinear correction term. In Section 7 we deduce global-in-time energy estimates for the nonlinear correction term. In Section 8 we put together the estimates for the linear part with the estimates for the nonlinear correction, obtaining a complete set of a priori estimates for velocity. In Section 9 we prove compactness in space-time, in Section 10 we perform the passage to the limit, in Section 11 we discuss uniqueness for the limit problem and in Section 12 we add comments and concluding remarks.

We conclude this introduction with a few remarks regarding notation. Given a vector $z = (z_1, z_2) \in \mathbb{R}^2$ we denote its orthogonal vector by $z^\perp = (-z_2, z_1)$. We use the subscript c in function spaces to denote compact support, as in C_c^∞, and we use standard notation for Sobolev spaces, $W^{k,p}$, where $1 \leq p \leq \infty$ and $k \in \mathbb{Z}$, with H^k standing for the case $p = 2$. We use the subscript loc in function spaces X to denote functions which are locally in X. In particular, $L^p_{loc}([0,\infty); W^{k,q})$ denotes functions $f = f(t,x) \in L^p([0,M]; W^{k,q})$ for any $M > 0$, whereas $L^p_{loc}((0,\infty); W^{k,q})$

denotes functions $f = f(t,x) \in L^p([\delta, M]; W^{k,q})$ for any $\delta > 0$ and any $M > 0$, but not necessarily for $\delta = 0$. Finally, $L^{2,\infty}$ denotes the Lorentz space of functions f whose distribution function satisfies $\lambda_f = \lambda_f(s) = |\{|f| > s\}| = \mathcal{O}(s^{-2})$.

2. Estimates for the Stokes semigroup

In this section we will put together several results on estimates for the Stokes semigroup on exterior domains. Let us begin by introducing some basic notation.

Let Ω be a bounded, open, simply connected subset of $\mathbb{R}^2$ with boundary Γ, a smooth Jordan curve. We denote by Π the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma$. Fix $\nu > 0$ and let $\mathbb{P}$ denote the Leray projector onto divergence-free vector fields on Π. Let $\mathbb{A} \equiv -\mathbb{P}\Delta$ be the Stokes operator on Π and denote the Stokes semigroup by $S_\nu(t) = e^{-\nu t \mathbb{A}}$. Given $v_0 \in C_c^\infty(\Pi)$, let $v(t,x) = S_\nu(t)v_0$ be the unique solution of the system

$$\begin{cases} \partial_t v - \nu \Delta v = -\nabla p, & \text{in } (0,\infty) \times \Pi \\ \text{div } v = 0, & \text{in } [0,\infty) \times \Pi \\ v = 0, & \text{on } (0,\infty) \times \Gamma \\ \lim_{|x|\to\infty} v(t,x) = 0, & \text{for all } t \geq 0 \\ v(0,x) = v_0(x), & \text{on } \{t = 0\} \times \Pi. \end{cases} \tag{2.1}$$

We denote by $X^p(\Pi)$ the closure of the space of divergence-free, $C_c^\infty(\Pi)$ vector fields with respect to the L^p-norm. The Stokes operator in X^p generates an analytic semigroup of class C^0 on $X^p(\Pi)$, for any $1 < p < \infty$, see [13], so that, in particular, problem (2.1) is well-posed in $X^p(\Pi)$.

We will require two kinds of estimates on the Stokes semigroup, L^p estimates and renormalized energy estimates. We first state the L^p estimates.

Theorem 2. *Let $1 < q < \infty$. Consider $v_0 \in X^q(\Pi)$ and $F \in L^q(\Pi; M_{2\times 2}(\mathbb{R}))$. Then we have the following estimates.*

(S1) *Let $q \leq p < \infty$. There exists $K_1 = K_1(\Pi, p, q) > 0$ such that*

$$\|S_\nu(t)v_0\|_{L^p} \leq K_1 (\nu t)^{\frac{1}{p} - \frac{1}{q}} \|v_0\|_{L^q},$$

for all $t > 0$.

(S2) *Let $q \leq p \leq 2$. There exists $K_2 = K_2(\Pi, p, q) > 0$ such that*

$$\|\nabla S_\nu(t)v_0\|_{L^p} \leq K_2 (\nu t)^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \|v_0\|_{L^q},$$

for all $t > 0$.

(S3) *Assume $q \geq 2$ and let $q \leq p < \infty$. Then there exists $K_3 = K_3(\Pi, p, q) > 0$ such that*

$$\|S_\nu(t)\,\mathbb{P}\ \text{div}\ F\|_{L^p} \leq K_3 (\nu t)^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \|F\|_{L^q},$$

for all $t > 0$, with the divergence taken along rows of the matrix F.

This theorem summarizes several results already contained in the literature, which we have collated above for convenience.

Proof. Estimates (S1) and (S2) were proved in [5, 20] (see also [6] for the case $p = \infty$). Estimate (S3) follows from (S2) by duality. Indeed, the adjoint of $S_\nu(t)$ on X^p is again $S_\nu(t)$, defined on $X^{p'}$, with $1/p+1/p' = 1$ and therefore the adjoint of $\nabla S_\nu(t)$ is $S_\nu(t)\,\mathbb{P}$ div . The dependence on the viscosity follows directly by rescaling time, since $S_\nu(t) = S_1(\nu t)$. □

Next we address a renormalized energy estimate for the Stokes semigroup. Our concerns include infinite energy solutions to the Navier-Stokes equations whose behavior at infinity is $\mathcal{O}(1/|x|)$. In the following result we will prove that solutions to the Stokes system retain the behavior at infinity of their initial data.

Proposition 3. *Let v_0 be a smooth divergence-free vector field on Π vanishing at the boundary Γ. We assume also that $v_0 \in X^p(\Pi)$ for some $p > 2$ and that $\nabla v_0 \in L^2(\Pi)$. Then*

$$S_\nu(t)v_0 - v_0 \in C^0([0,\infty); L^2(\Pi)) \cap L^2_{loc}([0,\infty); H^1(\Pi)).$$

Moreover the following inequality holds

$$\|S_\nu(t)v_0 - v_0\|^2_{L^2} + \nu \int_0^t \|\nabla[S_\nu(\tau)v_0 - v_0]\|^2_{L^2} d\tau \leq \nu t \|\nabla v_0\|^2_{L^2}. \tag{2.2}$$

Proof. Let $W = S_\nu(t)v_0 - v_0$. Then W satisfies:

$$\begin{cases} \partial_t W - \nu\Delta W = -\nabla p + \nu \Delta v_0, & \text{in } (0,\infty)\times\Pi \\ \text{div } W = 0, & \text{in } [0,\infty)\times\Pi \\ W = 0, & \text{on } (0,\infty)\times\Gamma \\ \lim_{|x|\to\infty} W(t,x) = 0, & \text{for all } t \geq 0 \\ W(0,x) = 0, & \text{on } \{t=0\}\times\Pi. \end{cases} \tag{2.3}$$

It is well-known that (2.3) admits a unique solution $\widetilde{W}$ in $C^0([0,\infty); L^2(\Pi)) \cap L^2_{loc}([0,\infty); H^1(\Pi))$, see, for instance, Theorem III.1.1 in [26]. The fact that $W - \widetilde{W} = 0$ follows from the well-posedness of (2.1) in X^p. The standard energy estimate gives (2.2). □

One consequence of the nontrivial topology of Π is the existence of harmonic vector fields, i.e. divergence-free and curl-free vector fields which are tangent to Γ and vanish at infinity. We denote by H_Π the unique harmonic vector field on the exterior domain Π which satisfies the condition

$$\oint_\Gamma H_\Pi \cdot ds = 1,$$

where the contour integral is taken in the counterclockwise sense. It is an elementary application of Hodge theory that the vector space of these harmonic vector fields on Π is one dimensional, and we can use H_Π as a basis. In the case where Π is the exterior of the unit disk centered at the origin, we will denote H_Π simply by H, and we have:

$$H = \frac{x^\perp}{2\pi|x|^2}. \tag{2.4}$$

We will require detailed information on the behavior of H_Π both at infinity and near the boundary Γ, which we obtain by means of a conformal mapping. We denote

$$\mathcal{U} \equiv \{|x| > 1\}$$

and switch to complex variables notation in the result below.

Lemma 4. *There exists a smooth biholomorphism $T : \Pi \to \mathcal{U}$, extending smoothly up to the boundary, mapping Γ to $\{|z| = 1\}$. Furthermore, there exists a nonzero real number β and a bounded holomorphic function $h : \Pi \to \mathbb{C}$ such that:*

$$T(z) = \beta z + h(z). \tag{2.5}$$

Additionally,

$$h'(z) = \mathcal{O}\left(\frac{1}{|z|^2}\right), \text{ as } |z| \to \infty. \tag{2.6}$$

This Lemma is an excerpt from [15]. Its proof is an exercise in complex analysis.

It was observed in [15] (see identity (2.10) in [15]) that

$$H_\Pi = H_\Pi(x) = \frac{1}{2\pi}\frac{DT^t(x)(T(x))^\perp}{|T(x)|^2}. \tag{2.7}$$

From Lemma 4, we see that $|H_\Pi|$ is $\mathcal{O}(1/|x|)$ for large $|x|$. This implies that H_Π belongs to the Lorentz space $L^{2,\infty}(\Pi)$.

We close this section with an estimate for the Stokes semigroup acting on infinite energy initial data.

Proposition 5. *Let $2 < p < \infty$ and let $v_0 \in L^{2,\infty}(\Pi) \cap X^p(\Pi)$. There exists a constant $K_5 > 0$ such that*

$$\|S_\nu(t)v_0\|_{L^p} \leq K_5(\nu t)^{\frac{1}{p}-\frac{1}{2}}\|v_0\|_{L^{2,\infty}}.$$

In particular, this estimate holds true for $v_0 = H_\Pi(x)$.

Proof. This estimate is contained in Proposition 2.2, item (4), of [17]. To see that it holds for H_Π, we first show that $H_\Pi \in X^p(\Pi)$ for any $p > 2$. This is easy to prove in the case $\Pi = \mathcal{U}$ because, for any function $\varphi \in C_c^\infty((0,\infty))$, $\varphi(|x|)H(x)$ is smooth, compactly supported and divergence-free, and, by taking φ^ε a sequence of cutoffs for the interval $(1+\varepsilon, 1/\varepsilon)$, it is easy to see that $\varphi^\varepsilon(|x|)H \to H$ in L^p, for $p > 2$. For general Π, we use the conformal mapping T, approximating H_Π by $\varphi^\varepsilon(|T(x)|)H_\Pi(x)$, where φ^ε is the same family of cutoffs used in the case of the exterior of the disk. This strategy works because $\varphi^\varepsilon(|T(x)|)H_\Pi(x)$ is also divergence-free. □

3. The evanescent obstacle

The purpose of this section is to set down a precise statement of the small obstacle problem. Many of the key issues regarding the small obstacle limit and incompressible flow have been discussed in detail in [15], so that we will focus on issues specifically related to viscous flow and briefly outline the rest.

As in [15], fix $\omega_0 \in C_c^\infty(\mathbb{R}^2)$ and assume that the origin does not belong to the support of ω_0. Let Ω be a bounded, open, connected and simply-connected subset of the plane whose boundary Γ is a C^∞ Jordan curve. The evanescent obstacle is the family of domains $\varepsilon\Omega$, with $0 < \varepsilon < \varepsilon_0$. The parameter ε_0 is chosen small enough so that the support of ω_0 does not intercept $\varepsilon\Omega$ for any $0 < \varepsilon < \varepsilon_0$.

Fix $0 < \varepsilon < \varepsilon_0$. Let $\Pi_\varepsilon \equiv \mathbb{R}^2 \setminus \overline{\varepsilon\Omega}$ and $\Gamma_\varepsilon = \partial\Pi_\varepsilon$. We use the conformal mapping $T : \Pi_1 \to \mathcal{U}$, given in Lemma 4, to define a family of smooth biholomorphisms

$$T^\varepsilon = T^\varepsilon(x) \equiv T\left(\frac{x}{\varepsilon}\right). \tag{3.1}$$

Throughout we write H^ε for H_{Π_ε} and $G^\varepsilon = G^\varepsilon(x,y)$ will be the Green's function of the Laplacian in Π_ε. Let $K^\varepsilon(x,y) = \nabla_x^\perp G^\varepsilon(x,y)$ be the kernel of the Biot-Savart law on Π_ε and denote the associated integral operator by $f \mapsto K^\varepsilon[f] = \int_{\Pi_\varepsilon} K^\varepsilon(x,y)f(y)\,dy$. Both K^ε and H^ε are related to $K_\mathcal{U}$ and $H_\mathcal{U}$ respectively, through the conformal mapping T^ε, in a way which was made explicit in [15]. The relevant fact is the way that both the Biot-Savart kernel and the basic harmonic vector field scale with ε, see identities (3.5) and (3.6) in [15].

Fix $\alpha \in \mathbb{R}$ and let

$$u_0^\varepsilon \equiv K^\varepsilon[\omega_0] + \alpha H^\varepsilon. \tag{3.2}$$

We consider the problem

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu\Delta u^\varepsilon = -\nabla p^\varepsilon, & \text{in } (0,\infty)\times\Pi_\varepsilon \\ \text{div } u^\varepsilon = 0, & \text{in } [0,\infty)\times\Pi_\varepsilon \\ u^\varepsilon = 0, & \text{on } (0,\infty)\times\Gamma_\varepsilon \\ \lim_{|x|\to\infty} u^\varepsilon(t,x) = 0, & \text{for all } t \geq 0 \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), & \text{on } \{t=0\}\times\Pi_\varepsilon. \end{cases} \tag{3.3}$$

We begin by observing that $u_0^\varepsilon \in L^{2,\infty}(\Pi_\varepsilon) \cap L^p(\Pi_\varepsilon)$ for any $2 < p \leq \infty$. Indeed, u_0^ε is smooth, and therefore locally bounded, so that we only require knowledge on the behavior of u_0^ε at infinity. By Lemma 4 and identity (2.7) $|H^\varepsilon|$ has $\mathcal{O}(1/|x|)$ behavior as $|x| \to \infty$, and therefore it belongs to $L^{2,\infty}(\Pi_\varepsilon) \cap L^p(\Pi_\varepsilon)$ for any $2 < p \leq \infty$. In fact, the $L^{2,\infty}$ bound on $|H^\varepsilon|$ is independent of ε, as can be readily seen by rescaling to a fixed domain and using the fact that H_Π belongs to $L^{2,\infty}$. In [15] it was shown that $|K^\varepsilon[\omega_0]|$ has behavior $\mathcal{O}(1/|x|^2)$ at infinity (see estimate (2.8) in [15]) and therefore it belongs to $L^p(\Pi_\varepsilon)$, for any $p \geq 2$, and, in particular, to $L^{2,\infty}(\Pi_\varepsilon)$.

Global-in-time well-posedness for problem (3.3) was established by Kozono and Yamazaki in [17]. The existence part of Kozono and Yamazaki's result requires that the initial velocity satisfy a smallness condition of the form

$$\limsup_{R\to\infty} R\,|\{x \in \Pi_\varepsilon \mid |u_0^\varepsilon(x)| > R\}|^{1/2} \ll 1.$$

Since u_0^ε is bounded, the limsup above is always zero, for any $\varepsilon > 0$. Uniqueness holds for divergence-free initial data in $L^{2,\infty}+X^p$ without any additional conditions.

The evanescent obstacle problem consists of understanding the asymptotic behavior of Kozono and Yamazaki's solution $u^\varepsilon(x,t)$ for small ε. More precisely we will show that, under appropriate assumptions, u^ε has a limit, and we will identify an equation satisfied by this limit.

Fix $\varphi : \mathbb{R} \to [0,1]$ a smooth, monotone function such that $\varphi(s) \equiv 0$ if $s \leq 2$ and $\varphi(s) \equiv 1$ if $s \geq 3$. For each $\varepsilon > 0$ and $\lambda > 0$ we introduce the adapted cut-off functions:

$$\varphi^{\varepsilon,\lambda}(x) \equiv \varphi\left(\frac{\varepsilon}{\lambda}|T^\varepsilon(x)|\right), \tag{3.4}$$

Note that the cutoff function $\varphi^{\varepsilon,\lambda}$ vanishes in a ball of radius $\mathcal{O}(\lambda)$ and it is identically equal to 1 outside a larger ball of radius $\mathcal{O}(\lambda)$, for large λ. Furthermore, the radii of the annulus where $\varphi^{\varepsilon,\lambda}$ is not constant can be made independent of ε. This follows easily from the fact that T is asymptotically affine at infinity, see (2.5).

We will now introduce a pair of parameters that are useful to describe the asymptotic behavior of u_0^ε when $\varepsilon \to 0$. Consider

$$m \equiv \int_{\mathbb{R}^2} \omega_0\,dx \qquad \text{and} \qquad \gamma \equiv \oint_{\Gamma_\varepsilon} u_0^\varepsilon \cdot ds. \tag{3.5}$$

By Stokes' Theorem we have that $\gamma = \alpha - m$, and therefore, the circulation γ does not the depend on ε, see the proof of Lemma 3.1 in [15].

For each $\lambda > 0$, we introduce a convenient decomposition of the initial velocity as

$$u_0^\varepsilon = \mathbf{b}_0^\varepsilon + \mathbf{i}_0^\varepsilon + \mathbf{o}_0^\varepsilon,$$

with

$$\mathbf{b}_0^\varepsilon \equiv K^\varepsilon[\omega_0] + m(1-\varphi^{\varepsilon,\lambda})H^\varepsilon,$$
$$\mathbf{i}_0^\varepsilon \equiv \gamma(1-\varphi^{\varepsilon,\lambda})H^\varepsilon,$$

and

$$\mathbf{o}_0^\varepsilon \equiv \alpha\varphi^{\varepsilon,\lambda}H^\varepsilon.$$

We need to understand the behavior of each of the components of this decomposition, in the limit $\varepsilon \to 0$. This is the content of our next result. The proof uses a large part of the work done in [15].

Lemma 6. *There exists $\lambda_0 > 0$, independent of ε, for which $\|\mathbf{b}_0^\varepsilon\|_{L^{p_1}}$, $\|\mathbf{i}_0^\varepsilon\|_{L^{p_2}}$ and $\|\mathbf{o}_0^\varepsilon\|_{L^{p_3}}$ are uniformly bounded in ε, for any $1 < p_1 \le \infty$, $1 \le p_2 < 2$ and $2 < p_3 \le \infty$. The vector fields $\mathbf{b}_0^\varepsilon$, $\mathbf{i}_0^\varepsilon$ and $\mathbf{o}_0^\varepsilon$ are divergence-free, the first two are tangent to Γ_ε and the last one vanishes on Γ_ε. Moreover, $\|\nabla\mathbf{o}_0^\varepsilon\|_{L^2(\Pi_\varepsilon)}$ is bounded independent of ε and*

$$\left\|\mathbf{o}_0^\varepsilon - \alpha\varphi\left(\frac{\beta|x|}{\lambda_0}\right)H\right\|_{L^2(\Pi_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0, \tag{3.6}$$

where $H = x^\perp/(2\pi|x|^2)$ and β is as in Lemma 4. We also have that

$$\|\mathbf{i}_0^\varepsilon\|_{L^2(\Pi_\varepsilon)} \le C|\log\varepsilon|^{\frac{1}{2}}.$$

Proof. Choose λ_0 such that the radii of the annulus where $\varphi^{\varepsilon,\lambda_0}$ is not constant are uniform in $\varepsilon < \varepsilon_0$.

The L^∞ bound on $\mathbf{b}_0^\varepsilon$ comes from Theorem 4.1 in [15]. The L^{p_1} bound on $\mathbf{b}_0^\varepsilon$, $1 < p_1 < \infty$ follows from the local L^∞ bound above, together with two facts: (1) $m(1-\varphi^{\varepsilon,\lambda_0})H^\varepsilon$ has support in a compact set independent of ε and (2) $|K^\varepsilon[\omega_0]| = \mathcal{O}(1/|x|^2)$ at infinity, uniformly in ε. As mentioned previously, fact (2) is estimate (2.8) in [15].

The L^{p_3} bound on $\mathbf{o}_0^\varepsilon$ follows from the fact that $\varphi^{\varepsilon,\lambda_0}$ is constant outside an annulus independent of ε, from formula (2.7), from the scaling $H^\varepsilon(x) = \frac{1}{\varepsilon}H_\Pi(x/\varepsilon)$ and from the behavior of T far from the obstacle given by Lemma 4.

For $\mathbf{i}_0^\varepsilon$, both the logarithmic estimate and the L^{p_2} estimate follow from adapting the argument used for estimate (3.7) of [15] in a straightforward manner.

To estimate $\nabla\mathbf{o}_0^\varepsilon$ we observe that $|\nabla\mathbf{o}_0^\varepsilon| = \mathcal{O}(1/|x|^2)$ near infinity, uniformly in ε. This estimate easily reduces to an estimate on DH^ε, which in turn reduces to calculating derivatives of the conformal mapping T using (2.5).

Finally, (3.6) reduces to showing that $H - H^\varepsilon$ goes to zero in L^2 near infinity, which can be done by a computation similar to the one carried out in the proof of Lemma 4.2 in [15].

□

In the remainder of this article, we will fix λ_0, independent of ε, as in Lemma 6, thereby fixing the bounded, inner and outer parts of the initial velocity, $\mathbf{b}_0^\varepsilon$, $\mathbf{i}_0^\varepsilon$ and $\mathbf{o}_0^\varepsilon$, respectively.

Let us denote the Stokes semigroup on Π_ε by $S_\nu^\varepsilon(t)$, so that $S_\nu^\varepsilon(t)[v_0^\varepsilon]$ is the solution to the Stokes system (2.1) on Π_ε with initial data v_0^ε. We introduce the notation $\tau^\varepsilon = \tau^\varepsilon(x) = \varepsilon x$, the contraction by ε. We observe the following fundamental relation between the Stokes system on Π_ε and on Π:

$$(S_\nu^1(t)[v_0^\varepsilon \circ \tau^\varepsilon])(x) = (S_\nu^\varepsilon(\varepsilon^2 t)[v_0^\varepsilon])(\varepsilon x), \text{ if } x \in \Pi. \tag{3.7}$$

Our strategy to study the small obstacle limit begins by considering the solution u^ε of (3.3) as a perturbation of $v^\varepsilon \equiv S_\nu^\varepsilon(t)u_0^\varepsilon$. The first thing we require is information on v^ε, which we deduce in the result below.

Lemma 7. *Let* $\mathbf{b}^\varepsilon \equiv S_\nu^\varepsilon(t)\mathbf{b}_0^\varepsilon$, $\mathbf{i}^\varepsilon \equiv S_\nu^\varepsilon(t)\mathbf{i}_0^\varepsilon$, $\mathbf{o}^\varepsilon \equiv S_\nu^\varepsilon(t)\mathbf{o}_0^\varepsilon$ *and let* $2 < p < \infty$. *Then there exists a constant* $K = K(p, \omega_0) > 0$ *such that for any* $\varepsilon > 0$ *we have:*

(i) $\|\mathbf{b}^\varepsilon\|_{L^p(\Pi_\varepsilon)} \leq K(\nu t)^{\frac{1}{p}-\frac{1}{2}}$,
(ii) $\|\mathbf{i}^\varepsilon\|_{L^p(\Pi_\varepsilon)} \leq K|\gamma|(\nu t)^{\frac{1}{p}-\frac{1}{2}}$,
(iii) $\|\mathbf{o}^\varepsilon\|_{L^p(\Pi_\varepsilon)} \leq K|\alpha|(\nu t)^{\frac{1}{p}-\frac{1}{2}}$.

Proof. By (3.7) we have that $\mathbf{b}^\varepsilon(\varepsilon^2 t, \varepsilon x) = (S_\nu^1(t)[\mathbf{b}_0^\varepsilon \circ \tau^\varepsilon])(x)$, for $x \in \Pi$. Now, by Theorem 2, item (S1), it follows that there exists $K_1 > 0$ such that

$$\|S_\nu^1(t)[\mathbf{b}_0^\varepsilon \circ \tau^\varepsilon]\|_{L^p(\Pi)} \leq K_1(\nu t)^{\frac{1}{p}-\frac{1}{2}}\|\mathbf{b}_0^\varepsilon \circ \tau^\varepsilon\|_{L^2(\Pi)}.$$

Item (i) above follows from this estimate, together with (3.7) and the fact that

$$\|\mathbf{b}_0^\varepsilon \circ \tau^\varepsilon\|_{L^2(\Pi)} = \frac{1}{\varepsilon}\|\mathbf{b}_0^\varepsilon\|_{L^2(\Pi_\varepsilon)} \leq C\frac{1}{\varepsilon},$$

where we have used Lemma 6 in the last inequality. Items (ii) and (iii) follow in an analogous manner using Proposition 5 together with the fact that

$$\|\mathbf{i}_0^\varepsilon \circ \tau^\varepsilon\|_{L^{2,\infty}(\Pi)} = \frac{1}{\varepsilon}\|\mathbf{i}_0^\varepsilon\|_{L^{2,\infty}(\Pi_\varepsilon)} \leq C\frac{|\gamma|}{\varepsilon}$$

and

$$\|\mathbf{o}_0^\varepsilon \circ \tau^\varepsilon\|_{L^{2,\infty}(\Pi)} = \frac{1}{\varepsilon}\|\mathbf{o}_0^\varepsilon\|_{L^{2,\infty}(\Pi_\varepsilon)} \leq C\frac{|\alpha|}{\varepsilon}.$$

We have used the scaling $H^\varepsilon(x) = (1/\varepsilon)H^1(x/\varepsilon)$ above, see identity (3.6) in [15]. □

Remark 8. Using the rescaling (3.7) we may deduce that the estimates (S1), (S2) and (S3) in Theorem 2 are valid in Π_ε with constants K_1, K_2 and K_3 independent of ε.

We will conclude this section with an observation on the amount of vorticity generated at the boundary in the initial layer. This is a "fixed ε" calculation, before we take the vanishing obstacle limit. We denote vorticity associated to the velocity u^ε, at time t, by $\omega^\varepsilon = \omega^\varepsilon(t,\cdot) \equiv$ curl $u^\varepsilon(t,\cdot)$. Let us recall the discussion of flow in an exterior domain found in [15]. It was shown there that, if div $u^\varepsilon(t,\cdot) = 0$, curl $u^\varepsilon(t,\cdot) = \omega^\varepsilon(t,\cdot)$, and if $u^\varepsilon(t,\cdot)$ is tangent to Γ_ε and vanishes at infinity, then there exists unique $a = a(t) \in \mathbb{R}$ such that one can write $u^\varepsilon(t,\cdot)$ as:

$$u^\varepsilon(t,\cdot) = K^\varepsilon[\omega^\varepsilon(t,\cdot)] + a(t)H^\varepsilon;$$

(see Section 3.1 of [15] for details). For the initial data (3.2) we have, of course, $a(0) = \alpha$. We will argue that $a(t) = \alpha$ for any $t > 0$. This fact relies on a result whose proof we defer to Section 8, see Corollary 17. Using the notation introduced above we have:

$$u^\varepsilon - \mathbf{o}_0^\varepsilon = K^\varepsilon[\omega^\varepsilon] + a(t)[1 - \varphi^{\varepsilon,\lambda_0}]H^\varepsilon + [a(t) - \alpha]\varphi^{\varepsilon,\lambda_0}H^\varepsilon.$$

It will be proved in Corollary 17 that $u^\varepsilon - \mathbf{o}_0^\varepsilon$ belongs to $L^\infty_{loc}([0,\infty); L^2(\Pi_\varepsilon))$, although the estimate blows up as $\varepsilon \to 0$. Next we recall that the harmonic vector field H^ε is smooth (because the conformal map T^ε is smooth and extends smoothly to Γ^ε) and has $\mathcal{O}(1/|x|)$ behavior near infinity. Therefore we find that $[1 - \varphi^{\varepsilon,\lambda_0}]H^\varepsilon \in L^2(\Pi_\varepsilon)$, but $\varphi^{\varepsilon,\lambda_0}H^\varepsilon \notin L^2(\Pi_\varepsilon)$. Hence, assuming that $K^\varepsilon[\omega^\varepsilon] \in L^\infty_{loc}([0,\infty); L^2(\Pi_\varepsilon))$, the only way for $u^\varepsilon - \mathbf{o}_0^\varepsilon$ to be square-integrable is for $a(t) - \alpha = 0$, as we wished.

Since the flow u^ε satisfies the no-slip condition at any positive time, the circulation around Γ_ε at $t > 0$ vanishes. We make use once more of Stokes' Theorem to conclude that $0 = a(t) - m^\varepsilon(t)$, where $m^\varepsilon(t) = \int_{\Pi_\varepsilon} \omega^\varepsilon(t,x)\,dx$. We can now account for the mass of vorticity produced at the boundary in the initial layer. We have:

$$m^\varepsilon(t) = \alpha.$$

4. Initial data asymptotics

The purpose of this section is to study the limit, as $\varepsilon \to 0$, of the initial velocity fields u_0^ε. We begin by introducing some notation.

For each function f defined on Π_ε, we introduce Ef, the extension of f to $\mathbb{R}^2$, by setting $Ef \equiv 0$ in $\varepsilon\Omega$.

Lemma 9. *If $f \in W^{1,1}_{loc}(\overline{\Pi}_\varepsilon)$ and if its trace vanishes on the boundary Γ_ε then $Ef \in W^{1,1}_{loc}$ and $E\nabla f = \nabla Ef$.*

The proof of this fact is elementary and we leave it to the reader. We will now introduce notation which will be used in the remainder of this paper. We denote by $\mathbb{P}$ the Leray projector on all of $\mathbb{R}^2$. Additionally, we introduce the cutoff

$$\eta^\varepsilon = \eta^\varepsilon(x) \equiv \varphi^{\varepsilon,\varepsilon}(x) = \varphi(|T^\varepsilon(x)|) = \varphi(|T(x/\varepsilon)|), \tag{4.1}$$

where T^ε, $\varphi^{\varepsilon,\varepsilon}$ and φ, were introduced in Section 3, see (3.1), (3.4). Note that there exists a constant $C > 0$ independent of ε such that $\eta^\varepsilon(x) \equiv 1$ in $\{|x| > C\varepsilon\}$.

Let

$$\mathcal{K}(x) = \frac{x^\perp}{2\pi|x|^2} \tag{4.2}$$

be the kernel of the Biot-Savart law in all of $\mathbb{R}^2$, $f \mapsto \mathcal{K} * f$. Note that we denoted the same vector field by H in (2.4). The different notations used for the same vector field are natural since $x^\perp/(2\pi|x|^2)$ plays two very different roles – one as the kernel for the Biot-Savart law for the full plane and another as a harmonic generator for the cohomology of the exterior of any disk centered at the origin.

Let u_0^ε be as in (3.2) and γ as in (3.5).

Lemma 10. *Let $u_0 = \mathcal{K} * \omega_0 + \gamma H$. Then we have that*

$$\mathbb{P}[\eta^\varepsilon E u_0^\varepsilon] \to u_0 \text{ in } \mathcal{D}'(\mathbb{R}^2)$$

as $\varepsilon \to 0$.

Proof. We split u_0^ε in a different way than before:

$$u_0^\varepsilon = (K^\varepsilon[\omega_0] + mH^\varepsilon) + \gamma H^\varepsilon \equiv v_0^\varepsilon + \gamma H^\varepsilon,$$

where m was defined in (3.5). By Lemma 4.2 in [15], $\eta^\varepsilon E H^\varepsilon \to H$ strongly in $L^1_{loc}(\mathbb{R}^2)$ as $\varepsilon \to 0$, and by Lemma 4.1, in [15] $\eta^\varepsilon E H^\varepsilon$ is divergence-free, so that $\mathbb{P}\eta^\varepsilon E H^\varepsilon = \eta^\varepsilon E H^\varepsilon$. Therefore,

$$\gamma\mathbb{P}[\eta^\varepsilon E H^\varepsilon] \to \gamma H \text{ in } \mathcal{D}'(\mathbb{R}^2).$$

All that remains to prove is that $\mathbb{P}[\eta^\varepsilon E v_0^\varepsilon] \to \mathcal{K} * \omega_0$ in $\mathcal{D}'(\mathbb{R}^2)$. To see this, we begin by observing that $\eta^\varepsilon E v_0^\varepsilon$ is uniformly bounded in $L^\infty(\mathbb{R}^2)$, see Theorem 4.1 in [15]. Furthermore, we have additional control over the behavior of $E v_0^\varepsilon$ at infinity, so that there exists a constant $C > 0$, independent of ε, such that $|\eta^\varepsilon E v_0^\varepsilon| \leq C/|x|$. This follows from the explicit expressions for K^ε, H^ε given in (3.5) and (3.6) of [15], from estimate (2.8) in [15] and from the compactness of the support of ω_0. Therefore, $\eta^\varepsilon E v_0^\varepsilon$ is also uniformly bounded in $L^p(\mathbb{R}^2)$ for all $p > 2$. Fix $p > 2$ and let $\zeta \in L^\infty(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ be a weak-$*$ limit of $\{\eta^\varepsilon E v_0^\varepsilon\}$.

Next we observe that div $\eta^\varepsilon E v_0^\varepsilon = \nabla\eta^\varepsilon \cdot E v_0^\varepsilon$ and curl $\eta^\varepsilon E v_0^\varepsilon = \nabla^\perp\eta^\varepsilon \cdot E v_0^\varepsilon + \eta^\varepsilon\omega_0$. The cutoff η^ε is such that $|\nabla\eta^\varepsilon|$ is bounded by C/ε and supported on a set of measure $C\varepsilon^2$. Thus, $\nabla\eta^\varepsilon$ (and $\nabla^\perp\eta^\varepsilon$) converges to zero strongly in $L^q(\mathbb{R}^2)$ for any $1 \leq q < 2$. Hence, div $\eta^\varepsilon E v_0^\varepsilon \to 0$ and curl $\eta^\varepsilon E v_0^\varepsilon \to \omega_0$ strongly in L^1. This, together with the convergence of a subsequence of $\eta^\varepsilon E v_0^\varepsilon$ to ζ weak in L^p implies that div $\zeta = 0$ and curl $\zeta = \omega_0$ in the sense of distributions. Using that $\zeta \in L^p$ for $p < \infty$, we obtain that $\zeta = \mathcal{K} * \omega_0$.

Since we identified the limit, we have actually proved that $\eta^\varepsilon E v_0^\varepsilon \rightharpoonup \mathcal{K} * \omega_0$ weakly in L^p, without the need to pass to subsequences. Therefore, as $\mathbb{P}$ is linear and continuous from L^p to itself, it follows that

$$\mathbb{P}[\eta^\varepsilon E v_0^\varepsilon] \rightharpoonup \mathbb{P}[\mathcal{K} * \omega_0] = \mathcal{K} * \omega_0,$$

which concludes the proof.

□

5. The impulsively stopped rotating cylinder

In this section we illustrate the physical meaning of the small obstacle problem by means of a concrete example.

Consider an infinite solid cylinder of radius $r > 0$ immersed in a viscous fluid occupying the whole space outside the cylinder. If the cylinder rotates with constant angular velocity λ, boundary friction will induce rotational motion in the surrounding fluid which, one expects, will settle to a steady flow with velocity u_0 of the form

$$u_0 = \frac{\lambda r^2 x^\perp}{|x|^2},$$

see [1] for a discussion of this example.

We consider viscous flow in the exterior of the cylinder with initial velocity u_0, imposing the standard no-slip condition $u = 0$ at $|x| = r$ for positive time. Physically, this corresponds to first "preparing" the initial data by rotating the cylinder for a long time, letting the flow settle into the steady configuration u_0, and then suddenly halting the motion of the cylinder at time $t = 0$. A shorthand description of this situation is that of the flow induced by an *impulsively stopped rotating cylinder*. The inconsistency between initial and boundary data in this problem generates a rather singular initial layer in the fluid motion. The symmetries of the problem allow us to reduce the equations to the Stokes equation in the exterior of a disk. One may find in [1] an explicit treatment of this problem, involving passing to polar coordinates, using separation of variables and expressing the solution by means of Fourier-Bessel integrals.

The problem we wish to consider is the small obstacle limit of the impulsively stopped rotating cylinder as posed above. This means that we consider $\Pi_\varepsilon = \{|x| >$

$\varepsilon\}$. We have $H^\varepsilon = x^\perp/(2\pi|x|^2)$, independent of ε. In the notation of the previous section we pick $\omega_0 = 0$ and

$$u_0^\varepsilon = \gamma \frac{x^\perp}{2\pi|x|^2}, \text{ in } \Pi_\varepsilon.$$

Note that, $\gamma = 2\pi\lambda\varepsilon^2$, so that fixing the circulation γ independent of ε means that the angular velocity λ of the obstacle must blow up as the obstacle becomes smaller.

We consider $u^\varepsilon = u^\varepsilon(x,t)$ and $p^\varepsilon = p^\varepsilon(x,t)$ solving (3.3) with initial data as above. It is a nice exercise, which we leave to the reader, to prove that the solution preserves circular symmetry. One consequence of circular symmetry is that $u^\varepsilon \cdot \nabla u^\varepsilon$ is a gradient field, so that we can absorb the nonlinearity into the pressure term. Therefore, u^ε satisfies the Stokes system on Π_ε.

We introduce the Lamb-Oseen vortex as the unique solution U of

$$\begin{cases} U_t + U\cdot\nabla U = -\nabla p + \nu\Delta U, & \text{in } (0,\infty)\times\mathbb{R}^2 \\ \operatorname{div} U = 0, & \text{in } [0,\infty)\times\mathbb{R}^2 \\ \lim_{|x|\to\infty} U(t,x) = 0, & \text{for all } t\geq 0 \\ U(0,x) = \frac{x^\perp}{2\pi|x|^2}, & \end{cases} \tag{5.1}$$

see [10, 12]. We have the following result.

Proposition 11. *Let $\gamma\in\mathbb{R}$. Then the extension of velocity, Eu^ε, converges weakly in $L^2_{loc}((0,\infty), L^2_{loc}(\mathbb{R}^2))$ to γU.*

This is a special case of our main result, so we do not include a proof here. We would like to use the impulsively stopped rotating cylinder as an illustration of what is taking place in the initial layer. We describe the events at time $t = 0$ for the flow generated by an impulsively stopped rotating cylinder noting that, for an (infinitesimally) small positive time, the fluid velocity vanishes at the boundary but has a nonvanishing limit as one approaches the boundary from inside the fluid. Tangential discontinuities in fluid velocity are called *vortex sheets* in hydrodynamics. The effect of impulsively stopping the rotation of the boundary amounts to placing a vortex sheet at the boundary and letting it diffuse into the bulk of the fluid through viscosity. The same rough picture describes what happens in the initial layer for our general problem.

6. Initial-layer and the nonlinear evolution

We have fixed an arbitrary initial vorticity ω_0 and a circulation γ, and hence we must understand solutions of the Navier-Stokes equations with initial data which does not satisfy the no-slip boundary condition. As we have seen in the previous Section, the effect of the consequent initial layer can be understood roughly as that of placing a vortex sheet at the boundary Γ_ε and letting it evolve, diffusing into the flow. The problem of resolving this initial layer for the Navier-Stokes system and obtaining uniform estimates for the small obstacle problem is rather delicate and it is the subject of the present section.

Let u^ε be the solution of the Navier-Stokes equations (3.3) with initial velocity u_0^ε given by (3.2) and let $v^\varepsilon = S_\nu^\varepsilon(t)[u_0^\varepsilon]$ as in Section 3. Let $W^\varepsilon \equiv u^\varepsilon - v^\varepsilon$. Let $\mathbb{P}^\varepsilon$ be the Leray projector on Π_ε. The evolution of W^ε is described by the following system:

$$W_t^\varepsilon - \nu \mathbb{P}^\varepsilon \Delta W^\varepsilon + \mathbb{P}^\varepsilon \text{ div } (W^\varepsilon \otimes W^\varepsilon + W^\varepsilon \otimes v^\varepsilon + v^\varepsilon \otimes W^\varepsilon + v^\varepsilon \otimes v^\varepsilon) = 0, \tag{6.1}$$

with the initial condition $W^\varepsilon(0,x) = 0$ and the boundary condition $W^\varepsilon = 0$ on Γ_ε.

We introduce the weighted-in-time norms. Let $p \geq 1$ and $f : (0,T) \to L^p(\Pi_\varepsilon)$ measurable. Let $T > 0$. We use the following notation:

$$\|f\|_{p,T} \equiv \sup_{0 \leq t \leq T} t^{\frac{1}{2} - \frac{1}{p}} \|f(t,\cdot)\|_{L^p(\Pi_\varepsilon)}.$$

The use of these norms for the Navier-Stokes equations was pioneered by H. Fujita and T. Kato, see for example [7].

Lemma 12. *Let $2 \leq p < \infty$. There exist positive constants C_0 and C_p such that, if $0 < T \leq C_0 \nu^3$ and $|\gamma| < C_0 \nu$ then*

$$\|W^\varepsilon\|_{p,T} \leq C_p \nu^{\frac{p+2}{2p}},$$

for every $0 < \varepsilon < \varepsilon_0$.

Proof. We use Duhamel's principle to write

$$W^\varepsilon(t) = -\int_0^t S_\nu^\varepsilon(t-\tau)\mathbb{P}^\varepsilon \text{ div } (W^\varepsilon \otimes W^\varepsilon + W^\varepsilon \otimes v^\varepsilon + v^\varepsilon \otimes W^\varepsilon + v^\varepsilon \otimes v^\varepsilon)(\tau)\, d\tau.$$

Take the L^p-norm and apply Theorem 2, estimate (S3), with $2 \leq q \leq p$ to obtain

$$\|W^\varepsilon\|_{L^p(\Pi_\varepsilon)} \leq \int_0^t \|S_\nu^\varepsilon(t-\tau)\mathbb{P}^\varepsilon \text{ div } (W^\varepsilon \otimes W^\varepsilon + W^\varepsilon \otimes v^\varepsilon + v^\varepsilon \otimes W^\varepsilon + v^\varepsilon \otimes v^\varepsilon)(\tau)\|_{L^p(\Pi_\varepsilon)}\, d\tau$$

$$\leq K_3 \int_0^t (\nu(t-\tau))^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} \|(W^\varepsilon \otimes W^\varepsilon + W^\varepsilon \otimes v^\varepsilon + v^\varepsilon \otimes W^\varepsilon + v^\varepsilon \otimes v^\varepsilon)(\tau)\|_{L^q(\Pi_\varepsilon)}\, d\tau$$

$$\leq K_3 \int_0^t (\nu(t-\tau))^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} (\|W^\varepsilon(\tau)\|_{L^{q_1}} \|W^\varepsilon(\tau)\|_{L^{q_2}} + \|W^\varepsilon(\tau)\|_{L^{q_1}} \|v^\varepsilon(\tau)\|_{L^{q_2}} + $$
$$+ \|v^\varepsilon(\tau)\|_{L^{q_1}} \|v^\varepsilon(\tau)\|_{L^{q_2}})\, d\tau,$$

where q_1 and q_2 are chosen so that $1/q = 1/q_1 + 1/q_2$ and where we have used Hölder's inequality. Next we use the definition of the (p,t)-norm to find

$$\|W^\varepsilon\|_{L^p(\Pi_\varepsilon)} \leq K_3(\|W^\varepsilon\|_{q_1,t}\|W^\varepsilon\|_{q_2,t} + \|W^\varepsilon\|_{q_1,t}\|v^\varepsilon\|_{q_2,t} +$$

$$+\|v^\varepsilon\|_{q_1,t}\|v^\varepsilon\|_{q_2,t})\int_0^t (\nu(t-\tau))^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}\tau^{\frac{1}{q}-1}\,d\tau.$$

We note that, for any $\alpha > -1$, $\beta > -1$, we have

$$\int_0^t (t-\tau)^\alpha \tau^\beta \, d\tau \leq C(\alpha,\beta)t^{\alpha+\beta+1}. \tag{6.2}$$

The proof of this inequality is an elementary calculation.

We wish to use (6.2) with $\alpha = -1/2 + 1/p - 1/q$ and $\beta = 1/q - 1$. Assume that:

$$2 \leq q \leq p. \tag{6.3}$$

Note that this condition implies

$$\alpha = -\frac{1}{2} + \frac{1}{p} - \frac{1}{q} > -1 \qquad \text{and} \qquad \beta = \frac{1}{q} - 1 > -1.$$

Therefore, we find

(6.4)

$$\|W^\varepsilon\|_{p,t} \leq C\nu^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}(\|W^\varepsilon\|_{q_1,t}\|W^\varepsilon\|_{q_2,t} + \|W^\varepsilon\|_{q_1,t}\|v^\varepsilon\|_{q_2,t} + \|v^\varepsilon\|_{q_1,t}\|v^\varepsilon\|_{q_2,t})$$

We divide the remainder of the proof in two steps: $p = 4$ and any $p \geq 2$.

First assume $p = 4$. Set $q_1 = q_2 = 4$, so that $q = 2$ and (6.3) is satisfied. In this situation, (6.4) gives that $X(t) \equiv \|W^\varepsilon\|_{4,t}$ satisfies

$$X^2 + \left(\|v^\varepsilon\|_{4,t} - \frac{\nu^{3/4}}{C}\right)X + \|v^\varepsilon\|_{4,t}^2 \geq 0. \tag{6.5}$$

Note that $X(0) = 0$ and the parabola described by (6.5) has, at $t = 0$, two distinct nonnegative roots. We observe that, *as long as this parabola has two distinct nonnegative roots* $r_1(t) < r_2(t)$, we have that inequality (6.5) together with the continuity in time of X, r_1 and r_2 imply that

$$0 \leq X(t) \leq r_1(t). \tag{6.6}$$

The condition for the polynomial above to have two distinct roots is

$$\left(\|v^\varepsilon\|_{4,t} - \frac{\nu^{3/4}}{C}\right)^2 - 4\|v^\varepsilon\|_{4,t}^2 > 0. \tag{6.7}$$

Since $\|v^\varepsilon\|_{4,t} \geq 0$ and $\nu^{3/4}/C > 0$ we find that (6.7) is equivalent to

$$\|v^\varepsilon\|_{4,t} < \frac{\nu^{3/4}}{3C}. \tag{6.8}$$

Furthermore, under the above assumption, the two distinct roots are also nonnegative. The $(4,t)$-norm is nondecreasing in t and hence, in order to guarantee that the polynomial in (6.5) have two distinct nonnegative roots, it is enough to verify (6.8) for $t = T$. Now we use the linear estimates from the previous section to find conditions under which (6.8) is valid at $t = T$.

First, recall that $v^\varepsilon = \mathbf{b}^\varepsilon + \mathbf{o}^\varepsilon + \mathbf{i}^\varepsilon$, and that both $\mathbf{b}_0^\varepsilon$ and $\mathbf{o}_0^\varepsilon$ belong to $L^p(\Pi_\varepsilon)$, for $p > 2$, with L^p-norms uniformly bounded in ε, see Lemma 6. We use estimate (S1) from Theorem 2 with $p = q = 4$, together with Lemma 7 to deduce:

$$\begin{aligned} \|v^\varepsilon\|_{4,T} &\leq \sup_{0\leq t\leq T} t^{\frac14}\|\mathbf{b}^\varepsilon(t) + \mathbf{o}^\varepsilon(t)\|_{L^4(\Pi_\varepsilon)} + \sup_{0\leq t\leq T} t^{\frac14}\|\mathbf{i}^\varepsilon(t)\|_{L^4(\Pi_\varepsilon)} \\ &\leq K_1 T^{\frac14}(\|\mathbf{b}_0^\varepsilon\|_{L^4(\Pi_\varepsilon)} + \|\mathbf{o}_0^\varepsilon\|_{L^4(\Pi_\varepsilon)}) + K|\gamma|\nu^{-\frac14} \\ &\leq C(T^{\frac14} + |\gamma|\nu^{-\frac14}). \end{aligned} \tag{6.9}$$

Choose $C_0 > 0$ so that the conditions

$$T \leq C_0\nu^3 \qquad \text{and} \qquad |\gamma| \leq C_0\nu \tag{6.10}$$

imply (6.8) with $t = T$.

Assuming now that (6.10) are valid, using (6.6) we have that

$$\|W^\varepsilon\|_{4,t} = X(t) \leq r_1(t) \leq C\nu^{3/4}, \tag{6.11}$$

for $0 \leq t \leq T$. This concludes the proof in the case $p = 4$.

For any $p \geq 2$ we bootstrap the $(4,T)$-estimate in the following way. We return to (6.4) and set $q_1 = q_2 = 4$. We then impose (6.10) to obtain

$$\|W^\varepsilon\|_{p,T} \leq C\nu^{-\frac12+\frac1p-\frac12}(\|W^\varepsilon\|_{4,T}^2 + \|W^\varepsilon\|_{4,T}\|v^\varepsilon\|_{4,T} + \|v^\varepsilon\|_{4,T}^2) \leq C(p)\nu^{\frac{p+2}{2p}}.$$

□

We conclude this section with the observation that

$$\|W^\varepsilon\|_{2,T} = \|W^\varepsilon\|_{L^\infty((0,T);L^2(\Pi_\varepsilon))},$$

so Lemma 12 actually provided a renormalized energy estimate on the initial layer.

7. Global-in-time nonlinear evolution

In the previous section we obtained a priori estimates for W^ε in the initial layer which are uniform in ε. We will now splice the information we already possess with a standard energy estimate, in order to obtain a result which is global in time. We retain the context introduced in the previous section.

Lemma 13. *Let $1 \leq p < 2$. Then $W^\varepsilon \in L^\infty_{loc}([0,\infty); L^2(\Pi_\varepsilon)) \cap L^2_{loc}((0,\infty); H^1(\Pi_\varepsilon))$, $W^\varepsilon \in L^p_{loc}([0,\infty); H^1(\Pi_\varepsilon))$, and the respective norms are bounded independently of ε.*

Remark 14. Note that the bound in $L^2_{loc}((0,\infty); H^1(\Pi_\varepsilon))$ means that W^ε is bounded in $L^2((\delta,T); H^1(\Pi_\varepsilon))$ for any $0 < \delta < T$, but not necessarily for $\delta = 0$.

Proof. We rewrite the evolution equation (6.1) for W^ε as

$$\begin{cases} W^\varepsilon_t - \nu\Delta W^\varepsilon + (W^\varepsilon + v^\varepsilon)\cdot\nabla W^\varepsilon + W^\varepsilon\cdot\nabla v^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon = -\nabla p, \text{ in } (0,\infty)\times\Pi_\varepsilon \\ \operatorname{div} W^\varepsilon = 0 \text{ in } [0,\infty)\times\Pi_\varepsilon \\ W^\varepsilon(0,\cdot) = 0 \text{ on } \{t=0\}\times\Pi_\varepsilon \\ W^\varepsilon(t,\cdot) = 0 \text{ on } [0,\infty)\times\Gamma_\varepsilon \end{cases}$$

We multiply the equation above by W^ε and integrate to obtain

$$\begin{aligned}\mathcal{E} \equiv \frac{1}{2}\frac{d}{dt}\|W^\varepsilon\|^2_{L^2} + \nu\|\nabla W^\varepsilon\|^2_{L^2} &= -\int_{\Pi_\varepsilon}[W^\varepsilon\cdot(W^\varepsilon\cdot\nabla v^\varepsilon) + W^\varepsilon\cdot(v^\varepsilon\cdot\nabla v^\varepsilon)]\,dx \\ &= \int_{\Pi_\varepsilon}[v^\varepsilon\cdot(W^\varepsilon\cdot\nabla W^\varepsilon) + v^\varepsilon\cdot(v^\varepsilon\cdot\nabla W^\varepsilon)]\,dx \\ &\leq \|W^\varepsilon\|_{L^4}\|\nabla W^\varepsilon\|_{L^2}\|v^\varepsilon\|_{L^4} + \|\nabla W^\varepsilon\|_{L^2}\|v^\varepsilon\|^2_{L^4}.\end{aligned}$$

We will use the following interpolation inequality:

$$\|W^\varepsilon\|_{L^4} \leq C\|W^\varepsilon\|^{1/2}_{L^2}\|\nabla W^\varepsilon\|^{1/2}_{L^2},$$

with a constant $C > 0$ independent of ε. This inequality in the case of $\mathbb{R}^2$ can be found in Chapter 1 of [18]. To obtain the corresponding inequality in Π_ε, one simply extends W^ε to $\mathbb{R}^2$ by setting it identically equal to zero inside $\varepsilon\Omega$. As W^ε vanishes on Γ_ε, the extension has H^1-norm in the plane identical to the H^1-norm of W^ε in Π_ε. Finally one uses the inequality in $\mathbb{R}^2$ on the extension.

We proceed with the estimate of $\mathcal{E}$:

$$\begin{aligned}\mathcal{E} &\leq C\|W^\varepsilon\|^{1/2}_{L^2}\|\nabla W^\varepsilon\|^{3/2}_{L^2}\|v^\varepsilon\|_{L^4} + \|\nabla W^\varepsilon\|_{L^2}\|v^\varepsilon\|^2_{L^4} \\ &\leq \frac{\nu}{2}\|\nabla W^\varepsilon\|^2_{L^2} + \frac{C}{\nu^3}\|W^\varepsilon\|^2_{L^2}\|v^\varepsilon\|^4_{L^4} + \frac{1}{\nu}\|v^\varepsilon\|^4_{L^4},\end{aligned}$$

where we used Young's inequality to estimate each of the products above. Next, we use Lemma 7 to deduce

$$\|v^\varepsilon\|^4_{L^4} \leq \frac{C}{\nu t}.$$

Hence,

$$\frac{d}{dt}\|W^\varepsilon\|^2_{L^2} + \nu\|\nabla W^\varepsilon\|^2_{L^2} \leq \frac{C}{\nu^4 t}\|W^\varepsilon\|^2_{L^2} + \frac{C}{\nu^2 t},$$

for some constant C independent of ε. Gronwall's inequality now gives, for any $0 < t_1 < t_2$,

$$\frac{\|W^\varepsilon(t_2,\cdot)\|^2_{L^2}}{t_2^{C/\nu^4}} + \nu\int_{t_1}^{t_2}\frac{\|\nabla W^\varepsilon(s,\cdot)\|^2_{L^2}}{s^{C/\nu^4}}\,ds \leq \frac{\nu^2}{t_1^{C/\nu^4}} - \frac{\nu^2}{t_2^{C/\nu^4}} + \frac{\|W^\varepsilon(t_1,\cdot)\|^2_{L^2}}{t_1^{C/\nu^4}}. \tag{7.1}$$

First choose $t_1 = C_0\nu^3/2$, with C_0 given in Lemma 12. It follows from Lemma 12 with $p = 2$ that

$$\|W^\varepsilon(t_1,\cdot)\|^2_{L^2} \leq C\nu^2.$$

Therefore,

$$\|W^\varepsilon(t,\cdot)\|_{L^2}^2 \le C\nu^2 \left(\frac{Ct}{\nu^3}\right)^{C/\nu^4}, \tag{7.2}$$

for any $t \ge t_1$, and we conclude that W^ε is uniformly bounded in $L^\infty_{loc}([0,\infty);L^2(\Pi_\varepsilon))$ as desired.

Next, we return to (7.1) for the derivative estimate. Let $a > 0$, multiply (7.1) by $t_1^{a+(C/\nu^4)-1}$ and integrate the resulting inequality with respect to t_1 from 0 to t_2. We obtain,

$$\int_0^{t_2} s^a \|\nabla W^\varepsilon(s,\cdot)\|_{L^2}^2\, ds \le \frac{a\nu^4 + C}{\nu^5}\left[\frac{\nu^2}{a} t_2^a + \int_0^{t_2} s^{a-1}\|W^\varepsilon(s,\cdot)\|_{L^2}^2\, ds\right]. \tag{7.3}$$

Since we already know that W^ε is uniformly bounded in $L^\infty_{loc}([0,\infty);L^2)$, this estimate implies that W^ε is bounded in $L^2_{loc}((0,\infty);H^1)$, uniformly in ε. Moreover, if $1 \le p < 2$, then the choice $a = (2-p)/2p$ above allows to conclude that W^ε is also bounded in $L^p_{loc}([0,\infty);H^1)$. □

8. Velocity estimates

In this section we derive global estimates on velocity using the analysis performed thus far. Before we begin, we require the following interpolation inequality.

Lemma 15. *Let $2 < p < \infty$, $q_0 = 2p/(p-2)$, $1 \le q \le q_0$. Let $r \ge q(p-2)/p$ and set*

$$\theta = \frac{2qr}{rp - q(p-2)}.$$

If $p > 2q$ assume further that $r \le q(p-2)/(p-2q)$. Then $\theta \ge 1$ and for any interval $I \subseteq \mathbb{R}$ and any $f \in L^r(I;H^1(\mathbb{R}^2)) \cap L^\theta(I;L^2(\mathbb{R}^2))$, we have

$$\|f\|_{L^q(I;L^p(\mathbb{R}^2))} \le C\|f\|_{L^r(I;H^1(\mathbb{R}^2))}^{(p-2)/p}\|f\|_{L^\theta(I;L^2(\mathbb{R}^2))}^{2/p}.$$

Proof. We start by recalling the following standard interpolation inequality: for any $g \in H^1(\mathbb{R}^2)$ we have

$$\|g\|_{L^{2/(1-s)}(\mathbb{R}^2)} \le C\|g\|_{H^s} \le C\|g\|_{L^2}^{1-s}\|g\|_{H^1}^s, \text{ for any } 0 \le s \le 1. \tag{8.1}$$

Fix exponents p, q, r and θ as in the statement of this lemma. Observe that, if $p > 2q$ then $\theta \ge 1$ if and only if $r \le q(p-2)/(p-2q)$; we hence assume this further restriction on r if $p > 2q$. In the other case, $p \le 2q$, there is no additional restriction on r to guarantee that $\theta \ge 1$.

Next, fix an interval $I \subseteq \mathbb{R}$ and let $f \in L^r(I; H^1(\mathbb{R}^2)) \cap L^\theta(I; L^2(\mathbb{R}^2))$. Let $s = (p-2)/p$, so that $2/(1-s) = p$. We use (8.1) and Hölder's inequality to obtain:

$$\begin{aligned}\|f\|^q_{L^q(I;L^p(\mathbb{R}^2))} = \int_I \|f(\tau,\cdot)\|^q_{L^p}\,d\tau \le C\int_I \|f(\tau,\cdot)\|^{q(p-2)/p}_{H^1}\|f(\tau,\cdot)\|^{2q/p}_{L^2}\,d\tau \\ \le C\|f\|^{q(p-2)/p}_{L^r(I;H^1(\mathbb{R}^2))}\|f\|^{2q/p}_{L^\theta(I;L^2(\mathbb{R}^2))},\end{aligned}$$

which concludes the proof. The condition $r \ge q(p-2)/p$ was used in Hölder's inequality when estimating the product of two functions in $L^{rp/q(p-2)}(I)$ and $L^{\theta p/(2q)}(I)$ above, so as to guarantee that $rp/q(p-2) \ge 1$.

□

Theorem 16. *Let u^ε be the solution of* (3.3) *with initial velocity u_0^ε as in* (3.2) *and recall that $u_0^\varepsilon = \mathbf{b}_0^\varepsilon + \mathbf{i}_0^\varepsilon + \mathbf{o}_0^\varepsilon$. Then the following hold true.*

1. *Let $2 < p < \infty$, $q_0 = 2p/(p-2)$ and $1 \le q < q_0$. Then $\{Eu^\varepsilon\}$ is bounded in $L^{q_0}_{loc}((0,\infty); L^p) \cap L^q_{loc}([0,\infty); L^p)$.*
2. *The family $\{Eu^\varepsilon - E\mathbf{o}_0^\varepsilon\}$ is bounded in $L^\infty_{loc}((0,\infty); L^2)$ and the family $\{Eu^\varepsilon - E\mathbf{o}_0^\varepsilon - E\mathbf{i}^\varepsilon\}$ is bounded in $L^\infty_{loc}([0,\infty); L^2)$.*
3. *The family $\{Eu^\varepsilon\}$ is bounded in $L^\infty_{loc}((0,\infty); L^2_{loc})$.*
4. *For any $1 \le p < 2$, we have $\{\nabla Eu^\varepsilon\}$ is bounded in $L^2_{loc}((0,\infty); L^2) \cap L^p_{loc}([0,\infty); L^2)$.*

Proof. Statement (1) involves two estimates: the first one on the open time interval $(0,\infty)$ and the second on the closed interval $[0,\infty)$. We begin by addressing the first estimate.

Fix $2 < p < \infty$. Fix $0 < \delta < T$ and set $I = (\delta, T)$. We first show that $\{Eu^\varepsilon - E\mathbf{o}_0^\varepsilon\}$ is bounded in $L^\infty((\delta,T); L^2(\mathbb{R}^2)) \cap L^2((\delta,T); H^1(\mathbb{R}^2))$. We write

$$Eu^\varepsilon - E\mathbf{o}_0^\varepsilon = E(\mathbf{i}^\varepsilon + \mathbf{b}^\varepsilon) + E(\mathbf{o}^\varepsilon - \mathbf{o}_0^\varepsilon) + EW^\varepsilon \equiv A_1 + A_2 + A_3. \tag{8.2}$$

We observe that A_1 is bounded in $L^\infty(I; H^1(\mathbb{R}^2))$. To see that, choose $1 < r < 2$ and use Theorem 2 together with Lemma 6 and Remark 8 to obtain

$$\begin{aligned}t^{\frac{1}{r}-\frac{1}{2}}\left(\|\mathbf{i}^\varepsilon\|_{L^2(\Pi_\varepsilon)} + \|\mathbf{b}^\varepsilon\|_{L^2(\Pi_\varepsilon)}\right) + t^{\frac{1}{r}}\left(\|\nabla\mathbf{i}^\varepsilon\|_{L^2(\Pi_\varepsilon)} + \|\nabla\mathbf{b}^\varepsilon\|_{L^2(\Pi_\varepsilon)}\right) \\ \le (K_1+K_2)\left(\|\mathbf{i}_0^\varepsilon\|_{L^r(\Pi_\varepsilon)} + \|\mathbf{b}_0^\varepsilon\|_{L^r(\Pi_\varepsilon)}\right) \le K(r),\end{aligned} \tag{8.3}$$

for some $K(r) > 0$, independent of ε. The estimate on A_1 follows from Lemma 9, together with the inequality above. For A_2, we use Proposition 3, together with Lemma 6 to conclude that

$$\|\mathbf{o}^\varepsilon(t,\cdot) - \mathbf{o}_0^\varepsilon\|^2_{L^2(\Pi_\varepsilon)} + \nu\int_0^t \|\nabla\mathbf{o}^\varepsilon(s,\cdot) - \nabla\mathbf{o}_0^\varepsilon\|^2_{L^2(\Pi_\varepsilon)}\,ds \le \nu t\|\nabla\mathbf{o}_0^\varepsilon\|^2_{L^2(\Pi_\varepsilon)} \le C, \tag{8.4}$$

for some $C > 0$ independent of ε. This, together with Lemma 9, implies that A_2 is uniformly bounded in $L^\infty(I; L^2(\mathbb{R}^2)) \cap L^2(I; H^1(\mathbb{R}^2))$. For the estimate on A_3, we simply use Lemma 13, together with Lemma 9, showing that A_3 is uniformly bounded in $L^\infty(I; L^2(\mathbb{R}^2)) \cap L^2(I; H^1(\mathbb{R}^2))$ as well.

We use Lemma 15 with $q = q_0$ and $r = 2$, so that $\theta = \infty$, to conclude that $\{Eu^\varepsilon - E\mathbf{o}_0^\varepsilon\}$ is bounded in $L^{q_0}(I; L^p(\mathbb{R}^2))$. Next we note that $\{E\mathbf{o}_0^\varepsilon\}$ is uniformly bounded in $L^p(\mathbb{R}^2)$ by Lemma 6, which concludes this portion of the proof.

We now address the second part of (1), which is an estimate on the closed time interval $[0, \infty)$. The difficulty here is that we do not have Leray-type estimates on the pieces of u^ε all the way down to $t = 0$, so that the result becomes more delicate.

Fix $2 < p < \infty$ and $1 \le q < q_0$. Let $T > 0$ and set $I = [0, T]$. We consider again the decomposition (8.2) and we estimate each piece. Estimate (8.3), together with Lemma 9 implies that A_1 is uniformly bounded on $L^{r_1}(I; H^1(\mathbb{R}^2)) \cap L^{\theta_1}(I; L^2(\mathbb{R}^2))$, for any $1 \le r_1 < 2$ and any $1 \le \theta_1 < \infty$. We use Lemma 15 with p and q as above. We need to find $r \in [1, 2)$ satisfying the restrictions in Lemma 15 in order to be able to use $r = r_1$. This is always possible because the restriction on r always includes $r \ge q(p-2)/p$, and $q < q_0$ is equivalent to $q(p-2)/p < 2$. This implies that A_1 is bounded in $L^q(I; L^p(\mathbb{R}^2))$. For A_2, we merely observe that (8.4) gives an uniform bound in $L^\infty(I; L^2(\mathbb{R}^2)) \cap L^2(I; H^1(\mathbb{R}^2))$ on A_2, which in turn yields the desired estimate. To treat A_3, we put together Lemma 13 and Lemma 9 to conclude that A_3 is uniformly bounded in $L^\infty(I; L^2(\mathbb{R}^2)) \cap L^{r_2}(I; H^1(\mathbb{R}^2))$, for any $1 \le r_2 < 2$. Clearly, this is enough to obtain the estimate in $L^q(I; L^p(\mathbb{R}^2))$ for A_3. The proof of (1) is concluded once we recall the observation that $\{E\mathbf{o}_0^\varepsilon\}$ is uniformly bounded in $L^p(\mathbb{R}^2)$, which we already used in the proof of the first part of (1).

We now address statement (2), which also consists of two estimates. The proof of the first estimate in (2) is contained in the proof of the first part of (1), the estimate on the open time interval. As for the second estimate in item (2), we write

$$Eu^\varepsilon - E\mathbf{o}_0^\varepsilon - E\mathbf{i}^\varepsilon = E\mathbf{b}^\varepsilon + E(\mathbf{o}^\varepsilon - \mathbf{o}_0^\varepsilon) + EW^\varepsilon.$$

We have already shown that the second and third terms in the decomposition above are bounded in $L^\infty_{loc}([0, \infty); L^2(\mathbb{R}^2))$. The first term satisfies

$$\|E\mathbf{b}^\varepsilon(t, \cdot)\|_{L^2} \le \|\mathbf{b}^\varepsilon(t, \cdot)\|_{L^2(\Pi_\varepsilon)} \le C\|\mathbf{b}_0^\varepsilon\|_{L^2(\Pi_\varepsilon)} \le C,$$

by Theorem 2 and Lemma 6.

The third item, statement (3), can be obtained from (2) by observing that, by Lemma 6 and Lemma 9, $E\mathbf{o}_0^\varepsilon$ is uniformly bounded in $L^r(\mathbb{R}^2)$, for any $r > 2$, which is contained in $L^2_{loc}(\mathbb{R}^2)$.

Statement (4) again consists of two estimates, one on the open time interval, the other on the closed interval. The estimates on the open time interval are trivially contained in the proof of the first estimate in item (1), once we observe that Lemma 6 and Lemma 9 give a uniform estimate in $L^2(\mathbb{R}^2)$ for $\nabla E\mathbf{o}_0^\varepsilon$. Similarly, the proof of the second part of (4) is contained in the proof of the second part of (1), together with the $L^2(\mathbb{R}^2)$ estimate for $\nabla E\mathbf{o}_0^\varepsilon$ which we have just derived.

This concludes the proof.

□

The last result in this section is an estimate for "fixed ε", which was already used to deduce that the amount of vorticity generated at the boundary in the initial layer, $m^\varepsilon(t) = \int_{\Pi_\varepsilon} \omega^\varepsilon(t,x)\,dx$ is the same for any ε and equals α.

Corollary 17. *For each fixed* $0 < \varepsilon < \varepsilon_0$ *we have:* $Eu^\varepsilon - E\mathbf{o}_0^\varepsilon \in L^\infty_{loc}([0,\infty); L^2(\mathbb{R}^2))$.

Proof. This is an immediate consequence of item (2) in Theorem 16, together with the observation that $E\mathbf{i}^\varepsilon \in L^\infty_{loc}([0,\infty); L^2(\Pi_\varepsilon))$ for each fixed ε. This last fact follows from the estimates on the Stokes semigroup in Theorem 2 and the fact that $E\mathbf{i}_0^\varepsilon \in L^2(\Pi_\varepsilon)$ with an L^2-norm that blows up as $|\log\varepsilon|^{\frac{1}{2}}$, see Lemma 6. □

9. Compactness in space-time

As far as a priori estimates go, the last ingredient we require is uniform control on how solutions evolve in time. This is often very easy to accomplish once spatial estimates are in place because ultimately the PDE itself is nothing more than an expression of time-derivatives of the solution in terms of spatial information. In our case, however, there are two difficulties that will make this step of the analysis somewhat involved: (i) the spatial estimates available are for Eu^ε, which does not satisfy a PDE: and (ii) the nonlinearity in our problem is quadratic, but the estimate up to time zero in Theorem 16 item (1) is L^p, $p > 2$, which entails problems at infinity. We deal with these difficulties through the following main ideas: we use the vorticity equation to describe the time evolution, we use the interplay of vorticity and velocity and we renormalize problem terms.

Let Φ be a smooth, compactly supported vector field and consider $\mathbb{P}$ the Leray projector for the plane. We consider the Hodge decomposition of the vector field Φ, given by $\Phi = \mathbb{P}\Phi + (\Phi - \mathbb{P}\Phi)$. The divergence-free part $\mathbb{P}\Phi$ is smooth, but not compactly supported. In fact,

$$|\mathbb{P}\Phi(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right), \text{ as } |x| \to \infty, \tag{9.1}$$

see the proof of Proposition 1.16 in [22]. Let $\psi = \psi(x)$ be the stream function associated with $\mathbb{P}\Phi$, so that $\nabla^\perp\psi = \mathbb{P}\Phi$. We assume that $\psi(0) = 0$, at the expense of having $\psi = \mathcal{O}(1)$ at ∞. Clearly, $|\psi(x)| \le |x| \|\nabla\psi\|_{L^\infty}$, so that using the Sobolev imbedding $H^2 \hookrightarrow L^\infty$ followed by the fact that $\nabla\psi = \nabla(\Delta)^{-1}\nabla^\perp \cdot \Phi$, a zeroth order singular integral operator acting on Φ, we obtain

$$|\psi(x)| \le C|x|\|\Phi\|_{H^2(\mathbb{R}^2)}. \tag{9.2}$$

We now observe that for $1 < q < \infty$ and $\chi \in C_c^\infty(\mathbb{R}^2)$, there exists a constant $C = C(q,\chi) > 0$ such that

$$\|\mathbb{P}[\chi\Phi]\|_{L^q(\mathbb{R}^2)} \le C\|\Phi\|_{H^2(\mathbb{R}^2)}. \tag{9.3}$$

Indeed, as $\mathbb{P}$ is a zero-th order singular integral operator we have

$$\|\mathbb{P}[\chi\Phi]\|_{L^q(\mathbb{R}^2)} \le C\|\chi\Phi\|_{L^q(\mathbb{R}^2)} \le C\|\chi\|_{L^q(\mathbb{R}^2)}\|\Phi\|_{L^\infty(\mathbb{R}^2)} \le C(q,\chi)\|\Phi\|_{H^2(\mathbb{R}^2)}.$$

Recall the extension operator E, introduced in beginning of Section 4. For each $\varepsilon > 0$, consider the cutoff η^ε introduced in (4.1). We are ready to state and prove the main result in this section.

Proposition 18. *The sequence $\{\mathbb{P}[\eta^\varepsilon E u^\varepsilon]\}$ is precompact in $L^\infty_{loc}([0,\infty); H^{-3}_{loc}(\mathbb{R}^2))$.*

Proof. Fix Φ a smooth, compactly supported vector field and let

$$\psi = \psi(x) = [(\Delta)^{-1} \text{ curl } \Phi](x) - [(\Delta)^{-1} \text{ curl } \Phi](0),$$

satisfying (9.2). For each $t \ge 0$, we introduce an auxiliary functional $F^\varepsilon = F^\varepsilon(t) \in H^{-2}(\mathbb{R}^2)$ defined by

$$\langle F^\varepsilon(t), \Phi\rangle = \int (Eu^\varepsilon(t,x) - E\mathbf{o}_0^\varepsilon(x)) \cdot (\nabla^\perp\eta^\varepsilon)(x)\psi(x)\,dx.$$

The proof will be divided into two steps. We will show that, for each $(t_1,t_2) \subset [0,\infty)$, we have $\{\mathbb{P}[\eta^\varepsilon E u^\varepsilon] + F^\varepsilon\}$ is bounded and equicontinuous as a function of (t_1,t_2) into $H^{-2}_{loc}(\mathbb{R}^2)$ and we will show that $F^\varepsilon \to 0$ strongly in $L^\infty_{loc}([0,\infty); H^{-2}(\mathbb{R}^2))$. The desired conclusion follows from these two steps by using Arzela-Ascoli's Theorem.

Let us begin by proving that $F^\varepsilon \to 0$ strongly in $L^\infty_{loc}([0,\infty); H^{-2}(\mathbb{R}^2))$. Indeed, we use Theorem 16, (9.2) and the properties of the cutoff η^ε to deduce

$$|\langle F^\varepsilon(t),\Phi\rangle| \le \|Eu^\varepsilon(t,\cdot) - E\mathbf{o}_0^\varepsilon\|_{L^2}\|\psi\nabla^\perp\eta^\varepsilon\|_{L^2} \tag{9.4}$$

$$\le C\|Eu^\varepsilon(t,\cdot) - E\mathbf{o}_0^\varepsilon\|_{L^2}\left(\int_{|x|<C\varepsilon} |x|^2\|\Phi\|^2_{H^2}|\nabla\eta^\varepsilon|^2\,dx\right)^{1/2}$$

$$\le C\varepsilon\|Eu^\varepsilon(t,\cdot) - E\mathbf{o}_0^\varepsilon\|_{L^2}\|\Phi\|_{H^2}$$

$$\le C\varepsilon(\|E\mathbf{i}^\varepsilon(t,\cdot)\|_{L^2} + \|Eu^\varepsilon(t,\cdot) - E\mathbf{o}_0^\varepsilon - E\mathbf{i}^\varepsilon(t,\cdot)\|_{L^2})\|\Phi\|_{H^2} \le C\varepsilon(C_1|\log\varepsilon|^{\frac{1}{2}} + C_2)\|\Phi\|_{H^2},$$

by Lemma 6, Theorem 2 and Theorem 16. Clearly, this proves our assertion. The proof of the other assertion is a bit more involved.

We introduce a cutoff for infinity. For each $R > 0$, let $\chi^R = \chi^R(x) = 1-\varphi(|x|/R)$. The vorticity $\omega^\varepsilon =$ curl u^ε satisfies the equation

$$\omega_t^\varepsilon + u^\varepsilon \cdot \nabla\omega^\varepsilon = \nu\Delta\omega^\varepsilon.$$

Let $0 \leq t_1 < t_2 < \infty$ and denote the interval $[t_1, t_2]$ by J. We multiply the vorticity equation by $\eta^\varepsilon\psi\chi^R$, integrate in space and time between t_1 and t_2, and integrate by parts to obtain

$$\int [Eu^\varepsilon(t_2,\cdot) - Eu^\varepsilon(t_1,\cdot)] \cdot \nabla^\perp(\eta^\varepsilon\psi\chi^R)\,dx \tag{9.5}$$

$$= \int_{t_1}^{t_2}\int (Eu^\varepsilon\cdot\nabla E\omega^\varepsilon)\eta^\varepsilon\psi\chi^R\,dxdt - \nu\int_{t_1}^{t_2}\int(\Delta E\omega^\varepsilon)\eta^\varepsilon\psi\chi^R\,dxdt \equiv I_1 - I_2.$$

We first estimate I_1. We integrate by parts and deduce

$$I_1 = -\int_{t_1}^{t_2}\int(Eu^\varepsilon\cdot\nabla\eta^\varepsilon)E\omega^\varepsilon\psi\chi^R\,dxdt + \int_{t_1}^{t_2}\int(Eu^\varepsilon\cdot(\mathbb{P}\Phi)^\perp)E\omega^\varepsilon\eta^\varepsilon\chi^R\,dxdt$$

$$-\int_{t_1}^{t_2}\int(Eu^\varepsilon\cdot\nabla\chi^R)E\omega^\varepsilon\psi\eta^\varepsilon\,dxdt \equiv -I_{11} + I_{12} - I_{13}.$$

Using Hölder's inequality first in space and then in time we have

$$|I_{11}| \leq \int_{t_1}^{t_2}\|Eu^\varepsilon\|_{L^4}\|E\omega^\varepsilon\|_{L^2}\|\psi\nabla\eta^\varepsilon\|_{L^4}\|\chi^R\|_{L^\infty}\,dt$$

$$\leq C\|\psi\nabla\eta^\varepsilon\|_{L^4}\|Eu^\varepsilon\|_{L^3(J;L^4)}\|E\omega^\varepsilon\|_{L^{9/5}(J;L^2)}|t_2-t_1|^{1/9} \leq C\|\Phi\|_{H^2}|t_2-t_1|^{1/9},$$

where in the last inequality we used Theorem 16 and we used again (9.2) along with the properties of η^ε.

Similarly, we have

$$|I_{12}| \leq \|\mathbb{P}\Phi\|_{L^4}\|Eu^\varepsilon\|_{L^3(J;L^4)}\|E\omega^\varepsilon\|_{L^{9/5}(J;L^2)}|t_2-t_1|^{1/9} \leq C\|\Phi\|_{H^2}|t_2-t_1|^{1/9},$$

since $\mathbb{P}$ is a zeroth order operator and $H^2 \hookrightarrow L^4$. Finally,

$$|I_{13}| \leq C\|\psi\eta^\varepsilon\|_{L^\infty}\|\nabla\chi^R\|_{L^4}\|Eu^\varepsilon\|_{L^3(J;L^4)}\|E\omega^\varepsilon\|_{L^{2/3}(J;L^2)} \leq C\|\Phi\|_{H^2}R^{-1/2},$$

as

$$|\psi(\infty)| = |[(\Delta)^{-1}\text{ curl }\Phi](0)| \leq C\|\Phi\|_{H^2}$$

and $\nabla\chi^R = \mathcal{O}(1/R)$, supported on a set of measure $\mathcal{O}(R^2)$.

Therefore,

$$\limsup_{R\to\infty}|I_1| \leq C\|\Phi\|_{H^2}|t_2-t_1|^{1/9}. \tag{9.6}$$

Next we treat I_2. We integrate by parts and use the fact that the supports of $\nabla\eta^\varepsilon$ and of $\nabla\chi^R$ are disjoint to obtain:

$$I_2 = \nu\int_{t_1}^{t_2}\int E\omega^\varepsilon(\Delta\eta^\varepsilon)\psi\chi^R\,dxdt + \nu\int_{t_1}^{t_2}\int E\omega^\varepsilon\eta^\varepsilon(\Delta\psi)\chi^R\,dxdt$$

$$+\nu \int_{t_1}^{t_2} \int E\omega^\varepsilon \eta^\varepsilon \psi (\Delta \chi^R)\, dxdt - 2\nu \int_{t_1}^{t_2} \int E\omega^\varepsilon (\chi^R \nabla \eta^\varepsilon + \eta^\varepsilon \nabla \chi^R)(\mathbb{P}\Phi)^\perp \, dxdt$$

$$= \nu I_{21} + \nu I_{22} + \nu I_{23} - 2\nu I_{24}.$$

By arguments similar to those used for I_1 we have

$$|I_{21}| \le \|\Delta \eta^\varepsilon \psi\|_{L^2} \|E\omega^\varepsilon\|_{L^{9/5}(J;L^2)} |t_2 - t_1|^{4/9} \le C\|\Phi\|_{H^2} |t_2 - t_1|^{4/9};$$

$$|I_{22}| \le \|\Delta \psi\|_{L^2} \int_{t^1}^{t_2} \|E\omega^\varepsilon\|_{L^2}\, dt \le C\| \text{ curl } \Phi\|_{L^2} |t_2 - t_1|^{4/9} \le C\|\Phi\|_{H^2} |t_2 - t_1|^{4/9};$$

$$|I_{23}| \le \|\Delta \chi^R\|_{L^2} \|\psi\|_{L^\infty} \int_{t_1}^{t_2} \|E\omega^\varepsilon\|_{L^2}\, dt \le C\|\Phi\|_{H^2};$$

$$|I_{24}| \le \|\mathbb{P}\Phi\|_{L^\infty} \|\chi^R \nabla \eta^\varepsilon + \eta^\varepsilon \nabla \chi^R\|_{L^2} \int_{t_1}^{t_2} \|E\omega^\varepsilon\|_{L^2}\, dt \le C\|\Phi\|_{H^2} |t_2 - t_1|^{4/9}.$$

Therefore,

$$\limsup_{R\to\infty} |I_2| \le C\|\Phi\|_{H^2} |t_2 - t_1|^{4/9}. \tag{9.7}$$

We expand the left hand side of identity (9.5) to find

$$\int [Eu^\varepsilon(t_2,\cdot) - Eu^\varepsilon(t_1,\cdot)] \cdot \nabla^\perp (\eta^\varepsilon \psi \chi^R)\, dx$$

$$= \int [Eu^\varepsilon(t_2,\cdot) - Eu^\varepsilon(t_1,\cdot)] \cdot (\nabla^\perp \eta^\varepsilon) \psi \chi^R \, dx + \int [Eu^\varepsilon(t_2,\cdot) - Eu^\varepsilon(t_1,\cdot)] \cdot \eta^\varepsilon (\nabla^\perp \psi) \chi^R \, dx$$

$$+ \int [Eu^\varepsilon(t_2,\cdot) - Eu^\varepsilon(t_1,\cdot)] \cdot \eta^\varepsilon \psi (\nabla^\perp \chi^R)\, dx = A_1 + A_2 + A_3.$$

We will show that each of the A_i's has a limit when $R \to \infty$. To see that, first note that

$$Eu^\varepsilon(t_2,\cdot) - Eu^\varepsilon(t_1,\cdot)$$

$$= [Eu^\varepsilon(t_2,\cdot) - E\mathbf{o}_0^\varepsilon - E\mathbf{i}^\varepsilon(t_2,\cdot)] - [Eu^\varepsilon(t_1,\cdot) - E\mathbf{o}_0^\varepsilon - E\mathbf{i}^\varepsilon(t_1,\cdot)] + E\mathbf{i}^\varepsilon(t_2,\cdot) - E\mathbf{i}^\varepsilon(t_1,\cdot),$$

which belongs to $L^2(\mathbb{R}^2)$, for each fixed $\varepsilon > 0$ and $0 \le t_1 < t_2 < \infty$. To see this note that the first two terms are bounded in L^2 by Theorem 16 whereas the last two terms were estimated in L^2, with a logarithmically growing norm as $\varepsilon \to 0$, in (9.4).

Therefore, since $\nabla^\perp \eta^\varepsilon$ and $\nabla^\perp \psi = \mathbb{P}\Phi$ are both square integrable functions, it follows by the Dominated Convergence Theorem that

$$\lim_{R\to\infty} A_1 = \int [Eu^\varepsilon(t_2,\cdot) - Eu^\varepsilon(t_1,\cdot)] \cdot (\nabla^\perp \eta^\varepsilon) \psi \, dx$$

and

$$\lim_{R\to\infty} A_2 = \int [Eu^\varepsilon(t_2,\cdot) - Eu^\varepsilon(t_1,\cdot)] \cdot \eta^\varepsilon (\nabla^\perp \psi)\, dx.$$

Furthermore, it is easy to see that $\nabla^{\perp}\chi^R$ converges to zero weakly in L^2 when $R \to \infty$. As $[Eu^\varepsilon(t_2,\cdot) - Eu^\varepsilon(t_1,\cdot)] \cdot \eta^\varepsilon \psi$ does not depend on R and belongs to L^2, we infer that $A_3 \to 0$ as $R \to \infty$.

We have found that the left hand side of identity (9.5) has a limit as $R \to \infty$. We can rewrite this limit as follows

$$\lim_{R\to\infty}(A_1 + A_2 + A_3) = \langle \mathbb{P}[\eta^\varepsilon Eu^\varepsilon](t_2,\cdot) + F^\varepsilon(t_2), \Phi\rangle - \langle \mathbb{P}[\eta^\varepsilon Eu^\varepsilon](t_1,\cdot) + F^\varepsilon(t_1), \Phi\rangle.$$

On the other hand, by identity (9.5), and using (9.6) and (9.7) we have

$$\lim_{R\to\infty}|A_1 + A_2 + A_3| \leq \limsup_{R\to\infty}|I_1| + |I_2| \leq C\|\Phi\|_{H^2}|t_2 - t_1|^{1/9}, \tag{9.8}$$

which shows that $\mathbb{P}[\eta^\varepsilon Eu^\varepsilon] + F^\varepsilon$ is equicontinuous as a function of time into H^{-2}.

We conclude this proof by showing that $\{\mathbb{P}[\eta^\varepsilon Eu^\varepsilon] + F^\varepsilon\}$ is uniformly bounded in $L^\infty(J; H^{-2}_{loc}(\mathbb{R}^2))$. We do not need to prove that $\{F^\varepsilon\}$ is bounded in this space because we have already shown that $F^\varepsilon \to 0$ as $\varepsilon \to 0$ in $L^\infty(J; H^{-2}(\mathbb{R}^2))$. The only thing left is to prove the boundedness of $\{\mathbb{P}[\eta^\varepsilon Eu^\varepsilon]\}$. To this end, let $\chi \in C_c^\infty(\mathbb{R}^2)$, fix $p > 2$, $1 < r < 2$ and write

$$|\langle \chi \mathbb{P}[\eta^\varepsilon Eu^\varepsilon], \Phi\rangle| = |\langle \eta^\varepsilon Eu^\varepsilon, \mathbb{P}[\chi\Phi]\rangle|$$

$$\leq |\langle \eta^\varepsilon(Eu^\varepsilon - E\mathbf{o}_0^\varepsilon - E\mathbf{i}^\varepsilon), \mathbb{P}[\chi\Phi]\rangle| + |\langle \eta^\varepsilon E\mathbf{o}_0^\varepsilon, \mathbb{P}[\chi\Phi]\rangle| + |\langle \eta^\varepsilon E\mathbf{i}^\varepsilon, \mathbb{P}[\chi\Phi]\rangle|$$

$$\leq \|Eu^\varepsilon - E\mathbf{o}_0^\varepsilon - E\mathbf{i}^\varepsilon\|_{L^2}\|\mathbb{P}[\chi\Phi]\|_{L^2} + \|E\mathbf{o}_0^\varepsilon\|_{L^p}\|\mathbb{P}[\chi\Phi]\|_{L^{p/(p-1)}} + \|E\mathbf{i}^\varepsilon\|_{L^r}\|\mathbb{P}[\chi\Phi]\|_{L^{r/(r-1)}}$$

$$\leq C(\chi, p, r, J)\|\Phi\|_{H^2},$$

where in the last inequality we used Theorem 2, Lemma 6, Theorem 16 and relation (9.3). Note that C is independent of $t \in J$.

It follows from Arzela-Ascoli that, for each $[t_1, t_2] \subset [0,\infty)$ and each $B_R \subset \mathbb{R}^2$, there is a subsequence of $\mathbb{P}[\eta^\varepsilon Eu^\varepsilon]$ which converges strongly in $L^\infty([t_1,t_2]; H^{-3}(B_R))$. By taking diagonal subsequences we may assume that there is a subsequence which converges strongly in $L^\infty_{loc}([0,\infty); H^{-3}_{loc}(\mathbb{R}^2))$.

This concludes the proof.

□

Remark 19. It follows from the proof of Proposition 18 that any strong limit of $\{\mathbb{P}[\eta^\varepsilon Eu^\varepsilon]\}$ in $L^\infty_{loc}([0,\infty); H^{-3}_{loc}(\mathbb{R}^2))$ in fact belongs to $C([0,\infty); H^{-3}_{loc}(\mathbb{R}^2))$. This is true as $\{\mathbb{P}[\eta^\varepsilon Eu^\varepsilon] + F^\varepsilon\}$ is equicontinuous and bounded as a function of time into H^{-2}_{loc} and therefore, by Arzela-Ascoli, its limits are continuous. Furthermore, $F^\varepsilon \to 0$, so that the limits of $\{\mathbb{P}[\eta^\varepsilon Eu^\varepsilon]\}$ and of $\{\mathbb{P}[\eta^\varepsilon Eu^\varepsilon] + F^\varepsilon\}$ are the same.

10. Passing to the limit

In this section we state and prove our main result. Let us begin with an improvement of the space-time compactness we have, which is a consequence of Proposition 18, obtained by means of interpolation.

Lemma 20. *The sequence* $\{Eu^\varepsilon\}$ *is precompact in* $L^2_{loc}((0,\infty)\times\mathbb{R}^2)$.

Proof. By Lemma 9, first order derivatives of functions that vanish on Γ_ε commute with the extension operator, and therefore, for any positive time, $Eu^\varepsilon = \mathbb{P}[Eu^\varepsilon]$. We write

$$Eu^\varepsilon = \mathbb{P}[(1-\eta^\varepsilon)Eu^\varepsilon] + \mathbb{P}[\eta^\varepsilon Eu^\varepsilon] \equiv B_1 + B_2.$$

First we note that $B_1 \to 0$ strongly in $L^2_{loc}([0,\infty)\times\mathbb{R}^2)$. Indeed, let us fix $0 \le t_1 < t_2 < \infty$ and set $J = [t_1, t_2]$. By Theorem 16, properties of the cutoff η^ε and the fact that the Leray projector is continuous from L^2 to itself we have

$$\|\mathbb{P}[(1-\eta^\varepsilon)Eu^\varepsilon]\|_{L^2(J\times\mathbb{R}^2)} \le C\|(1-\eta^\varepsilon)Eu^\varepsilon\|_{L^2(J\times\mathbb{R}^2)}$$

$$\le C\|1-\eta^\varepsilon\|_{L^4(\mathbb{R}^2)}\|Eu^\varepsilon\|_{L^2(J;L^4(\mathbb{R}^2))} \le C\varepsilon^{1/2},$$

which proves the desired estimate on B_1.

Next we work on B_2. We know from Proposition 18 that B_2 is precompact in $L^\infty_{loc}((0,\infty);H^{-3}_{loc}(\mathbb{R}^2))$. We will show that, for any $1 < q < 2$, B_2 is bounded in $L^2_{loc}((0,\infty);W^{1,q}_{loc}(\mathbb{R}^2))$. The result will follow by interpolation. Fix $1 < q < 2$ and let $q^* = 2q/(2-q) > 2$. By Theorem 16, $\{Eu^\varepsilon\}$ is bounded in $L^4_{loc}((0,\infty);L^4(\mathbb{R}^2))$. Since $|\eta^\varepsilon| \le 1$ and since $\mathbb{P}$ is continuous from $L^4(\mathbb{R}^2)$ into itself, it follows that B_2 is bounded in $L^4_{loc}((0,\infty);L^4(\mathbb{R}^2))$ which can be continuously imbedded into $L^2_{loc}((0,\infty);L^q_{loc}(\mathbb{R}^2))$.

What remains is to show that derivatives of B_2 are also uniformly bounded in $L^2_{loc}((0,\infty);L^q_{loc}(\mathbb{R}^2))$. Since the gradient and the Leray projector $\mathbb{P}$ are both Fourier multipliers, the gradient commutes with $\mathbb{P}$. Therefore,

$$D(\mathbb{P}[\eta^\varepsilon Eu^\varepsilon]) = \mathbb{P}[\eta^\varepsilon(DEu^\varepsilon)] + \mathbb{P}[(D\eta^\varepsilon)Eu^\varepsilon] \equiv B_{21} + B_{22}.$$

By Theorem 16, DEu^ε is bounded in $L^2_{loc}((0,\infty);L^2(\mathbb{R}^2))$. Since $|\eta^\varepsilon| \le 1$ and $\mathbb{P}$ is continuous from $L^2(\mathbb{R}^2)$ to itself, we immediately obtain the desired estimate for B_{21}. As for the term B_{22}, we use Theorem 16 once again to obtain that Eu^ε is bounded in $L^{2q^*/(q^*-2)}_{loc}((0,\infty);L^{q^*}(\mathbb{R}^2))$ and we recall that $D\eta^\varepsilon$ is uniformly bounded in $L^2(\mathbb{R}^2)$. With this, we have that $(D\eta^\varepsilon)Eu^\varepsilon$ is bounded in $L^{2q^*/(q^*-2)}_{loc}((0,\infty);L^q(\mathbb{R}^2))$, continuously imbedded into $L^2_{loc}((0,\infty);L^q(\mathbb{R}^2))$. This concludes the proof.

□

We will prove that limits of the sequence $\{Eu^\varepsilon\}$ are solutions of the Navier-Stokes equations in a suitable weak sense. To be precise, we formulate the notion of weak solution we will use.

Definition 21. *Let $u \in L^2_{loc}((0,\infty)\times\mathbb{R}^2)\cap C([0,\infty);\mathcal{D}'(\mathbb{R}^2))$. We say that u is a weak solution of the incompressible Navier-Stokes equations with initial velocity u_0 if, for any divergence-free test vector field $\Psi \in C^\infty_c((0,\infty)\times\mathbb{R}^2)$, we have*

$$\int_0^\infty \int_{\mathbb{R}^2} \big(u\cdot\Psi_t + [(u\cdot\nabla)\Psi]\cdot u + \nu u\cdot\Delta\Psi\big)\,dxdt = 0.$$

Furthermore, for every $t \geq 0$, $\operatorname{div} u(t,\cdot) = 0$ in the sense of distributions and $u(t,\cdot) \rightharpoonup u_0$ in the sense of distributions as $t \to 0^+$.

Recall that $\mathcal{K}$ denotes the kernel of the Biot-Savart law, as introduced in (4.2). We are finally ready to state and prove the main result of this work.

Theorem 22. *There exists $\gamma_0 > 0$ such that, if $|\gamma| < \gamma_0$, then any strong limit u of $\{Eu^\varepsilon\}$ in $L^2_{loc}((0,\infty)\times\mathbb{R}^2)$ is a weak solution of the incompressible Navier-Stokes equations in $\mathbb{R}^2$ with initial velocity given by $u_0 = \mathcal{K}*\omega_0 + \gamma H$.*

Remark 23. As $\{Eu^\varepsilon\}$ is precompact, by virtue of Lemma 20, there exists at least one such strong limit.

Proof. For each ε sufficiently small, choose $0 < \delta < 1$ such that $\{|x| > 2\delta\} \subseteq \Pi_\varepsilon$. Clearly, if $\{|x| > 2\delta\} \subseteq \Pi_{\varepsilon_0}$ then $\{|x| > 2\delta\} \subseteq \Pi_\varepsilon$, for all $\varepsilon \leq \varepsilon_0$. Also consider $R > 2 > 2\delta$. We use the cutoff φ introduced in Section 3 (see (3.4)) to define:

$$\varphi^\delta = \varphi^\delta(x) \equiv \varphi(|x|/\delta) \text{ and } \chi^R = \chi^R(x) \equiv 1 - \varphi(|x|/R).$$

As in the proof of Proposition 18 we let Φ be a smooth, compactly supported vector field in $\mathbb{R}^2$, which, in addition, we assume to be divergence-free. We define $\psi = \psi(x) = [(\Delta)^{-1} \text{ curl } \Phi](x) - [(\Delta)^{-1} \text{ curl } \Phi](0)$. Recall that $\nabla^\perp\psi = \Phi$ and that ψ satisfies (9.2). We also consider $\theta = \theta(t) \in C^\infty_c((0,\infty))$.

We use the test function $\varphi^\delta\theta\psi\chi^R$, which belongs to $C^\infty_c((0,\infty)\times\Pi_\varepsilon)$ in the weak form of the vorticity equation. We can rewrite the integrals on Π_ε as full plane integrals using the extension operator to obtain the following integral identity

$$\int_0^\infty \int_{\mathbb{R}^2} E\omega^\varepsilon\theta_t\varphi^\delta\psi\chi^R\,dxdt + \int_0^\infty \int_{\mathbb{R}^2} Eu^\varepsilon E\omega^\varepsilon\cdot\theta\nabla(\varphi^\delta\psi\chi^R)\,dxdt$$

$$+\nu\int_0^\infty \int_{\mathbb{R}^2} E\omega^\varepsilon\theta\Delta(\varphi^\delta\psi\chi^R)\,dxdt = 0. \tag{10.1}$$

Our first step is to pass to the limit $\varepsilon \to 0$ in this identity, while keeping δ and R fixed. Let u be a strong limit in $L^2_{loc}((0,\infty)\times\mathbb{R}^2)$ of a subsequence Eu^{ε_k} of Eu^ε. We observe that $E\omega^{\varepsilon_k} \to$ curl $u \equiv \omega$ strongly in $L^2_{loc}((0,\infty); H^{-1}_{loc}(\mathbb{R}^2))$. Similarly, we may also deduce that u is divergence-free in the sense of distributions. The passage to the limit is immediate in the linear terms of (10.1). For the nonlinear term we recall that, by Theorem 16, $\{E\omega^{\varepsilon_k}\}$ is uniformly bounded in $L^2_{loc}((0,\infty); L^2(\mathbb{R}^2))$. Hence a subsequence of $\{E\omega^{\varepsilon_k}\}$ converges weakly in $L^2_{loc}((0,\infty); L^2(\mathbb{R}^2))$. Using the convergence $E\omega^{\varepsilon_k} \to \omega$ strong in $L^2_{loc}((0,\infty); H^{-1}_{loc}(\mathbb{R}^2))$ and uniqueness of weak limits we conclude that $E\omega^{\varepsilon_k} \rightharpoonup \omega$ weakly in $L^2_{loc}((0,\infty); L^2(\mathbb{R}^2))$, without passing to further subsequences. Now $Eu^{\varepsilon_k} E\omega^{\varepsilon_k}$ is a weak-strong pair, so that we can pass to the limit in the nonlinear term as well. We arrive at the identity

$$J_1 + J_2 + J_3 \equiv \int_0^\infty \int_{\mathbb{R}^2} \omega\, \theta_t \varphi^\delta \psi \chi^R \, dxdt + \int_0^\infty \int_{\mathbb{R}^2} u\, \omega \,\cdot \theta \nabla(\varphi^\delta \psi \chi^R) \, dxdt$$

$$+\,\nu \int_0^\infty \int_{\mathbb{R}^2} \omega\, \theta \Delta(\varphi^\delta \psi \chi^R) \, dxdt = 0. \tag{10.2}$$

Now we pass to the limit both $\delta \to 0$ and $R \to \infty$ in each separate term in (10.2). We begin with J_1.

First we observe that

$$u - \alpha\varphi\left(\frac{\beta|x|}{\lambda_0}\right) H \equiv u - F \in L^\infty_{loc}((0,\infty); L^2(\mathbb{R}^2)), \tag{10.3}$$

where α was introduced in (3.2), β in Lemma 4 and λ_0 in Lemma 6 and H is the harmonic vector field introduced in (2.4). Estimate (10.3) follows from the convergence $Eu^{\varepsilon_k} \to u$, from the fact that $Eu^\varepsilon - E\mathbf{o}_0^\varepsilon$ is bounded in $L^\infty_{loc}((0,\infty); L^2(\mathbb{R}^2))$, see Theorem 16 and from the fact that $E\mathbf{o}_0^\varepsilon \to \alpha\varphi\left(\frac{\beta|x|}{\lambda_0}\right) H$ strongly in $L^2(\mathbb{R}^2)$ by Lemma 6.

Next, write $\omega = \nabla^\perp \cdot u$ and integrate by parts to obtain

$$J_1 = -\int_0^\infty \int_{\mathbb{R}^2} u \cdot \theta_t \nabla^\perp(\varphi^\delta \psi \chi^R) \, dxdt = -\int_0^\infty \int_{\mathbb{R}^2} (u - F) \cdot \theta_t \nabla^\perp(\varphi^\delta \psi \chi^R) \, dxdt,$$

where we have used the fact that F does not depend on time, so that the additional integral vanishes.

We write

$$J_1 = -\int_0^\infty \int_{\mathbb{R}^2} (u-F) \cdot \theta_t (\nabla^\perp \varphi^\delta) \psi \chi^R \, dxdt - \int_0^\infty \int_{\mathbb{R}^2} (u-F) \cdot \theta_t \varphi^\delta \Phi \chi^R \, dxdt$$

$$- \int_0^\infty \int_{\mathbb{R}^2} (u-F) \cdot \theta_t \varphi^\delta \psi (\nabla^\perp \chi^R) \, dxdt \equiv -J_{11} - J_{12} - J_{13}.$$

It is easy to see that $\nabla \varphi^\delta$ converges to zero weakly in $L^2(\mathbb{R}^2)$ when $\delta \to 0$ and therefore,

$$\lim_{R\to\infty} \lim_{\delta \to 0} J_{11} = 0. \tag{10.4}$$

On the other hand, $\nabla\chi^R$ also converges to zero weakly in $L^2(\mathbb{R}^2)$ when $R \to \infty$. Furthermore, $\varphi^\delta \to 1$ pointwise when $\delta \to 0$, so by dominated convergence, we find that

$$\lim_{R\to\infty}\lim_{\delta\to 0} J_{13} = 0. \tag{10.5}$$

We also note that $\varphi^\delta\chi^R$ converges pointwise to 1 as $\delta \to 0$ and $R \to \infty$ (no matter which order), so that, by dominated convergence, we deduce that

$$\lim_{R\to\infty}\lim_{\delta\to 0} J_{12} = \int_0^\infty \int_{\mathbb{R}^2} (u-F)\cdot\theta_t\Phi\,dxdt = \int_0^\infty \int_{\mathbb{R}^2} u\cdot\theta_t\Phi\,dxdt. \tag{10.6}$$

Putting together (10.4), (10.5) and (10.6) we obtain

$$\lim_{R\to\infty}\lim_{\delta\to 0} J_1 = -\int_0^\infty \int_{\mathbb{R}^2} u\cdot\theta_t\Phi\,dxdt. \tag{10.7}$$

Next we treat the nonlinear term J_2. First note that the uniform estimates on Eu^ε contained in Theorem 16 imply that $u \in L^4_{loc}((0,\infty);L^4(\mathbb{R}^2))$. The argument is the same we used to prove $\omega \in L^2_{loc}((0,\infty);L^2(\mathbb{R}^2))$

We write

$$J_2 = \int_0^\infty \int_{\mathbb{R}^2} u\,\omega\,\cdot\theta(\nabla\varphi^\delta)\psi\chi^R\,dxdt - \int_0^\infty \int_{\mathbb{R}^2} u\,\omega\,\cdot\theta\varphi^\delta\Phi^\perp\chi^R\,dxdt$$
$$+\int_0^\infty \int_{\mathbb{R}^2} u\,\omega\,\cdot\theta\varphi^\delta\psi(\nabla\chi^R)\,dxdt \equiv J_{21} - J_{22} + J_{23}.$$

We have that

$$|J_{21}| \le \int_0^\infty |\theta|\|u\|_{L^4}\|\omega\|_{L^2}\|\psi\nabla\varphi^\delta\|_{L^4}\,dt = \mathcal{O}(\sqrt{\delta})$$

and

$$|J_{23}| \le \|\psi\|_{L^\infty}\int_0^\infty |\theta|\|u\|_{L^4}\|\omega\|_{L^2}\|\nabla\chi^R\|_{L^4}\,dt = \mathcal{O}(R^{-1/2}).$$

We conclude that

$$\lim_{R\to\infty}\lim_{\delta\to 0}(J_{21}+J_{23}) = 0. \tag{10.8}$$

In addition, by dominated convergence we have that

$$\lim_{R\to\infty}\lim_{\delta\to 0} J_{22} = \int_0^\infty \int_{\mathbb{R}^2} u\,\omega\,\cdot\theta\Phi^\perp\,dxdt = -\int_0^\infty \int_{\mathbb{R}^2} (u\cdot\nabla)u\theta\cdot\Phi\,dxdt, \tag{10.9}$$

where this last equality follows from the identity $u\cdot\nabla u - (u\omega)^\perp = \nabla(|u|^2/2)$, together with the fact that Φ is divergence free.

Therefore, using (10.8) and (10.9) and integrating by parts we find

$$\lim_{R\to\infty}\lim_{\delta\to 0} J_2 = -\int_0^\infty \int_{\mathbb{R}^2} [(u\cdot\nabla)\theta\Phi]\cdot u\,dxdt. \tag{10.10}$$

Lastly we treat J_3. Once again, we write

$$J_3 = \nu\int_0^\infty \int_{\mathbb{R}^2} \omega\,\theta(\Delta\varphi^\delta)\psi\chi^R\,dxdt + \nu\int_0^\infty \int_{\mathbb{R}^2} \omega\,\theta\varphi^\delta(\Delta\psi)\chi^R\,dxdt$$

$$+\nu\int_0^\infty\int_{\mathbb{R}^2}\omega\,\theta\varphi^\delta\psi(\Delta\chi^R)\,dxdt+2\nu\int_0^\infty\int_{\mathbb{R}^2}\omega\,\theta\nabla\psi\cdot((\nabla\varphi^\delta)\chi^R+\varphi^\delta(\nabla\chi^R))\,dxdt$$
$$\equiv J_{31}+J_{32}+J_{33}+J_{34}.$$

Using, similarly to what we have already done, that: $\omega\in L^2_{loc}((0,\infty);L^2(\mathbb{R}^2))$, $\psi\Delta\varphi^\delta\rightharpoonup 0$ weakly in $L^2(\mathbb{R}^2)$ as $\delta\to 0$, $\Delta\chi^R\to 0$ strongly in $L^2(\mathbb{R}^2)$ as $R\to\infty$, $\nabla\varphi^\delta\rightharpoonup 0$ weakly in $L^2(\mathbb{R}^2)$ as $\delta\to 0$, $\nabla\chi^R\rightharpoonup 0$ weakly in $L^2(\mathbb{R}^2)$ as $R\to\infty$ and dominated convergence, we deduce that

$$\lim_{R\to\infty}\lim_{\delta\to 0}(J_{31}+J_{33}+J_{34})=0. \tag{10.11}$$

Therefore we obtain, integrating by parts,

$$\lim_{R\to\infty}\lim_{\delta\to 0}J_3=\nu\int_0^\infty\int_{\mathbb{R}^2}\omega\,\theta(\Delta\psi)\,dxdt=-\nu\int_0^\infty\int_{\mathbb{R}^2}u\cdot\,\theta\Delta\Phi\,dxdt. \tag{10.12}$$

Recall that $J_1+J_2+J_3=0$, so that, adding (10.7) with (10.10) and (10.12) we find

$$\int_0^\infty\int_{\mathbb{R}^2}\big(u\cdot\theta_t\Phi+[(u\cdot\nabla)\theta\Phi]\cdot u+\nu u\cdot\theta\Delta\Phi\big)\,dxdt=0. \tag{10.13}$$

We observe that linear combinations of products of smooth, compactly supported functions of the form $\theta\,\Phi$ are dense in $C_c^\infty((0,\infty)\times\mathbb{R}^2)$. With this observation and (10.13) we find that u satisfies the integral identity in Definition 21. We have already noted that u is divergence-free in the sense of distributions. All that remains is to show that $u\in C([0,\infty);\mathcal{D}'(\mathbb{R}^2))$ and that $u(t,\cdot)\rightharpoonup u_0$ in $\mathcal{D}'$ as $t\to 0$.

First note that in the proof of Lemma 20 we showed that $Eu^\varepsilon-\mathbb{P}[\eta^\varepsilon Eu^\varepsilon]\to 0$ strongly in $L^2_{loc}((0,\infty)\times\mathbb{R}^2)$. Therefore, $\mathbb{P}[\eta^{\varepsilon_k}Eu^{\varepsilon_k}]\to u$ in $L^2_{loc}((0,\infty)\times\mathbb{R}^2)$. By Remark 19, we have a subsequence of $\{\mathbb{P}[\eta^{\varepsilon_k}Eu^{\varepsilon_k}]\}$ which converges strongly in $L^\infty_{loc}([0,\infty);H^{-3}_{loc}(\mathbb{R}^2))$ to a limit $v\in C([0,\infty);H^{-3}_{loc}(\mathbb{R}^2))$. It follows by uniqueness of limits (in $L^2_{loc}((0,\infty);H^{-3}_{loc})$, for example) that $u(t,\cdot)=v(t,\cdot)$ for almost all $t\in(0,\infty)$, which implies that u can be identified with v. This in turn implies that $u\in C([0,\infty);\mathcal{D}'(\mathbb{R}^2))$. Furthermore, as $\mathbb{P}[\eta^{\varepsilon_k}Eu^{\varepsilon_k}]$ converges to u uniformly in time with values in H^{-3}_{loc}, one has that $\mathbb{P}[\eta^{\varepsilon_k}Eu_0^{\varepsilon_k}]$ converges to u_0 in H^{-3}_{loc}. On the other hand, Lemma 10 says that $\mathbb{P}[\eta^{\varepsilon_k}Eu_0^{\varepsilon_k}]$ converges to $\mathcal{K}*\omega_0+\gamma H$ in the sense of distributions. By uniqueness of the limit in $\mathcal{D}'$, we conclude that $u_0=\mathcal{K}*\omega_0+\gamma H$. This concludes this proof. □

Remark 24. At the end of the proof above we showed that the initial data for the limit problem is attained in $C([0,\infty);\mathcal{D}'(\mathbb{R}^2))$. We can actually prove a stronger statement, namely that there exists a positive constant $C>0$ such that

$$|\langle u(t)-u_0,\phi\rangle|\le C\|\phi\|_{H^2}\,t^{1/9},$$

for all $\phi \in \mathcal{D}$. Indeed, to see this fix $\phi \in C_c^\infty(\mathbb{R}^2)$ and let $\psi = \psi(x) = [\Delta^{-1} \text{ curl } \phi](x) - [\Delta^{-1} \text{ curl } \phi](0)$. Consider the sequence of approximations $\{u^{\varepsilon_k}\}$ constructed in the proof above. For each ε_k recall the auxiliary functional F^{ε_k} used in the proof of Proposition 18, given by

$$\langle F^{\varepsilon_k}(t), \phi \rangle = \int (Eu^{\varepsilon_k}(t,x) - E\mathbf{o}_0^{\varepsilon_k}(x)) \cdot (\nabla^\perp \eta^{\varepsilon_k})(x)\psi(x)\, dx.$$

Write

$$\langle u(t) - u_0, \phi \rangle \equiv L_1 + L_2 + L_3 + L_4, \tag{10.14}$$

where

$$L_1 = \langle u(t) - \mathbb{P}[\eta^{\varepsilon_k} Eu^{\varepsilon_k}](t) - F^{\varepsilon_k}(t), \phi \rangle;$$

$$L_2 = \langle \mathbb{P}[\eta^{\varepsilon_k} Eu^{\varepsilon_k}](t) + F^{\varepsilon_k}(t) - \mathbb{P}[\eta^{\varepsilon_k} Eu_0^{\varepsilon_k}] - F^{\varepsilon_k}(0), \phi \rangle;$$

$$L_3 = \langle F^{\varepsilon_k}(0), \phi \rangle;$$

$$L_4 = \langle \mathbb{P}[\eta^{\varepsilon_k} Eu_0^{\varepsilon_k}] - u_0, \phi \rangle.$$

By Remark 19, we have that $\lim_{\varepsilon_k \to 0} L_1 = 0$ uniformly in time. By the argument in the proof of Proposition 18, see estimate (9.8), we find that $|L_2| \leq C\|\phi\|_{H^2} t^{1/9}$. By estimate (9.4) we know that $\lim_{\varepsilon_k \to 0} L_3 = 0$. Finally, by Lemma 10 we obtain that $\lim_{\varepsilon_k \to 0} L_4 = 0$. Therefore, using (10.14) we deduce

$$|\langle u(t) - u_0, \phi \rangle| \leq C\|\phi\|_{H^2} t^{1/9},$$

as desired.

11. Uniqueness for the limit problem

Our result above provides strong compactness of viscous flows around a small obstacle but does not address actual convergence. The passage from compactness to convergence is clearly reduced to the issue of uniqueness for the limit problem. The issue of uniqueness of solutions for the 2D incompressible Navier-Stokes equations with initial data of the form u_0 is classical and delicate. Let us briefly review the related literature. The first relevant results are due to G. Benfatto, R. Esposito and M. Pulvirenti, see [2], who showed uniqueness for initial flows of the form $\sum \gamma_i H(x - x_i)$ if $\sum |\gamma_i|$ is sufficiently small, and to G.-H. Cottet, see [4], who showed uniqueness for small initial vorticities which are general bounded measures. Later, Y. Giga, T. Miyakawa and H. Osada generalized this uniqueness result for initial flows of the form $\mathcal{K} * \omega$ with ω a Radon measure with sufficiently small atomic part, see [14, 16]. As we observed in the Introduction, this smallness condition is closely related, in a technical sense, to the smallness condition on γ which we also had to impose, see (6.10). Recently the uniqueness assumption on the atomic part of ω was removed, first by T. Gallay and C. E. Wayne for initial flow of the form γH in [12], see also [10], and then for general $\mathcal{K} * \omega_0$ initial flows with ω_0 an arbitrary Radon measure by I. Gallagher and T. Gallay in [9]. These results are a byproduct of the remarkable large-time asymptotics results obtained by Gallay and Wayne in [11].

We now address the problem of uniqueness of the limit flow. Trying to prove that our solution verifies the conditions imposed in [9] is not the shortest way to deal with this issue since in that result, the vorticity is supposed to be bounded in L^1 and we have no L^1 *a priori* bounds for vorticity. On the other hand, since the circulation γ is small, our solution falls within the setting of the small data uniqueness results of [4, 14, 16]. There, among other things, uniqueness is proved in the class of small solutions belonging to the weighted in time space used in Section 6. Adapting those proofs to our case requires to prove that the limit solution is a mild solution, *i.e.* a solution of the integral version of the Navier-Stokes equations; this is what we are dealing with in the sequel.

We first collect additional information on the limit flow u. Recall that (10.3) claims that

$$u - F \in L^\infty_{loc}((0,\infty); L^2(\mathbb{R}^2)), \tag{11.1}$$

where F is a time-independent divergence-free smooth vector field which belongs to L^p for all $p \in (2,\infty]$ and such that ∇F and $\triangle F$ belong to $L^2 \cap L^\infty$. Next, the a priori estimates given in Theorem 16 imply immediately that

$$\nabla u \in L^2_{loc}((0,\infty); L^2). \tag{11.2}$$

Next, we go back to the proof of Lemma 12 and, with the notations of that Lemma, we observe that $\|u^\varepsilon - v^\varepsilon\|_{4,T} \le \|v^\varepsilon\|_{4,T}$. Indeed, the estimate (6.11) claims that $\|u^\varepsilon - v^\varepsilon\|_{4,T}$ is bounded by r_1, the smallest root of $X^2 + \left(\|v^\varepsilon\|_{4,t} - \frac{\nu^{3/4}}{C}\right) X + \|v^\varepsilon\|^2_{4,t} = 0$. On the other hand, one easily checks that under the assumption (6.8), $\|v^\varepsilon\|_{4,T}$ lies between the two roots of the above equation, so $\|u^\varepsilon - v^\varepsilon\|_{4,T} \le r_1 \le \|v^\varepsilon\|_{4,T}$. Using also (6.9), we finally deduce that the limit velocity verifies that

$$\exists T_0 > 0, \quad \|u\|_{4,T} \equiv \sup_{0<t\le T} t^{\frac12 - \frac14} \|u(t,\cdot)\|_{L^4(\mathbb{R}^2)} \le C(\gamma + o(1)) \quad \forall 0 < T \le T_0, \tag{11.3}$$

where $o(1)$ denotes a quantity that goes to 0 as $T \to 0$ independently of γ. We now state the uniqueness result that completes the proof of Theorem 1.

Proposition 25. *There exists a constant $\gamma_0 > 0$ such that if $|\gamma| < \gamma_0$, then there exists at most one global solution in the sense of Definition 21, verifying the regularity assumptions* (11.1), (11.2), (11.3) *and with initial velocity $u_0 = \mathcal{K} * \omega_0 + \gamma H$.*

Proof. As explained above, we need to prove that u is a mild solution. The argument is pretty standard, so we will omit some of the details. From Definition 21, we know that

$$\partial_t u - \nu\triangle u + \mathbb{P}(u\cdot\nabla u) = 0 \tag{11.4}$$

in the sense of distributions. Here $\mathbb{P}$ denotes the Leray projector. From (11.3) we deduce that $\mathbb{P}(u\cdot\nabla u)\in L^1\big((0,T_0);H^{-1}\big)$ and $\triangle u\in L^1\big((0,T_0);W^{-2,4}\big)$. Since $H^{-1}\hookrightarrow W^{-2,4}$, we infer from (11.4) that $\partial_t u\in L^1\big((0,T_0);W^{-2,4}\big)$ so that $u\in C^0\big([0,T_0];W^{-2,4}\big)$.

Let $t_0\in(0,T_0)$ and set $\widetilde{u}=u-F$. Relations (11.1) and (11.2) clearly imply that $\widetilde{u}\in L^\infty_{loc}\big((0,\infty),L^2\big)\cap L^2_{loc}\big((0,\infty),H^1\big)$. By interpolation, we also have that $\widetilde{u}\in L^4_{loc}\big((0,\infty),H^{\frac12}\big)$. Since a product of two $H^{\frac12}$ functions is an L^2 function, we now deduce that $\partial_t\widetilde{u}=\nu\triangle u-\mathbb{P}(u\cdot\nabla u)\in L^2_{loc}\big((0,\infty),H^{-1}\big)$. Combining this with the information that $\widetilde{u}\in L^\infty_{loc}\big((0,\infty),L^2\big)\cap L^2_{loc}\big((0,\infty),H^1\big)$, we deduce in a classical manner that $\widetilde{u}\in C^0\big((0,\infty),L^2\big)$.

Next we write (11.4) under the form

$$\partial_t\widetilde{u}-\nu\triangle\widetilde{u}=G\equiv\nu\triangle F-\mathbb{P}(u\cdot\nabla u).$$

Clearly $G\in L^2\big((t_0,T_0);\dot{H}^{-1}\big)$, where $\dot{H}^{-1}$ is the usual homogeneous Sobolev space. We consider G to be given and solve the above equation. The standard L^2 theory for the heat equation on the time interval $[t_0,T_0]$ says that there is exactly one solution $\widetilde{u}\in C^0\big([t_0,T_0],L^2\big)\cap L^2\big((t_0,T_0),H^1\big)$ and this solution verifies the integral form of the equation

$$\widetilde{u}(t)=e^{(t-t_0)\nu\triangle}\widetilde{u}(t_0)+\int_{t_0}^t e^{(t-s)\nu\triangle}G(s)\,ds.$$

Going back to u, we deduce that u verifies the integral form of (11.4):

$$u(t)=e^{(t-t_0)\nu\triangle}u(t_0)-\int_{t_0}^t \operatorname{div}\,e^{(t-s)\nu\triangle}\ \mathbb{P}(u\otimes u)(s)\,ds. \tag{11.5}$$

Above, the Leray projector $\mathbb{P}$ is applied along the rows of the matrix $u\otimes u$. We use standard estimates for the fundamental solution of the heat operator to conclude that

$$\|\operatorname{div}\,e^{(t-s)\nu\triangle}\ \mathbb{P}(u\otimes u)(s)\|_{L^2}\le C(t-s)^{-\frac12}\|\mathbb{P}(u\otimes u)(s)\|_{L^2}\le C(t-s)^{-\frac12}s^{-\frac12}\|u\|_{4,t}^2$$

Since the function $(0,t)\ni s\mapsto(t-s)^{-\frac12}s^{-\frac12}$ is integrable, we deduce that the integral $\int_0^t \operatorname{div}\,e^{(t-s)\nu\triangle}\ \mathbb{P}(u\otimes u)(s)\,ds$ is convergent in L^2 so that

$$\lim_{t_0\searrow 0}\int_{t_0}^t \operatorname{div}\,e^{(t-s)\nu\triangle}\ \mathbb{P}(u\otimes u)(s)\,ds=\int_0^t \operatorname{div}\,e^{(t-s)\nu\triangle}\ \mathbb{P}(u\otimes u)(s)\,ds\quad\text{ in } L^2.$$

Next, denoting by $E(t)$ the convolution kernel of $e^{t\nu\triangle}$, we clearly have that, as $t_0 \searrow 0$, $E(t-t_0)$ converges to $E(t)$ in the class of Schwarz $\mathcal{S}$. Recalling that $u(t_0)$ converges to u_0 in $W^{-2,4}$, we deduce that, when $t_0 \searrow 0$, $e^{(t-t_0)\nu\triangle}u(t_0)$ converges to $e^{t\nu\triangle}u_0$ in $\mathcal{S}'$, the space of tempered distributions. Letting $t_0 \searrow 0$ in (11.5) we infer that

$$u(t) = e^{t\nu\triangle}u_0 - \int_0^t \operatorname{div} e^{(t-s)\nu\triangle}\ \mathbb{P}(u\otimes u)(s)\,ds.$$

In other words, u is a mild solution of the Navier-Stokes equations on $[0,T_0]$. Local uniqueness now follows with similar arguments as in [14]. We briefly recall the argument for the convenience of the reader. Let u^1 and u^2 be two solutions. Subtracting the equations for u^1 and u^2 one obtains

$$(u^1-u^2)(t) = \int_0^t \operatorname{div} e^{(t-s)\nu\triangle}\ \mathbb{P}[(u^2-u^1)\otimes u^2 + u^1\otimes(u^2-u^1)](s)\,ds.$$

With an argument analogous to that of the proof of Lemma 12, we get

$$\|u^1-u^2\|_{4,t} \le C_0\|u^1-u^2\|_{4,t}(\|u^1\|_{4,t}+\|u^2\|_{4,t}),$$

for some constant C_0. According to (11.3), if γ and T are sufficiently small, one has that $\|u^1\|_{4,T}+\|u^2\|_{4,T} \le \frac{1}{2C_0}$. We deduce that $\|u^1-u^2\|_{4,T} \le \frac{1}{2}\|u^1-u^2\|_{4,T}$ which of course implies that $u^1=u^2$ up to time T if the circulation γ is sufficiently small.

Uniqueness starting from time T is similar to the uniqueness of finite energy solutions of the Navier-Stokes equations. Indeed, given that $u^1-u^2 \in C^0\big([T,\infty);L^2\big)\cap L^2_{loc}\big([T,\infty);H^1\big)$, one can multiply the equation of u^1-u^2 by u^1-u^2 and integrate in space and from T to some $t>T$ to obtain that

$$\begin{aligned}\|u^1-u^2\|_{L^2}^2 + 2\nu\int_T^t \|\nabla(u^1-u^2)\|_{L^2}^2 &= -2\int_T^t\int_{\mathbb{R}^2}(u^1-u^2)\cdot\nabla u^2\cdot(u^1-u^2)\\ &\le 2\int_T^t \|u^1-u^2\|_{L^4}^2\|\nabla u^2\|_{L^2}\\ &\le C\int_T^t \|u^1-u^2\|_{L^2}\|\nabla(u^1-u^2)\|_{L^2}\|\nabla u^2\|_{L^2}\\ &\le \nu\int_T^t \|\nabla(u^1-u^2)\|_{L^2}^2 + C\int_T^t \|u^1-u^2\|_{L^2}^2\|\nabla u^2\|_{L^2}^2.\end{aligned}$$

Global uniqueness now follows from the Gronwall lemma. □

12. Conclusions

The purpose of this section is to interpret what we have done in a broader context and to point out some directions for improvement and further work. Our basic problem was to find conditions under which the presence of a single small obstacle could be ignored in the modelling of large scale flow. The precise formulation we used, working in the unbounded exterior domain and fixing the large scale flow by choosing an initial vorticity ω_0 and a circulation γ, was convenient from the mathematical point of view, although perhaps not physically natural.

One natural issue to explore is the possibility that the circulation γ might depend on the size of the obstacle. The circulation is a passive parameter in our analysis, and it is easy to see that the limit flow would depend on $\gamma(\varepsilon)$ only through its limit as $\varepsilon \to 0$. From the physical interpretation of the case $\gamma(\varepsilon) \neq 0$, given in Section 5, together with the natural scaling of this problem, we can see that the cases $\gamma(\varepsilon) = 0$ or $\gamma(\varepsilon) \to 0$ are by far the most physically interesting situations. However, there would have been no substantial simplification of the argument by restricting our problem to the case $\gamma(\varepsilon) = o(1)$. Moreover, the situation in which $\gamma(\varepsilon) = \mathcal{O}(1)$ is mathematically very interesting. Indeed, the smallness condition on the initial circulation only appears when we assume $\gamma(\varepsilon) = \mathcal{O}(1)$. Also, there is a discrepancy between the results obtained for the inviscid and viscous cases when $\gamma(\varepsilon) = \mathcal{O}(1)$, which suggests that the limits $\varepsilon \to 0$ and $\nu \to 0$ do not commute.

We have assumed throughout that ω_0 was smooth and compactly supported. How much regularity on ω_0 did we really use? The answer is none. We actually needed u_0 bounded in L^2_{loc} and $L^{2,\infty}$ and nothing else. We contrast this with the inviscid argument, where we needed ω_0 in L^p, $p > 2$.

Let us turn to some problems which arise naturally from our work. One particularly interesting question is the issue of considering both the viscosity and the obstacle small. This should be a difficult problem, because the wake due to an obstacle becomes more pronounced and turbulent as viscosity vanishes. It is well-known that, for full plane flow, one can take the vanishing viscosity limit, obtaining solutions of the incompressible 2D Euler equations, see [3, 8, 21]. In the presence of a material boundary, the vanishing viscosity limit is a classical open problem, even if the flow is very smooth. The difficulty is due to the boundary layer. The problem of taking the vanishing viscosity limit outside a very small obstacle interpolates nicely between the full plane result and the open problem of taking the vanishing viscosity limit in the presence of a fixed material boundary. In fact, this question is one of the main motivations of the present work and it is still under consideration by the authors. Taking into account the result obtained in this paper it is clear that one should first pursue the small viscosity problem in the case $\gamma(\varepsilon) = 0$ or $\gamma(\varepsilon) = o(1)$ as $\varepsilon \to 0$, since the smallness condition in our convergence result gets more restrictive as viscosity vanishes. With this future work in mind we have included the specific dependence of our estimates on viscosity for as long as it was practical.

A second problem is to extend our analysis to velocity fields which are constant at infinity, in order to include the classical case of a material body moving in a fluid with roughly constant speed.

Yet another problem that arises from our work is to remove the smallness condition on the initial circulation. The parallel between our convergence problem and uniqueness for the limit flow suggests a strategy. Is it possible to adapt the entropy-entropy flux techniques used by Gallay and Gallagher for the uniqueness problem to the small obstacle asymptotics?

A fourth problem is to obtain an asymptotic description of the correction term in the small obstacle limit, i.e. a description of the "wake" associated with the small obstacle. Finally, one can consider a whole host of related problems, described loosely as the study of limit flows in singularly perturbed domains. For instance, one can study limit flows in a bounded domain with one or more small obstacles, or in a domain composed of a small neck joining two fat domains, or in a domain having a long thin tail, etc.

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