van den Dries answers, Chapters 1-7

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1.2.5.1. Let $A \subseteq R^m$ and let $f = (f_1, \ldots, f_n) : A \to R^n$ be a map with component functions $f_i : A \to R$. Show that f belongs to $\mathcal{S}$ if and only if each f_i belongs to $\mathcal{S}$.

f is to be considered a subset of R^{m+n}, and each f_i a subset of R^{m+1}. Let $\bar{f}_i = \{(a, x_1, \ldots, x_n) \in R^{m+n} \mid x_i = f_i(a)\}$. $\bar{f}_i$ is in $\mathcal{S}$ through the axioms allowing cartesian multiplication by R and permutations of variables. Then it is easily seen that $f = \bigcap \bar{f}_i$. Thus, $f \in \mathcal{S}$.

Conversely, each f_i is just the projection of f onto some version of R^{m+1}, so each is definable.

1.2.5.2. (Sheaf property) Let I be a finite index set, let $A \in \mathcal{S}_m$ be the union of the sets $A_i \in \mathcal{S}_m$ $(i \in I)$. Show that a map $f : A \to R^n$ belongs to $\mathcal{S}$ if and only if all its restrictions $f \mid A_i$ belong to $\mathcal{S}$.

Clearly, if $f \in S$, then just taking $f \cap (A_i \times R^n)$, $f \mid A_i$ is in $\mathcal{S}$.

Conversely, suppose each restriction is in $\mathcal{S}$, and let $g_i = f \mid A_i$. $\bigcup_i g_i = f$, since g_i and g_j must agree on any points A_i and A_j have in common. Thus $f \in \mathcal{S}$.

1.2.5.3. Given $A \subseteq R^{m+n}$ and $x \in R^m$ we put $A_x := \{y \in R^n \mid (x, y) \in A\}$. Show that if $A \in \mathcal{S}_{m+n}$ and $k \in \mathbb{N}$, then the sets $\{x \in R^m \mid |A_x| \leq k\}$ and $\{x \in R^m \mid |A_x| = k\}$ belong to $\mathcal{S}$.

Let $B_i = \{(x, y_1, \ldots, y_{k+1}) \in R^{m+(k+1)n} \mid (x, y_i) \in A\}$. Let $B = \bigcap_i B_i$. Let $Q = \{(x, y_1, \ldots, y_{k+1}) \in R^{m+(k+1)n} \mid \exists i \neq j(y_i = y_j)\}$. Then the first set is $(B \setminus Q)^C$, projected down to R^m. The second is the first set, with the set for $k-1$ removed.

1.2.5.4. Let the sets A, B, C and the function $f : A \times B \to C$ belong to $\mathcal{S}$. Show that the set $\{a \in A \mid f(a, \cdot) : B \to C$ is injective$\}$ belongs to $\mathcal{S}$, and show also that the set $\{a \in A \mid f(a, \cdot) \mid B \to C\}$ is surjective$\}$ belongs to $\mathcal{S}$.

Let $\bar{f} = \{(a, c, b) \mid f(a, b) = c\}$. If a function is injective, every point in the image has at most one pre-image, so the set of a's where f is not injective corresponds to the set $\{a \in A \mid \exists c |\bar{f}|_{(a,c)} > 1\}$. This set is in $\mathcal{S}$, by the previous problem, so its complement is as well.

Likewise, if f is surjective, then every point in C has at least one preimage, so the set where f is not surjective is the set $\{a \in A \mid \exists c \in C |\bar{f}|_{(a,c)} = 0\}$ which is in $\mathcal{S}$, and so its complement is too.

1.2.5.5. Let $P \subseteq R$ be a nonempty subset belonging to $\mathcal{S}_1$. For $m = 0, 1, 2, \ldots$ put

$$(\mathcal{S}|P)_m := \{A \cap P^m \mid A \in \mathcal{S}_m\},$$

a boolean algebra of subsets of P^m. Show that $\mathcal{S}|P := ((\mathcal{S}|P)_m)_{m \in \mathbb{N}}$ is a structure on P. (The "restriction of $\mathcal{S}$ to P".)

Verification that $(\mathcal{S}|P)$ satisfies the definition of a structure is routine.

1.2.5.6. Suppose $\mathcal{S}$ contains binary operations $+ : R^2 \to R$ and $\cdot : R^2 \to R$ with respect to which R is a ring (always associative with 1 in this book). Show that $\mathcal{S}$ contains $\{0\}$ and $\{1\}$, and that if S contains $A \subseteq R^m$ and the functions $f, g : A \to R$, then it contains the functions $-f$, $f + g$, and $f \cdot g$ from A to R.

$\{0\}$ is the set defined by $+ \cap \{(x_1, x_2, x_3) \in R^3 \mid x_2 = x_3\}$, projected down to the first coordinate. $\{1\}$ is the same thing with $\cdot$. $-f$ is defined by taking $f \times R \times R$ and intersecting it with $A \times \{(x_1, x_2) \mid x_1 + x_2 = 0\}$, which is in $\mathcal{S}$ since the graph of addition is, and then projecting away the second coordinate. $f + g$ is defined by taking $f \times R \times R$, $\{(a, x, g(a), y) \mid a \in A\}$, $A \times +$, and intersecting them. $f \cdot g$ is defined similarly.

1.2.5.7. Suppose $R = \mathbb{R}$ and $\mathcal{S}$ contains the order relation $\{(x, y) \in \mathbb{R}^2 \mid x < y\}$. Show that the topological closure $\mathrm{cl}(A)$ of a set $A \in \mathcal{S}_m$ also belongs to $\mathcal{S}$. Show that if a function $f : \mathbb{R}^{m+1} \to \mathbb{R}$ belongs to $\mathcal{S}$, then the set

$$A := \{a \in \mathbb{R}^m \mid f(a, t) \text{ tends to a limit } l(a) \in \mathbb{R} \text{ as } t \to +\infty\}$$

belongs to $\mathcal{S}$, and the limit function $l : A \to \mathbb{R}$ so defined belongs to $\mathcal{S}$.

Let $(a, b]$ denote the half-open interval for any $a, b \in \mathbb{R}$, even if $a > b$. The set $\mathrm{cl}(A)$ is defined to be $\{(b_1, \ldots, b_m) \in \mathbb{R}^m \mid \forall c_1, \ldots, c_m \exists d_1, \ldots, d_m (\bigwedge_i d_i \in (c_i, b_i] \wedge \bar{d} \in A)\}$. Take the set $\mathbb{R}^{2m} \times A$ and intersect with the set $\{(\bar{b}, \bar{c}, \bar{d}) \mid \bar{b} \neq \bar{c} \wedge \forall i \leq n (d_i \in (c_i, b_i]\}$ (obtained from $<$ by repeated crossings and then permutations), then project it down to $\mathbb{R}^{2m}$. This consists of all pairs of points in $\mathbb{R}^m$ which have a point of A "between" them. Taking the complement of this set gets all pairs of points which do not have a point of A between them. Projecting down to $\mathbb{R}^m$ gives $\mathrm{cl}(A)^C$, which is therefore in $\mathcal{S}$, and so $\mathrm{cl}(A)$ is.

For the second part, the set A is $\{\bar{a} \in \mathbb{R}^m \mid \exists b \forall c \exists T \forall t (t > T \to f(\bar{a}, t) \in (c, b])\}$. Using the fact that all logical connectives and quantifiers preserve membership in $\mathcal{S}$, this set is in $\mathcal{S}$. The limit function is $\{(\bar{a}, b) \mid \bar{a} \in A \wedge \forall c \exists T \forall t (t > T \to f(\bar{a}, t) \in (c, b])\}$, which is likewise in $\mathcal{S}$.

1.2.5.8. Suppose $R = \mathbb{R}$ and $\mathcal{S}$ contains the graphs of addition and multiplication. Show that $\mathcal{S}$ contains the order relation $\{(x, y) \in \mathbb{R}^2 \mid x < y\}$, and each singleton $\{q\}$, with q a rational number. Show that if $\mathcal{S}$ contains a function $f : I \to \mathbb{R}$, with open $I \subseteq \mathbb{R}$, then it contains the set $I' := \{x \in I \mid f \text{ is differentiable at } x\}$, and the derivative $f' : I \to \mathbb{R}$.

We can define $<$ to be the set $\{(x, y) \mid \exists z (x + z^2 = y)\}$. Define 1 to be the unique element in $\{x \mid \forall y (x \cdot y = 1)\}$. Then each singleton $q = a/b$, $a, b \in \mathbb{Z}$, to be $\{x \mid x \cdot \bar{b} = \bar{a}\}$, where $\bar{b} = 1 + \ldots + 1$ (b copies of 1).

A function f is differentiable at a point a if the limit $(f(x) - f(a))/(x - a)$ exists as x goes to a, or in other words, if the two limits $(f(a + 1/t) - f(a))/(1/t)$ as $t \to \infty$ and $(f(a) - f(a - 1/t))/(1/t)$ as

$t \to \infty$ exist and are equal to each other. The set on which they exist is in $\mathcal{S}$ by the previous problem, and the set where they are equal is easily seen to be in $\mathcal{S}$ as well. On this set, I', the derivative is defined to be this limit, and since the limit at a point is in $\mathcal{S}$, the derivative will be too.

1.3.6.1. The definably connected subsets of R are the following: the empty set, the intervals, the sets $[a, b)$ with $-\infty < a < b \leq +\infty$, the sets $(a, b]$ with $-\infty \leq a < b < +\infty$, and the sets $[a, b]$ with $-\infty < a \leq b < +\infty$.

The empty set is clearly definably connected. We take the remaining cases together. Let I be any such "interval," and assume it is not definably connected. Then there would be U, V, disjoint open sets such that $U \cap I$ and $V \cap I$ are non-empty and cover I. By o-minimality, U and V are both finite unions of intervals. Write $U = (a_0, b_0) \cup \ldots \cup (a_n, b_n)$, with $a_i < b_i < a_j < b_j$, for $i < j$ (a_O possibly $-\infty$, b_n possibly ∞). Let $B = \{a_i, b_i \mid i \leq n\}$. I claim $I \cap B \neq \emptyset$. For if not, then for some i, $a_i \leq a < b \leq b_i$, or $b_i \leq a < b \leq a_{i+1}$, or $b \leq a_0$ or $b_n \leq a$. In the first case, $I \subseteq U$, except possibly for one or two endpoints. However, then V must contain such a point, and since it is open, contains an interval around that point, and so intersects U. This is impossible, so $I \subseteq U$, but then $V \cap I$ is empty, which is again impossible, so the first case is out. In the other cases, I is disjoint from U, which is impossible too.

1.3.6.2. The image of a definably connected set $X \subseteq R^m$ under a definable continuous map $f : X \to R^n$ is definably connected.

If f is continuous, it is easy to see that the inverse image of an open set is open. Thus, if $f(X)$ were not definably connected, we could take the inverse images of U and V, witnesses to the failure of definable connectedness, and X would not be definably connected.

1.3.6.3. If X and Y are definable subsets of R^m, $X \subseteq Y \subseteq cl(X)$, and X is definably connected, then Y is definably connected.

Suppose Y is not definably connected. Let U and V be open subsets of Y witnessing this. Consider $U' = U \cap X$ and $V' = V \cap X$. They are certainly disjoint and open in X, so if they are non-empty, then we are done. Assume $V' = \emptyset$. Then $V \subseteq \mathrm{cl}(X) \setminus X$. Consider any open set around $a \in V$. Since $a \in \mathrm{cl}(X)$, there is a point $x \in X$ which is in the open set. But since this is true for any open set, V cannot be open, which is impossible. Thus $V' \neq \emptyset$, likewise for U', and so X is not definably connected.

1.3.6.4. If X and Y are definably connected subsets of R^m and $X \cap Y \neq \emptyset$, then $X \cup Y$ is definably connected.

Suppose $X \cup Y$ is not definably connected, and let U and V be subsets of $X \cup Y$ witnessing this. Then $U \cap X$ and $V \cap X$ are open and disjoint, so one must be empty. WLOG, let $X \subseteq U$. By the same argument, we have $Y \subseteq U$ or $Y \subseteq V$, but in the first case, U covers $X \cup Y$, which is impossible, so $Y \subseteq V$. But then $X \cap Y$ is in both U and V, so they are not disjoint.

1.3.7.1. Let $S \subseteq R^{m+n}$ be definable. Show: (i) $\{x \in R^m \mid S_x \text{ is open}\}$ is definable. (ii) $\{(x,y) \in R^{m+n} \mid y \in \int(S_x)\}$ is definable.

(i) S_x is defined to be $\{y \mid (x,y) \in S\}$. The statement "S_x is open" is equivalent to the statement $\forall \bar{y} \in S_x \exists u_1, v_1, \ldots, u_n, v_n (\bigwedge_i y_i \in (u_i, v_i) \wedge \forall \bar{z} (\bigwedge_i z_i \in (u_i, v_i) \to z_i \in S_x))$ – that there is a box around each point in S_x wholly contained in S_x. Combining the above two definitions gives a formula in x true only when S_x is open.

(ii) $\int(S_x)$ is definable by Lemma 3.4(i), so it is easy to see that the set (x,y) such that $y \in \int(S_x)$ is definable. The statement is essentially that there exists a box around y entirely contained in S_x.

1.5.9.1. Let $\mathcal{R} = (R, \ldots)$ be a model-theoretic structure, $C \subseteq R$, and $A \subseteq R^n$. Show that A is definable in $\mathcal{R}$ using constants from C if and only if there exist a set $S \subseteq R^{m+n}$ definable in $\mathcal{R}$, and elements $c_1, \ldots, c_m \in C$ such that for all $(x_1, \ldots, x_n) \in R^n$

$$(x_1, \ldots, x_n) \in A \leftrightarrow (c_1, \ldots, c_m, x_1, \ldots, x_n) \in S$$

.

If A is definable in $\mathcal{R}$ using constants from C, then, using the fact that the operations which preserve membership in a structure have logical analogues, we can write A as $\varphi(x_1, \ldots, x_n)$, where φ is a formula in the language of $\mathcal{R}_C$. Let $c_1, \ldots, c_m$ be the elements of C which appear in φ. Replace the occurrences of $c_1, \ldots, c_m$ with $y_1, \ldots, y_m$ to get $\varphi(y_1, \ldots, y_m, x_1, \ldots, x_n)'$. By the equivalence of structure membership and logical formulas, we can find S such that $\varphi(y_1, \ldots, y_m, x_1, \ldots, x_m)' \leftrightarrow (y_1, \ldots, y_m, x_1, \ldots, x_m) \in S$. So $(c_1, \ldots, c_m, x_1, \ldots, x_n) \in S \leftrightarrow \varphi(c_1, \ldots, c_m, x_1, \ldots, x_n)'$. But $\varphi(c_1, \ldots, c_m, x_1, \ldots, x_n)'$ by definition is just $\varphi(x_1, \ldots, x_n)$, and $\varphi(x_1, \ldots, x_n) \leftrightarrow (x_1, \ldots, x_n) \in A$.

1.5.9.2. Let $\mathcal{R} = (R, \ldots)$ be a model-theoretic structure and $S \subseteq R^{m+n}$ definable in $\mathcal{R}$. Show that if $a \in R^m$ is definable in $\mathcal{R}$, then the set $S_a := \{b \in R^n \mid (a,b) \in S\}$ is definable in $\mathcal{R}$.

Let $\varphi(x)$ define a, and $\psi(x,y)$ define S. Then $\exists x(\psi(x,y) \wedge \varphi(x))$ defines S_a.

1.5.9.3. Let $\mathcal{R} = (R, <, 0, 1, -, +, \cdot)$ be the ordered field of real numbers. Show that each function $f : \mathbb{R}^m \to \mathbb{R}$ defined by a polynomial $f(X_1, \ldots, X_m) \in \mathbb{R}X_1, \ldots, X_m$ is definable in $\mathcal{R}$ using constants. Derive that each semialgebraic set in $\mathbb{R}^m$ (see "Introduction and Overview") is definable in $\mathcal{R}$ using constants.

Let $f(X_1, \ldots, X_M)$ be the polynomial giving any such function. Note that the coefficients of the terms may be arbitrary real numbers. The formula $y = f(x_1, \ldots, x_M)$ gives a definable set in $\mathcal{R}$ using constants (since exponentiation to a constant is just repeated multiplication), which is clearly a function. Since a semi-algebraic set is a finite union of such sets and sets of the form $y > f(x_1, \ldots, x_M)$, it is definable.

1.7.8. The subsets of R^m definable in the L_F-structure R using constants are exactly the semilinear sets in R^m ($L_F := \{<, 0, -, +\} \cup \{\lambda \cdot \mid \lambda \in F$, so R is a vector space over F).

Note that in L_F, any term is of the form $\lambda_1 x_1 + \ldots + \lambda_n x_n$ (any $-$ signs can be subsumed into the λ_i's). Thus, atomic formulas are of the form $\lambda_1 x_1 + \ldots + \lambda_n x_n (> \mid =) 0$, which are the basic semilinear sets. Boolean combinations of these yield the semilinear sets. The only question then is whether all formulas, not just atomic ones, yield semilinear sets. The answer is yes, since by Corollary 7.6, the set of all semilinear sets is a structure, and thus closed under quantification, the only remaining logical operation.

1.7.12. Let R be a nontrivial ordered vector space over an ordered division ring F, and consider R as an L_F-structure. Show that the maps $R \to R$ that are additive and definable using constants are constants are exactly the scalar multiplications by elements of F. (Hence these maps are actually definable without using constants.)

We know that R is o-minimal, so any definable function, f, is semilinear. Since f is a function, each basic semilinear set must include an equation of the form $\lambda_1 x + \lambda_2 y + a = 0$. Since f is a finite union of basic semilinear sets, one basic semilinear set, U, must contain $(x, f(x))$ for all sufficiently large x. Let M be such that $x > M \to (x, f(x)) \in U$. Let U's equation be $\lambda x + a = y$. However, since $x + x > x$, $\lambda(x + x) + a = \lambda x + \lambda x + a + a$, showing that $a = 0$. Now, for any other x, let $\lambda' x + a' = y$ be the equation in the basic semilinear set which includes $(x, f(x))$ (there is exactly one, since f is a function). Take $y > M$ such that $x + y > M$. Then $f(x) = f(x + y) - f(y) = \lambda(x + y) - \lambda y = \lambda x$. Thus, $f(x) = \lambda x$. Since this is true for any $x \in R$, $f(x) = \lambda x$.

2.1.4.1. Let $f = f(X_1, \ldots, X_m) \in F[X_1, \ldots, X_m]$ be a polynomial with coefficients in the field F, and let $d_1, \ldots, d_m \in \mathbb{N}$ be such that $\deg_{X_i} f \leq d_i$ for $i = 1, \ldots, m$. Show that if f vanishes identically on a cartesian product $A_1 \times \cdots \times A_m$ with $|A_1| > d_1, \ldots, |A_m| > d_m$ (all $A_i \subseteq F$), then $f = 0$.

By induction. For the case $m = 1$, note that if $f(a) = 0$ for some $a \in F$, then $(X - a) | f$ in $F(X)$, $(X - a) \not| (X - b)$ if $a \neq b$, and $\deg(gh) = \deg(g) + \deg(h)$ unless $g = 0$ or $h = 0$. Thus, if f has degree less than d, it can have no more than d roots, unless the factorization of f includes 0, or in other words, if $f = 0$.

If $m = n + 1$, let $d = d_{n+1}$ and write f as $T_d X_n^d + \cdots + T_0$. Fix $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$. Then $f(a_1, \ldots, a_n, X)$ is a polynomial in one variable. By the case $m = 1$, $f(a_1, \ldots, a_n, X) = 0$, and so each $T_i = 0$. This is true for every $(x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n$. But now each T_i falls under the induction assumption, and so each is 0, and thus $f = 0$.

2.1.4.2. Let F be an ordered field and $f \in F[X_1, \ldots, X_m]$, $f \neq 0$. Show that the zero set $Z(f) := \{a \in F^m \mid f(a) = 0\}$ is a closed subset of F^m with empty interior.

It is easy to see that $Z(f)^C$ is open, since f is continuous. If $Z(f)$ had non-empty interior, then it would contain a box, but then, since F is dense, it would contain a cartesian product in which each set had infinitely many elements, which is impossible by the previous problem.

2.3.3. With the same assumptions as in Lemma 2.3.2, set $\zeta_0 := -\infty, \zeta_{r+1} := +\infty$. Fix an $i \in \{0, \ldots, r\}$ and put $\epsilon(n) = \text{sign}\,(f_n|(\zeta_i, \zeta_{i+1}))$. Then we have (a) $(\zeta_i, \zeta_{i+1}) = \{(x,t) \in C \times \mathbb{R} \mid \text{sign}\,(f_n(x,t)) = \epsilon(n) \text{ for } n = 1, \ldots, N\}$, (b) $\text{cl}(\zeta_i, \zeta_{i+1}) = \{(y,t) \in \text{cl}(C) \times \mathbb{R} \mid \text{sign}\,(f_n(y,t)) \in \{\epsilon(n), 0\} \text{ for } n = 1, \ldots, N\}$

(a) Let $B' = \{(x,t) \in C \times \mathbb{R} \mid \text{sign}\,(f_n(x,t)) = \epsilon(n) \text{ for } n = 1, \ldots, N\}$. Fixing x, let $B'(x)$ be the projection to the t coordinate. By Thom's lemma, $B'(x)$ is either empty, a point, or an interval. It is thus an interval. If for all f, $f(x,T) = f(\zeta_i(x))$ for $\zeta_i(x) < T < \zeta_{i+1}(x)$, then, since $\Gamma(\zeta_i)$ is contained in the zero set of some f_n, f_n must be 0 on this interval, which is impossible. Thus, one endpoint of the interval is $\zeta_i(x)$. A similar argument shows the other end is $\zeta_{i+1}(x)$. This is true for all $x \in C$, and so $B' = \bigcup B'(x) = (\zeta_i, \zeta_{i+1})$.

(b) Note that if f_n is positive on (ζ_i, ζ_{i+1}) it cannot be negative on $\Gamma(zeta_i)$, since, fixing x, by the intermediate value theorem it would have to be 0 for some (x,t), $\zeta_i(x) < t < \zeta_{i+1}(x)$, which is impossible, since f_n is positive on the entire interval. Thus, for any x, $\text{sign}(f_n(\zeta_i(x))) \in \{\epsilon(n), 0\}$.

Now, let $B' = \{(y,t) \in \text{cl}(C) \times \mathbb{R} \mid \text{sign}\,(f_n(x,t)) \in \{\epsilon(n), 0\} \text{ for } n = 1, \ldots, N\}$, and let $B'(x)$ be as before. By lemma 3.2(b), each ζ extends uniquely to a continuous function η on $\text{cl}(C)$. By the second half of Thom's lemma, $B'(x) = \text{cl}(\zeta_i(x), \zeta_{i+1}(x))$ for $x \in C$. For $y \in \text{cl}(C)$, we use the same argument as used in the proof of 3.2(b): for any fixed $t \in (\eta_i(y), \eta_{i+1}(y))$, we first restrict to a neighborhood U of y such that for $x \in U \cap C$, $t \in (\zeta_i(x), \zeta_{i+1}(x))$. $\text{sign}(f_n(x,t)) = \epsilon(n)$ for all $x \in U \cap C$, so $\text{sign}(f_n(y,t)) \in \{\epsilon(n), 0\}$. This is true for each n and each t, so $[\eta_i(y), \eta_{i+1}(y)] \subseteq B'(y)$. Using part (a), and Thom's lemma (that $B'(y)$ is connected), it is easy to see that equality holds. Since $B = \bigcup_{y \in \text{cl}(C)} B' = \bigcup_{y \in \text{cl}(C)} (\eta_i(y), \eta_{i+1}(y)) = (\eta_i, \eta_{i+1})$.

2.3.7.1. Let $Q(X,Y) \in \mathbb{R}[X,Y]$ be a nonzero polynomial in two variables. Show that there are $d \in \mathbb{N}$ and $M > 0$ such that if $(x,y) \in \mathbb{R}^2$, $x > M$ and $Q(x,y) = 0$, then $|y| \leq x^d$.

Write Q as a polynomial in y with coefficients in x, $A_n y^n + \ldots + A_0$, where each A_i is a polynomial in x. Let $d = \max(\{\deg(A_i) - \deg(A_n) \mid i < n\}, 0) + 1$. Let M be such that for all $x > M$ and $i < n$, $x^d > 1 + (A_i(x)/A_n(x))$. Then by lemma 1.1, we are done.

2.3.7.2. Show that if $g : \mathbb{R} \to \mathbb{R}$ is semialgebraic, then there are $d \in \mathbb{N}$ and $M > 0$ such that $|g(x)| \leq x^d$ for all $x > M$.

$\Gamma(g)$ is semialgebraic, but since g is a function, it must just be piecewise the zero set of a polynomial. Restricting to the last piece (the interval $N < x < \infty$ for some $N \in \mathbb{N}$), we are back in the previous problem.

2.3.7.3. Suppose a continuous function $g : \mathbb{R}^m \to \mathbb{R}$ satisfies $Q(x, g(x)) = 0$ for all $x \in \mathbb{R}^m$ and some nonzero polynomial $Q(X,T) \in \mathbb{R}[X,T]$, $X = (X_1, \ldots, X_m)$. Show that then g is semialgebraic.

Treating g as a set, we know that $g \subseteq Q(x,y)$. For any fixed x, either $Q(x,y)$ is 0 or not. If not, then it has only finitely many roots, and $g(x)$ is one of them. Let $Q(x,y)$ have e roots. We know from lemma 2.6 that the set $K_e\{z \mid Q(z,y)$ has e roots$\}$ is semilinear. We know that $g(z)$'s place in the ordering of the roots does not change on this set. Let it be the kth root. Consider the set $g_e = \{(z,y) \mid Q(z,y) = 0 \wedge \exists^{=k-1} y'(Q(z,y') = 0)\}$. It is definable, and therefore semilinear, and defines g on K_e. Thus, we can define g for all x such that $Q(x,y)$ has only finitely many roots. This leaves the cases where $Q(x,y) = 0$. If we write $Q(X,T) = A_n T^n + \ldots + A_0$, with the A_i's polynomials in X, $Q(x,T) = 0$ means that $A_n(x) = \cdots = A_0(x) = 0$. By exercise 2.1.4.1, we know that the set $Z = \{x \mid Q(x,T) = 0\}$ is a closed subset with empty interior. Thus, for each $z \in Z$ and U an open set containing z, we can find $x \notin Z$, $x \in U$. By o-minimality, for sufficiently small U we can find such x belonging to a single K_e. Define $x \in (v,w)$, where $x, v, w \in \mathbb{R}^m$, to mean $x_i \in (v_i, w_i)$, for $i \leq m$. We can then use a limit function, defined as $\bar{g}_e = \{(x,y) \mid x \in \mathrm{cl}(K_e) \wedge \forall y' < y, y'' > y \exists v, w \forall u (x \in (v,w) \wedge (u \in (v,w) \wedge u \in K_e \rightarrow g_e(u) \in_(y', y'')))\}$, which, since g is continuous, will yield $g|\,\mathrm{cl}(K_e)$. Thus, $\bar{g}_e$ extends g_e to $\mathrm{cl}(K_e)$, which includes z. We can therefore extend each of these finitely many definable functions, which together cover $\mathbb{R}^m$ (they will be compatible on their common domains because of their definitions as limits of the same continuous function), which yields a definition for g.

3.1.9.1. Suppose the function $f : (a,b) \to R$ on the interval (a,b) is definable. Show there exist elements $a_1, \ldots, a_k$ withe property of the monotonicity theorem such that $a_1, \ldots, a_k$ are definable in the model-theoretic structure $(R, <, \Gamma(f))$.

By the monotonicity theorem, we can find $a'_1 = a, \ldots, a'_m = b$ such that between these points, f is constant, or monotone and continuous. Note that a and b are definable from $\Gamma(f)$. Set $a_1 = a'_1$. Suppose on (a'_1, a'_2), f is monotone increasing. Then the set $\{x \mid x \in (a,b) \wedge \forall a_1 < y < x(\text{f is continuous at y } \wedge \text{ f is increasing at y}) \wedge (\text{f is not continuous at x} \vee \text{ f is not increasing at x})\}$ has a unique element. Let it be a_2. Then on (a_1, a_2), f is monotone and continuous. Since for all $y < a_2$, f is monotone increasing and continuous, and f is not monotone increasing and continuous at a_2, a_2 must be a'_i for some $i > 1$. Suppose f is constant on (a'_i, a'_{i+1}). Then consider the set $\{x \mid x \in (a_2, b) \wedge \forall a_2 < y < x(\text{f is constant}) \wedge \text{f is not constant at x}\}$. It has one element, a_3. Repeating this procedure yields $a_1, \ldots, a_k$ with the desired properties.

3.1.9.2. Let I and J be intervals and $f : I \to R$ and $g : J \to R$ strictly monotone definable functions such that $f(I) \subseteq g(I)$ and $\lim_{s \to r(I)} f(s) = \lim_{t \to r(J)} g(t)$ in R_∞, where $r(I)$ and $r(J)$ are the right endpoints of the intervals I and J in R_∞. Show that near these right endpoints f and g are reparametrizations of each other, that is, there are subintervals I' of I and J' of J, with $r(I) = r(I')$, $r(J) = r(J')$ and a strictly increasing definable bijection $h : I' \to J'$ such that $f(s) = g(h(s))$ for all $s \in I'$.

Let $h(s) = \mu t(g(t) = f(s))$, where μ signifies "the least." h is clearly definable, and is a function since g is strictly monotone and g's range includes f's. h is a bijection since f is strictly monotone:

if $h(s_1) = h(s_2)$, then $f(s_1) = f(s_2)$, which is impossible. h has domain I and range $J' = \{j \mid \exists i \in I(f(i) = g(j))\}$.

Assume f and g are monotone in opposite directions, say decreasing and increasing, respectively. Then for t sufficiently close to $r(I)$, s sufficiently close to $r(J')$, $f(x) < g(y)$, for all $r(I) < x < t$, $r(J) < y < s$, since they have the same limit, and thus must approach it from different sides. But then $f(x) < g(y)$ for all $x \in I$, $y \in J'$, which is impossible, since $f(I) \subseteq g(J)$. The same argument works when f is increasing and g is decreasing, so they must both be going in the same direction. Given this, h is clearly increasing, so we are done.

3.2.19.1. Let $C \subseteq R^m$ be a regular open cell and $f : C \to R$ a regular definable function. Show that the open cells $(-\infty, f)$, $(f, +\infty)$, and $C \times R = (-\infty, +\infty)$ in R^{m+1} are regular. Show that if $g : C \to R$ is a second regular definable function with $f < g$, then the open cell (f, g) in R^{m+1} is regular.

First we check regularity for $i < m + 1$. Let x and y differ only on the ith coordinate, say $x_i < y_i$, and let z agree with x and y on everything but the ith coordinate. By regularity, the projection of z to R^m, $\pi(z)$, is in C. This takes care of $C \times R$. Now consider $(-\infty, f)$. f is regular, assume it is strictly increasing on the ith coordinate, so $f(\pi(z)) \in (f(\pi(x)), f(\pi(y)))$. But $z_{m+1} = x_{m+1}$. $x_{m+1} < f(\pi(x))$, so $z_{m+1} < f(\pi(z))$, and $z \in (-\infty, f)$. If f is strictly decreasing, the same argument with y works. If f is the left endpoint, a similar argument works, and of course g as the right endpoint has the same argument.

Now we check for $i = m + 1$. But this is trivial, since if z differs from x and y only on the last coordinate, then $\pi(z) = \pi(x) = \pi(y)$, so $\pi(z) \in C$, and g_{m+1} lies on the interval between x_{m+1} and y_{m+1}, and thus g must be in the open cell.

3.2.19.2. Prove by induction on m the regular cell decomposition theorem.

Going through the proof of the cell decomposition theorem, we see that in the proof of I_{m+1}, that any definable set has a decomposition, first cells in R^m are constructed (obtained through the inductive assumption), along with some functions on these cells. By induction, these cells can be made regular. As well, they can be decomposed so that the functions are regular on these cells. Then the actual decomposition, formed by using these functions on these cells, will be regular by the previous problem.

For II_{m+1}, by the cell decomposition theorem, we can decompose f so that it is continuous. By the first part here, the decomposition can be regular. We can thus restrict to the case of a continuous function f on a regular cell, $C \subset R^{m+1}$. On each $x \in R^m$ such that $(x, t) \in C$ for some $t \in R$, the monotonicity theorem applies to $f(x, \cdot)$, which is defined on some interval by the regularity of C. Let $k(x)$ denote the number of points at which $f(x, \cdot)$ changes its type (from constant to monotone, etc.), and let the set of such points be $A(x)$. By uniform finiteness, there is an upper bound to $k(x)$ on C, so we can partition C so that $k(x)$ is constant. We now restrict to a cell in this new partition. Let $a_1(x), \ldots, a_k(x)$ denote the points of $A(x)$. Restricting, we can assume that the a_i's are continuous and regular, by induction. Restricting further, we can assume that f is continuous on a regular cell C and $f(x,\cdot)$ is monotone in the same way for every $x \in \pi(C)$. Now repeat this for each coordinate (there was nothing special about the last coordinate) to make f monotone on every coordinate, and thus regular.

3.2.19.3. Let C be a cell in R^m, $D = (\alpha, \beta)_C$ a cell in R^{m+1} and $f : D \to R$ a definable function such that for all $x \in C$ the function $f(x,\cdot) : (\alpha(x), \beta(x)) \to R$ is continuous. Show that C can be partitioned into cells $C_1, \ldots, C_k$ such that, with $\alpha_i = \alpha|C_i$, $\beta_i = \beta|C_i$, each restriction $f|(\alpha_i, \beta_i) : (\alpha_i, \beta_i) \to R$ is continuous.

By cell decomposition, we can partition D so that f is continuous on each cell in D. Let $\mathcal{D}$ be the decomposition. Note that if $B, B' \in \mathcal{D}$, then $\pi(B)$ and $\pi(B')$ are either equal or disjoint, where π is the projection map from R^{m+1} to R^m. Consider the decomposition of D given by $\{E \times R \mid E \in \pi(\mathcal{D})\}$. Denote it $\mathcal{E}$. Let $E \in \mathcal{E}$ be any cell. We show $f|E$ is continuous. Let $E = B_0 \cup \ldots \cup B_l$, $B_i \in \mathcal{D}$. Let $g_1, \ldots, g_k : R^m \to R$ be the boundary functions. We need only check continuity on the boundaries, since it is a local property. But this is easy – given (x,t) on a boundary, suppose we want $|f(\bar{u}) - f(x,t)| < \epsilon$. Let $t = g(x)$, where $g(x)$ is some boundary function. By the continuity of x on the boundary, we can find a definable open set U such that if $y \in \pi(E)$ and $(y, g(y)) \in U$, then $|f(y, g(y)) - f(x,t)| < \epsilon/2$. Let V be the set of such y's. By continuity of $f(x,\cdot)$, for each $y \in V$, there exists an interval $I(y)$ about $g(y)$ such that for $r \in I(y)$, $|f(y,r) - f(y,g(y))| < \epsilon/2$. $I(y)$ is clearly definable. Thus, if we consider the set $W = \{(y,s) \mid y \in V \wedge s \in I(y)\}$, for every $u \in W$, $|f(u) - f(x,t)| < \epsilon$, by the triangle inequality. W is easily seen to contain the intersection of an open set with E: $\pi(U)$ contains the intersection of an open set with $\pi(E)$, B, and then the construction of W keeps it open, since each point is crossed with an interval (in Van den Dries' notation, if B is a $(\ldots)$ cell, W is a $(\ldots, 1)$ cell).

3.2.19.4. Improve the cell decomposition theorem as follows: (I_m) If the sets $A_1, \ldots, A_k \subset R^m$ are definable, then there is a decomposition of R^m partitioning each set A_i, all of whose cells are definable in the model-theoretic structure $(R, <, A_1, \ldots, A_k)$. (II_m) Let the function $f : A \to R$, $A \subseteq R^m$, be definable. Then there is a decomposition $\mathcal{D}$ of R^m partitioning A, such that the restriction $f|B$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous, and each cell in $\mathcal{D}$ is definable in the model-theoretic structure $(R, <, \Gamma(F))$.

Going through the proof of I_m in the usual cell decomposition theorem, we see that the boundaries of the sets $A_1, \ldots, A_k$ are definable from them, so Y is definable, so its fibers are, so the B_i's are. The functions f_ij are definable, since they are definable from the Y_x's. Likewise, the $C_{\lambda ij}$'s and $D_{\lambda ij}$'s are definable. Thus, the decomposition $\mathcal{D}$ (which by our inductive hypothesis, modified for this problem, can be defined solely in terms of the B's, C's, D's, and f's) is definable from the A's. The final decomposition, D^*, comes from D through the f's, which were definable.

For II_m, note that the domain of a definable function is definable. Thus A is definable. The homeomorphism $p_A : A \to p(A)$ is definable from A (although not uniformly so). Then the inductive assumption, modified to this problem, finishes things if the cell is not open.

The set of well-behaved points, A^*, is definable from A and f, since boxes are always definable, and continuous and monotone are always definable. The fact that A^* is dense in A does not depend on our language. Then the decomposition, $\mathcal{D}$, which partitions A^* and A is definable from them by induction, and thus from f, since they are. Finally, the proof that f is continuous on $\mathcal{D}$ is language-independent.

3.2.19.5. Let $X_1, \ldots, X_k \subseteq R^m$ be distinct nonempty definably connected sets and X their union. Define a graph with vertex set $\{X_1, \ldots, X_k\}$ by putting an edge between X_i and X_j $(i \neq j)$ if $X_i \cap \mathrm{cl}(X_j) \neq \emptyset$ or $\mathrm{cl}(X_i) \cap X_j \neq \emptyset$. Show that if $X_{i(1)}, \ldots, X_{i(r)}$ are the vertices of a connected component of this graph, then $X_{i(1)} \cup \ldots \cup X_{i(r)}$ is a definably connected component of X, and that all definably connected components of X are of this form.

Define $A\ B$ to be the relation $A \cap \mathrm{cl}(B) \neq \emptyset$ or $B \cap \mathrm{cl}(A) \neq \emptyset$. Suppose $A\ B$, and A, B are definably connected. WLOG, let $A \cap \mathrm{cl}(B) \neq \emptyset$. Let $Y = A \cup B$. Suppose U, V witnessed Y's being not definably connected. Since A is definably connected, it is covered by either U or V. WLOG, let it be U. Then by a similar argument, V covers B. Since U and V are disjoint, $U = A$, $V = B$. Then $U \cap \mathrm{cl}(V) \neq \emptyset$. Let u be in this intersection. Since $u \in U$, we can find an open $W \subseteq U$, $u \in W$. But since $u \in \mathrm{cl}(V)$, any open set containing u intersects V. Then U and V intersect non-trivially, which is impossible. Thus $A \cup B$ is still definably connected.

Note that if $A = A_1 \cup A_2$, $A\ B \leftrightarrow A_1\ B \vee A_2\tilde{B}$, by the properties of cl(). Thus, the above result shows that any connected component of the graph is definably connected.

We now show that among the 2^k sets formed from the unions of $X_1, \ldots, X_k$, these connected components of the graph are maximal with respect to being definably connected. Consider any X_i, $X_i \not\subseteq S$. Since S comes from a connected component of the graph, we know that $\mathrm{cl}(S) \cap X_i = \emptyset$ and $S \cap \mathrm{cl}(X_i) = \emptyset$. Thus, S and X_i are open in $S \cup X_i$, so $S \cup X_i$ is not definably connected.

Such sets are connected components of X: Let S be such a set and consider any Y, definably connected, such that $S \cap Y \neq \emptyset$. Since $X_1, \ldots, X_k$ cover X, Y is covered as well. Let $P = \{X_i \mid X_i \cap Y \neq \emptyset$. Then $S \cup \bigcup P$ is definably connected (because Y is). But since S is maximal among the 2^k sets, and $S \cup \bigcup P$ is one of these, $S \cup \bigcup P = S$, so $Y \subseteq S$, and so S is maximal.

It remains to prove that these are all the connected components. But this is trivial – any connected component intersects some X_i's, which can be extended to form some S, a connected component from the graph, which intersects the connected component we already had and therefore contains it by the previous result.

3.2.19.6. Suppose $\mathcal{S}'$ is an o-minimal structure on $(R, <)$ with $\mathcal{S} \subseteq \mathcal{S}'$, and let $X \subseteq R^m$ belong to $\mathcal{S}$, so X also belongs to $\mathcal{S}'$. Show that X is definably connected in the sense of $\mathcal{S}$ if and only if X is definably connected in the sense of $\mathcal{S}'$.

In the forward direction, let $\mathcal{D}$ be a decomposition partitioning X in $\mathcal{S}$. If the graph (as defined above) of $\mathcal{D}$ is connected, then X is too, assuming that the members of $\mathcal{D}$ are all connected. So the question is whether a cell in $\mathcal{S}$ is necessarily definably connected in $\mathcal{S}'$. But any cell in $\mathcal{S}$ is a cell in $\mathcal{S}'$, and any cell in $\mathcal{S}'$ is definably connected in $\mathcal{S}'$. Thus, X is definably connected in $\mathcal{S}'$. The reverse implication is trivial.

3.2.19.7. Suppose $\mathcal{S}$ is an o-minimal structure on the ordered set $(\mathbb{R}, <)$ of real numbers. Show that for a definable set $X \subseteq \mathbb{R}^m$ the following are equivalent: (a) X is definably connected, (b), X is connected in the usual topological sense.

(b) implies (a) trivially, so assume X is definably connected. We show X is connected. Since X is definably connected, it has exactly one connected component. But this implies that the graph of a cell-decomposition of X is connected, as done above. Cells are certainly connected in the usual topology, so their union will be connected, because of the closure property used in constructing the graph.

3.2.19.8. With the same hypothesis as in exercise 7, show that each definable set $X \subseteq \mathbb{R}^m$ is locally connected, that is, for each $x \in X$ and each open subset U of X containing x there is a connected open subset V of X containing x and contained in U.

Since any open set in X contains a box intersected with X, we can restrict the discussion to definable sets. So we have a definable open set U containing x in X. Take a cell decomposition partitioning U and X. Take the connected component of U which contains x. Then this component is open in U, and thus in X (since U is open in X), and it is connected and contains x.

3.3.8. Let $\mathcal{R} = (R, <, \ldots)$ be an o-minimal L-structure and $\mathcal{R}' = (R', <', \ldots)$ an L-structure elementarily equivalent to $\mathcal{R}$. Show that $\mathcal{R}'$ is also o-minimal.

For any $\varphi(x, \bar{y})$, let k be the most boundary points $\varphi(x, \bar{a})$ ever has for any choice of $\bar{a} \in R^m$. Given $x_1, \ldots, x_k, x$, let $\sigma(x_1, \ldots, x_k, x)$ denote an ordering (possibly with some equalities) of $x_1, \ldots, x_k, x$. Let $\sigma_{\bar{y}}$ denote the ordering such that $\varphi(x, \bar{y})$ holds, with $x_1, \ldots, x_k$ the boundary points, with $x_l = x_{l+1} = \ldots = x_k$ if there are $l < k$ boundary points. Let $\Sigma = \{\sigma_{\bar{y}} \mid \bar{y} \in R^m\}$. Then

$$\mathcal{R} \models \forall \bar{y} \exists x_1, \ldots, x_k (\bigvee_{\Sigma} \forall x (\varphi(x, \bar{y}) \leftrightarrow \sigma(x_1, \ldots, x_k, x)))$$

Since $\mathcal{R}' \equiv \mathcal{R}$, $\mathcal{R}'$ satisfies the same sentence. But the above sentence says that φ defines an o-minimal set, for any parameters. Since φ was arbitrary, this means every definable set in $\mathcal{R}'$ must be o-minimal.

4.1.17.1. Let $\mathfrak{A} \subseteq R^m$ be definable and $) \leq d \leq m$. Show that $\dim A \geq d$ if and only if there is a d-tuple $i = (i(1), \ldots, i(d))$ with $1 \leq i(1) < \cdots < i(d) \leq m$ such that the projection map $p_i : R^m \to R^d$ given by $p_i(x_1, \ldots, x_m) = \big(x_{i(1)}, \ldots, x_{i(d)}\big)$ has the property that $p_i(A)$ has nonempty interior in R^d.

If $\dim(A) \geq d$, then A contains an $(\epsilon_1, \ldots, \epsilon_m)$-cell, where $epsilon_i \in \{0,1\}$, and $\sum_i \epsilon_i \geq d$. Then let $i(1)$ be the least i such that $\epsilon_i = 1$, $i(2)$ the next i such that $\epsilon_i = 1$, and so on. Then the projection of this cell to R^d yields an open cell, and thus $p_i(A)$ will have nonempty interior.

Conversely, if we have an A with the stated property, let A' be the image of A under the definable bijection which sends $x_{i(j)}$ to x_j. Then we know that $p(A')$, where p projects down to the first d coordinates, has nonempty interior. Let $\mathcal{D}$ be a decomposition of A'. $p(\mathcal{D}')$ is a decomposition of $p(A')$, so for some $C \in \mathcal{D}'$, $p(C)$ has nonempty interior in R^d, so it is open, so it is a $(1, \ldots, 1)$ cell. Then C must have dimension at least d, so A' does, so A does.

4.1.17.2. Let $A \subseteq R^m$ be a definable set and $a \in R^m$. Show there is a number $d \in \{-\infty, 0, \ldots, \dim A\}$ such that $\dim(U \cap A) = d$ for all sufficiently small definable neighborhoods U of a in R^m, that is, for all definable neighborhoods of a in R^m that are contained in some fixed definable neighborhood of a in R^m.

Let $\mathcal{D}$ be a decomposition partitioning A. Let $E = \{D \in \mathcal{D} \mid a \in \mathrm{cl}(D)\}$. Then I claim that $e = \max(\{\dim(D) \mid D \in E\})$ is the required constant. Let U be any open set containing a. Then $U \cap D \neq \emptyset$ for any $D \in E$. It is easy to see that, since U contains a box, that $\dim(U \cap D) = \dim D$. Thus, for any open U, $\dim(U \cap A)$ is certainly at least e. As well, $\dim(U \cap A) = \max(\{\dim(U \cap D) \mid D \in \mathcal{D}\})$. Taking V open such that $V \cap D = \emptyset$ for $D \notin E$, we see that for $U \subseteq V$, $\dim U \cap A \leq e$, and thus $= e$.

4.1.17.3*. Show that if A is a d-dimensional cell, then $\dim_a(A) = d$ for all $a \in \mathrm{cl}(A)$.

This follows from the previous problem by taking $\mathcal{D}$ to be $\{A\}$.

4.1.17.4. Let $A \subseteq R^m$ be a definable set and $d \in \{0, \ldots, \dim A\}$. Show that the set $\{a \in R^m \mid \dim_a(A) \geq d\}$ is a definable closed subset of $\mathrm{cl}(A)$. Show also that if $A \neq \emptyset$, then $\dim(\{a \in \mathrm{cl}(A) \mid \dim_a(A) < d\}) < d$.

Let $\mathcal{D}$ be a cell decomposition of A (note $\mathcal{D}$ is definable). By the arguments of 17.2, an element x is in the set defined by $\dim_x(A) = d$ if and only if x is in the closure of D for some D in $\mathcal{D}$ with $\dim(D) = d$. Since there are only finitely many $D \in \mathcal{D}$, and each D is definable, along with its dimension, this set is definable.

For the second part, note that the desired set consists of $\bigcup\{\mathrm{cl}(D) \mid \dim(D) < d\}$, and since cl preserves dimension, and it is a finite union, the union has dimension less than d.

5.2.14.1. Finish the proof of (2.12) by showing that $V(\mathrm{pos}(\mathcal{L})) \geq m + 1$.

Choose $f_1, \ldots, f_m$, and $a_1, \ldots, a_m \in X$, $f_i(a_j) = 0$ for $i \neq j$, $f_i(a_j) = 1$ for $i = j$. (These can be chosen by basic linear algebra.) Then for $u \subseteq \{1, \ldots, m\}$, let $f_u = \Sigma_i \delta_i f_i$, where $\delta_i = 1$ if $i \in u$, $\delta_i = 0$ if $i \in u$. Then $\mathrm{pos}(f_u) = \{a_i \mid i \in u\}$.

5.2.14.2. Show that $f^{\Phi \vee \Psi} \leq f^{\Phi} \cdot f^{\Psi}$, and derive that $f^{\Phi \wedge \Psi} \leq f^{\Phi} \cdot f^{\Psi}$.

Note that $(\Phi \vee \Psi)_x = \Phi_x \cup \Psi_x$. For any $x(1), \ldots, x(n) \in X$, $B((\Phi \vee \Psi)_{x(1)}, \ldots, (\Phi \vee \Psi)_{x(1)}) \subseteq B(\Phi_{x(1)}, \ldots, \Phi_{x(n)}, \Psi_{x(1)}, \ldots, \Psi_{x(n)})$. But the second boolean algebra is generated by the atoms in the boolean algebras for Φ and Ψ. There are at most $p_{f^\Phi}(n)$ and $p_{f^\Psi}(n)$ atoms in those, respectively. Thus, the number of atoms in the boolean algebra is the product of these two. The degrees of the polynomials are $f^\Phi - 1$ and $f^\Psi - 1$, so their product has degree $f^\Phi + f^\Psi - 2$. Thus, the number of atoms is given by a polynomial of degree $\leq f^\Phi + f^\Psi - 2$, which will be less than $p_{f^\Phi + f^\Psi - 1}(n)$, so $f^{\Phi \vee \Psi} \leq f^\Phi + f^\Psi - 1 \leq f^\Phi \cdot f^\Psi$.

Using the fact that $f^{\neg\Phi} = f^\Phi$, it is easy to show that $f^{\Phi \wedge \Psi} \leq f^\Phi \cdot f^\Psi$.

5.2.14.3. Show that if Φ and Ψ are dependent, then $\Phi \vee \Psi$ and $\Phi \wedge \Psi$ are dependent.

Trivial from the previous problem.

5.2.14.4. Derive from the previous two exercises and proposition (2.12) that every semi-algebraic relation $\Phi \subseteq \mathbb{R}^M \times \mathbb{R}^N$ $(M, N > 0)$ is dependent.

Given any Φ, let $f_1(X_1, \ldots, X_M, Y_1, \ldots, Y_N), \ldots, f_n(X_1, \ldots, X_M, Y_1, \ldots, Y_N)$ be the polynomials used to define the set. Let f_i have degree m_i in the Y_j's. Then the set of positive values of polynomials of degree $\leq m_i$ has finite VC-index, by proposition (2.12). So $\{\mathrm{pos}(f_i(a, Y_1, \ldots, Y_N)) \mid a \in \mathbb{R}^M\}$ has finite VC-index. Φ is the result of a finite boolean combination of these sets, and so by the above problems, is dependent.

6.1.15.1. Show that the definable set of representatives indicated in the proof of (1.2) is definable in the model-theoretic structure $(R, <, 1, +, E)$. (Recall that this set of representatives is given by $T := \{e(A) \mid A \text{ is an equivalence class}\}$.)

Note that $e(X)$, for $X = \varphi(R, y)$, is given by a formula which uses $\varphi(x, y)$. (It is

$$
\begin{aligned}
&\varphi(x,y) \\
&\wedge (\forall z < x(\neg\varphi(z,y)) \\
&\quad \vee (\forall z(\varphi(z,y) \to \exists u < z(\varphi(u,y))) \\
&\qquad \wedge ((x = 0 \wedge \forall z \exists u < z, w > z(\varphi(u,y) \wedge \varphi(w,y))) \\
&\qquad\quad \vee (\exists z(\forall u < z(\varphi(u,y)) \wedge \neg\varphi(z,y) \wedge x = z - 1)) \\
&\qquad\quad \vee (\exists z(\forall u > z(\varphi(u,y)) \wedge \neg\varphi(z,y) \wedge x = z + 1)) \\
&\qquad\quad \vee (\exists z_1, z_2 > z_1(\forall w < z_1(\neg\varphi(w,y)) \wedge \forall z_1 < u < z_2(\varphi(u,y)) \wedge \neg\varphi(z_2,y) \wedge x + x = z_1 + z_2)))))
\end{aligned}
$$

.)

Let it be $e_\varphi(x, y)$. Let $\varphi(x, y) = E(x, y)$. Then T can be defined as $\exists y(e_\varphi(x, y))$.

6.1.15.2. Show that E has only finitely many equivalence classes of dimension $\dim(X)$, and that each of them is definable in the model-theoretic structure $(R, <, 1, +, E)$.

Consider the set $S = e_\varphi(x, y) \subset X \times X$, as defined above. By a result on dimension (proposition 4.1.5), the fibers S_x which have dimension $\dim(X)$ form a definable set, definable from S, and thus definable in this structure. This set has to have dimension 0, from another part of the same result. Thus, it is finite. Since each equivalence class with dimension $\dim(X)$ is mapped to a distinct point in this set, there can be only finitely many such classes.

6.1.15.3. Suppose all equivalence classes of E have the same Euler characteristic e. Show that then the Euler characteristic of X is a multiple of e. (In particular, this shows that for $e > 1$ there is no definable equivalence relation on R^m all of whose equivalence classes have exactly e elements.)

Consider the definable injective map $f : X \to X \times X$ given by $a \to (e_\varphi(x, a), a)$. Then the Euler characteristic of X is equal to the Euler characteristic of $f(X)$. The projection map has fibers with Euler characteristic e. and so by a previous result (corollary 4.2.11), the Euler characteristic of $f(X)$ is $eE(\pi(e_\varphi(x, y)))$, and so it is a multiple of e.

6.1.15.4. (Uniform continuity) Let $X \subseteq R^m$ be a closed and bounded definable set and $f : X \to R^n$ a continuous definable map. Show that there is for each $\epsilon > 0$ a $\delta > 0$ such that whenever $|x - y| < \delta$, $x, y \in X$, we have $|f(x) - f(y)| < \epsilon$.

Suppose not, for some ϵ. Then for each $t > 0$, there is some $x \in X$ such that $\exists y(|x - y| < t \wedge |f(x) - f(y)| > \epsilon)$. Through definable curve selection, we can choose a curve, $\gamma : (0, a) \to X$, which picks out such an x for each $t \in (0, a)$ and is continuous. $\lim_{t\to 0} \gamma(t)$ exists, since each component of γ is eventually monotonic, and thus has a limit, since X is bounded. Since X is closed, the limit is in X. Then at this point, f will not be continuous. Since this is impossible, our assumption is false, so f is uniformly continuous.

6.1.15.5. (Fixed point theorem) Let X be a nonempty closed bounded definable subset of R^m and $f : X \to X$ a definable map such that $|f(x) - f(y)| < |x - y|$ for all distinct points $x, y \in X$. Show that f has a unique fixed point.

Define $\gamma : (0, 1) \to X$ by $|f(\gamma(t)) - \gamma(t)| = t$ (using definable choice). Clearly the domain of γ cannot have a minimum element, since $|f(f(x)) - f(x)| < |f(x) - x|$. So assume $(0, \epsilon] \notin \operatorname{dom}(\gamma)$ for some $\epsilon > 0$. Then define $\gamma'(t) = \gamma(t + \epsilon)$, and we have a map which is defined in a neighborhood of 0. Thus, since X is closed and bounded, the limit of γ' is in X, and so there is an element of X such that $|f(x) - x| = \epsilon$, which contradicts our assumption about ϵ, so actually γ is defined on a neighborhood of 0, and so its limit, which is in X, is a fixed point.

6.1.15.6. (Uniform curve selection) Let $X \subseteq R^m$ be definable. Show there are definable maps $\epsilon : \partial X \to (0, \infty)$ and $\Gamma : \partial X \times (0, \epsilon) \to X$ such that for each $a \in \partial X$ the function $t \to \Gamma(a, t) : (0, \epsilon(a)) \to X$ is continuous, injective and satisfies $\lim_{t\to 0} \Gamma(a, t) = a$.

For each $a \in \partial X$ and $t \in (0, \infty)$, we have the set $|x - a| = t$, which can be written as $\varphi(x, a, t) = x - a = t \vee a - x = t$. Then, using notation from the first problem, we can define $e_\varphi(x, y, t)$. For a given choice of y and t, $e_\varphi(x, y, t)$ picks out a unique element such that $|x - y| = t$, if it exists. The set $e_\varphi(X, a, t)$ is nonempty for some interval near 0, so we can define $\psi(y) = \exists x e_\varphi(x, y, t)$, which defines a subset of R. Then let $\epsilon(y) = e_\psi(y)$, and $\Gamma(y, t)$ be the unique x such that $e_\varphi(x, y, t)$.

6.1.15.7. Given a map $f : A \to R^n$, $A \subseteq R^m$, we call f locally bounded if each point $a \in A$ has a neighborhood U in A such that $f(U)$ is bounded. Let $A \subseteq R^m$ be definable and $f : A \to R^n$ definable. Prove the following equivalence:

$$f \text{ is continuous} \leftrightarrow f \text{ is locally bounded and } \Gamma(f) \text{ is closed in } A \times R^n$$

.

It is clear that if f is continuous, it is locally bounded. Moreover, if f is continuous, then take $(a, b) \in \mathrm{cl}(\Gamma(f))$. We can find, for any $\epsilon > 0$, U containing a such that if $a' \in U$, $|(a, b) - (a', f(a'))| < \epsilon$. But we can also find V containing a such that if $a' \in V$, $|(a, f(a)) - (a', f(a'))| < \epsilon$ (since we are using the supnorm, V is just the usual open set guaranteed by continuity intersected with an ϵ-box around a). Thus, for any ϵ we can get that $|b - f(a)| < 2\epsilon$, and thus that $b = f(a)$, so $\Gamma(f)$ is closed.

For the converse, suppose f is not continuous at $a \in A$. Let $\epsilon > 0$ be a counterexample to continuity. Thus, for any open U with $a \in U$, we can find $a' \in U$ with $|f(a) - f(a')| > \epsilon$. Assume f is locally bounded, so fix V open and bounded, $a \in V$, such that $f(V)$ is bounded. By definable choice, for each $t > 0$, we can find such an a' in the box $B(t) \subset R^m$, centered at a with sides of length t. Let c be such that $B(c) \subseteq V$. Then we have a definable curve $\gamma : (0, c) \to B(c)$ with the properties $\gamma(t) \in B(t)$ and $|f(\gamma(t)) - f(a)| > \epsilon$. Now consider the curve $\gamma' : (0, c) \to B(c) \times f(B(c))$ defined by $\gamma'(t) = (\gamma(t), f(\gamma(t))$. Clearly $\gamma(t)$ goes to a, but clearly $f(\gamma(t))$ does not go to $f(a)$, so $\gamma'(t)$ cannot go to $(a, f(a))$, and so its limit is in fact not in $\Gamma(f) \cap \mathrm{cl}(V, f(V))$, if the limit exists. If $\Gamma(f)$ were closed, then γ' would have a limit in $\Gamma(f) \cap \mathrm{cl}(V, f(V))$ as $t \to 0$, since $\Gamma(f) \cap \mathrm{cl}(V, f(V))$ contains the image of γ' and would be closed and bounded. Since this is not true, $\Gamma(f)$ cannot be closed.

6.1.15.8. Consider the o-minimal model-theoretic structure $(\mathbb{R}, <)$ and the set

$$X := \left\{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\, 0 < y < 2\right\},$$

which is definable in $(\mathbb{R}, <)$ using the constants $0, 1, 2$. Note taht $(1, 2) \in \mathrm{cl}(X)$ and show that there is no subset Y of X such that Y is definable in $(\mathbb{R}, <)$ using constants, $\dim(Y) = 1$ and $(1, 2) \in \mathrm{cl}(Y)$.

By elimination of quantifiers for dense linear orders, we have that Y must be defined by some quantifier-free formula $\varphi(x, y, \bar{a})$, with $\bar{a}$ some constants from $\mathbb{R}$, and consists of a disjunction of orderings of $x, y, \bar{a}$, of the form $a_1 < x = y < a_5$ (or $x < a_2 < y = a_3$, etc.) Since Y is 1-dimensional, it is easy to see that every such ordering must have either $x = a_i$ or $y = a_i$ for some i. Thus, geometrically,

Y is the finite union of points and horizontal and vertical lines. Since it is a finite union, we can take the closure of each of these components individually. The closures of the points are the points themselves, and since $Y \subseteq X$, they cannot include $(1,2)$. The closure of a horizontal or vertical line is the line along with the endpoints, which necessarily have one coordinate the same as the points on the line. Since no horizontal or vertical line contained in X can have x-coordinate 1 or y-coordinate 2, the endpoints cannot either, and so $(1,2)$ is not in the closure of any of these lines, and hence not in $\mathrm{cl}(Y)$.

6.2.5.1. Assume $(R, <, \mathcal{S})$ expands an ordered field. Show that then (2.4) holds even without the assumption that f is locally bounded. (2.4 states: Let $S \subset R^{m+n}$ be definable, $f : S \to R^k$ a locally bounded definable map, and $A \subseteq R^m$ a definable set such that for all $a \in A$ the map $f_a : S_a \to R^k$ is continuous, where $f_a(y) := f(a, y)$. Then there is a partition of A into definable subsets $A_1, \dots, A_M$ such that each restriction

$$f|S \cap (A_i \times R^n) : S \cap (A_i \times R^n) \to R^k$$

is continuous.)

We assume $A = \pi S$, since if not, redefine S. We need to show that we can partition πS so that f is locally bounded on the restrictions.

Define $s : S \times R \to R$ by $s(a, r) = \sup\{|f(x)| \mid x \in S \wedge \pi x = \pi a \wedge |x - a| < r\}$. Define $h : S \to R$ by $h(a) = \min(\sup\{r/2 \mid s(a, r) \text{ is defined}\}, 1)$. f is fiberwise continuous, so $s(a, r)$ is defined on some interval near 0, so h is defined on all of S. Define $g : S \to R$ by $g(a) = s(a, h(a))$. Take a decomposition, $\mathcal{D}$, of R^{m+n} which makes g and h continuous.

Let $C \in \mathcal{D}$ be a cell with $C \subseteq S$. Let $C' = \mathrm{cl}(S) \cap (\pi C \times R^n)$. We wish to show that $f|C'$ is locally bounded. Choose any $a \in C$. Take a box, B, containing a and small enough that in every "open" coordinate of C, B is contained in C. In other words, if C were made open by deleting all equality constraints in its cell definition, we would have $\mathrm{cl}(B) \subseteq C$. Note that $\mathrm{cl}(B) \cap C$ is closed. Define the set $W = \{w \mid \exists c \in \mathrm{cl}(B) \cap C(\pi w = \pi c \wedge |w - c| < h(c))\}$. W contains an open set in C'. (See below for argument.) Since g is continuous on C, g is continuous on $\mathrm{cl}(B) \cap C$, and since $\mathrm{cl}(B) \cap C$ is closed and bounded, g attains a maximum on it, which means that $|f|$ attains a maximum on W. Then in a neighborhood of a in C', f is bounded.

We construct a cell open in C', U, such that $U \subseteq W$. Begin with $\pi B \cap \pi C$. For each $b \in \pi B \cap \pi C$, there is a $y \in R^n$ such that $(b, y) \in C_b$. So $e(C_b)$ is defined for every $b \in \pi B \cap \pi C$ and gives some $c \in C$, where e is the definable choice function. Then for coordinates x_{m+1} to x_{m+n}, the equations for U are $e(C_b)_j - h(e(C_b)) < x_j < e(C_b)_j + h(e(C_b))$. This cell is clearly open in C', and for each $u \in U$, $|u - e(C_{\pi u})| < h(e(C_{\pi u}))$, so $u \in W$.

6.2.5.2. Assume $(R, <, \mathcal{S})$ expands an ordered field. Let $S \subseteq R^{m+n}$ be definable, $f : S \to R^k$ a definable map, and $A \subseteq R^m$ a definable set such that $f|S \cap (A \times R^n)$ is injective and $f_a : S_a \to R^k$

is a homeomorphism from S_a onto $f_a(S_a)$ for all $a \in A$. Show that there is a partition of A into definable subsets $A_1, \ldots, A_M$ such that each restriction $f|S \cap (A_i \times R^n) : S \cap (A_i \times R^n) \to R^k$ is a homeomorphism from $S \cap (A_i \times R^n)$ onto $f(S \cap (A_i \times R^n))$.

Denote f^{-1} by p. Define $s : f(S) \times R \to R$ by $s(a,r) = \sup\{|p(x)| \mid x \in f(S) \wedge \pi p(x) = \pi p(a) \wedge |x - a| < r\}$. Define $h : f(S) \to R$ by $h(a) = \min(\sup\{r \mid s(a,r) \text{ is defined}\}, 1)$. p is fiberwise continuous, so $s(a,r)$ is defined on some interval near 0, so h is defined on all of $f(S)$. Define $g : f(S) \to R$ by $g(a) = s(a, h(a))$. Take a decomposition of R^k which partitions $f(S)$ and makes g and h continuous, $\mathcal{D}_1$. Map the cells in $\mathcal{D}_1$ which are also in $f(S)$ to R^{m+n} using p, and decompose R^{m+n}, partitioning these cells, with $\mathcal{D}$, such that for $C \in \mathcal{D}$, $C \subseteq S$, $f|S \cap (\pi C \times R^n)$ is continuous. All we need to show is that p is locally bounded on the images of these sets, since the graph of p is just a permutation of the graph of f, and is therefore closed.

Let $C \in \mathcal{D}$ be a cell with $C \subseteq S$. Let $C' = \mathrm{cl}(S) \cap (\pi C \times R^n)$. We wish to show that $p|f(C')$ is locally bounded. Choose any $a \in f(C)$. Take a box, B, containing $p(a)$ and small enough that in every "open" coordinate of C, B is contained in C. Note that $\mathrm{cl}(B) \cap C$ is closed. Define the set $W = \{w \mid \exists c \in f(\mathrm{cl}(B) \cap C)(\pi p(w) = \pi p(c) \wedge |w - c| < h(c))\}$. W contains an open set in $f(C')$. Since g is continuous on $f(C)$ (as $f(C) \subseteq C_1$, for some $C_1 \in \mathcal{D}_1$), g is continuous on $f(\mathrm{cl}(B) \cap C)$, and since $\mathrm{cl}(B) \cap C$ is closed and bounded, $f(\mathrm{cl}(B) \cap C)$ is also closed and bounded. Thus, g attains a maximum on it, which means that $|p|$ attains a maximum on W. Then in a neighborhood of a in $f(C')$, p is bounded, so p is locally bounded on $f(C')$.

6.3.9.1. Show that in the model-theoretic structure $(\mathbb{R}, <, 0, 1, 2)$ the definable set $\{(x,y) \mid 0 < x < 1, 0 < y < 2\} \cup \{(1,2)\}$ is definably connected but not definably path connected.

Denote the set by X. As shown in problem 6.1.15.8, there is no definable path to $(1,2)$ from any point in the rest of X, so X is not definably path connected. However, X is definably connected: suppose U and V are open sets partitioning X. Since the rectangle is certainly definably connected, either U or V must contain the whole rectangle. But then the other one must be just $(1,2)$ if it is non-empty. But $\{(1,2)\}$ is not open in X, since any open set containing $\{(1,2)\}$ must also intersect the rectangle. So the other set is empty, and thus X is definably connected.

6.4.8.1. Let $f : X \to Y$ be a definably proper map. (This includes the assumption that X and Y are definable sets and f is definable and continuous.) Show that if $A \subseteq X$ is definable and closed in X, then $f(A)$ is closed in Y. Show that if $f' : X' \to Y'$ is a definably proper map, then $f \times f' : X \times X' \to Y \times Y'$ is definably proper. Show that if $g : Y \to Z$ is definably proper, then $g \circ f : X \to Z$ is definably proper.

For the first claim, let $y \in \mathrm{cl}(f(A))$. Then for each open box $B(t)$ with length t containing y, we can find $a \in A$ such that $f(a) \in B(t)$. Using definable choice, we can then make a map $\gamma : (0,c) \to A$ (for some $c > 0$), with $f(\gamma(t)) \in B(t)$. $f(\gamma)$ is completable in Y, with completion y. Thus it is completable in X, but since A is closed, its completion is in A, and it is clear that this completion's image is y, so

$y \in f(A)$, and so $f(A)$ is closed.

For the second claim, let A be a closed bounded subset of $Y \times Y'$. Let $A_1 = \{y \in Y \mid \exists y'((y, y') \in A)\}$, and A_2 the corresponding set for Y'. Consider $B = (f \times f')^{-1}(A)$. Let B_1 and B_2 be the similar sets for X and X'. B is clearly bounded, since $B_1 = f^{-1}(A_1)$ and $B_2 = {f'}^{-1}(A_2)$ must be, since A_1 and A_2 are closed and bounded. Since f and f' are continuous, their cartesian product is as well, and thus the inverse image of a closed set is closed, so B is closed and bounded.

For the third claim, if $L \subseteq Z$ is closed and bounded, then $K = g^{-1}(L)$ is closed and bounded, so $f^{-1}(K)$ is closed and bounded, so $(f \circ g)^{-1}(L) = f^{-1}(g^{-1}(L)) = f^{-1}(K)$ is closed and bounded.

6.4.8.2*. Let $(R, <, \mathcal{S}')$ be an o-minimal structure with $\mathcal{S} \subseteq \mathcal{S}'$ and let $f : X \to Y$ be a definable continuous map between definable sets X and Y in R^m and R^n, where "definable" is taken in the sense of $\mathcal{S}$. Assume also that Y is locally closed in R^n. Show f is definably proper with respect to $(R, <, \mathcal{S})$ if and only if f is definably proper with respect to $(R, <, \mathcal{S}')$.

The reverse direction is trivial. In the forward direction, let $K \subseteq Y$ be closed and bounded, with $K \in \mathcal{S}' \setminus \mathcal{S}$. $f^{-1}(K)$ is certainly closed, so the question is whether it is bounded. But if K is bounded, we can take a box which contains it, intersect it with Y, and take the closure to get a set definable in $\mathcal{S}$, containing K, and closed and bounded. Then its inverse image will be bounded, and so K's will be as well.

6.4.8.3*. Let $f : X \to Y$ be a definable continuous map between definable sets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$. Show that f is proper if and only if f is definably proper.

First, the reverse direction. By the first problem $f(A)$ is closed for any definable closed A. Since any closed B is the intersection of definable closed B's, and the images are all closed, $f(B)$ will be closed. $\{y\}$ is closed and bounded, so $f^{-1}(\{y\})$ is closed and bounded, and thus compact.

In the forward direction, Bourbaki has apparently proved that if f is proper, the inverse image of a compact set is compact, which is more than we need.

7.2.12.1. (L'Hôpital's rule) Let I be an interval and $f, g : I \to R$ definable functions, and let a be one of the endpoints of the interval, possibly $a = +\infty$ or $a = -\infty$. Suppose that $g'(x) \neq 0$ for all $x \in I$ in some neighborhood of a, and that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, or $\lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)| = +\infty$. Then

$$\lim_{x \to a} (f(x)/g(x)) = \lim_{x \to a} (f'(x)/g'(x)) .$$

(Note that both limits exist in R_∞, by Chapter 3, (1.6).)

(From Rudin) Assume a is the left endpoint (the right endpoint case is precisely analogous). Let the right-hand limit be A. If possible, choose a real q such that $q > A$, and choose r, $A < r < q$. We show the left-hand limit must be less than q. A similar argument will show that if $q' < A$, the left-hand limit is greater than q'. Thus, the limit exists and is equal to A (when $A = +\infty$, there is no q, but we can choose q' any real number, showing the limit is $+\infty$). Since the right-hand limit is A, we can find $c \in I$ such that $a < x < c$ implies $f'(x)/g'(x) < r$. If $a < x < y < c$, then by the generalized mean value theorem (Rudin 5.9), we can find $t \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r.$$

Now consider the different possibilities for $\lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)|$. If this limit is 0, then letting $x \to a$ in the above expression, we see that $f(y)/g(y) \le r < q$, for $a < y < c$.

If the limits of $|f|$ and $|g|$ are ∞, then choose $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ for $x \in (a, c_1)$. Multiplying the above equation by $[g(x) - g(y)]/g(x)$, we have

$$\frac{f(x) - f(y)}{g(x)} < r \frac{g(x) - g(y)}{g(x)}$$

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}.$$

As we let $x \to a$, since the left-hand side goes to r, this shows that for some $c_2 \in (a, c_1)$, $f(x)/g(x) < q$ for $x \in (a, c_2)$.

Thus, in any case, we have that the left-hand limit is less than q. A precisely similar argument for $q' < A$ shows that it is greater than q', and so is A. The case where a is a right endpoint is analogous.

7.2.12.2. (Taylor's formula) Suppose the definable function $f : I \to R$ is $(n+1)$ times differentiable on the interval I, and let $a, b \in I$, $a < b$. Then

$$f(b) = f(a) + f'(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(b-a)^{n+1}$$

for some z with $a < z < b$.

(From Rudin) Let the polynomial above be $P(b)$. Let M be defined by $f(b) = P(b) + M(b-a)^{n+1}$, and let $g(t) = f(t) - P(t) - M(t-a)^{n+1}$. We must show that $(n+1)!M = f^{(n+1)}(z)$ for some $z \in (a, b)$. Differentiating this expression for g, we get $g^{(n+1)}(t) = f^{(n+1)}(t) - (n+1)!M$, for $t \in (a, b)$. So we will be done if we know that $g^{(n+1)}(z) = 0$ for some $z \in (a, b)$. Since $P^{(k)}(a) = f^{(k)}(a)$ for $k \le n$, we have $g(a) = g'(a) = \ldots = g^{(n)}(a) = 0$. Our choice of M means that $g(b) = 0$, so there is some $z_1 \in (a, b)$ such that $g'(z_1) = 0$ by the mean value theorem, and then since $g'(a) = 0$, again there is some $z_2 \in (a, z_1)$ such that $g''(z_2) = 0$, and so on. This will give a z_{n+1} with $g^{(n+1)}(z_{n+1}) = 0$, which is the desired z.

7.3.3.1. Show that the remarks at the end of (3.1) go through with "C^1" replaced by "C^k." [The remarks note that inclusion maps are C^1, that compositions preserve C^1, and components are C^1 iff the function is too.]

The identity map is infinitely differentiable, the kth derivative of a composition involves only kth derivatives of each function, and the kth derivative of a function is defined componentwise.

7.3.3.2. State and prove the C^k-Cell Decomposition Theorem, $k \geq 1$.

The statement is the same as for the C^1 case (Theorem 7.3.2), with C^k replacing C^1. Note that, by first decomposing using the C^1 version of the theorem, we can consider only cells. So I_m^k can just say that, given C, there is a decomposition partitioning C such that each cell is C^k, and II_m^k can consider functions just on cells. For the proof, we go by induction on k and the dimension of the cells we are working with. We show I_{m+1}^{k+1} given I_m^{k+1}, I_{m+1}^k, and II_m^{k+1}. First, using I_{m+1}^k, decompose C into C^k-cells. Each function used to define these C^k cells is defined on (at most R^m). Thus, by II_m^{k+1}, there is a decomposition partitioning these cells such that all of these functions are C^{k+1} and the cells they are defined on are C^{k+1}, and hence this refinement of the original decomposition is C^{k+1}.

Now we show II_{m+1}^{k+1} given I. We have a function f on a cell, C. By II_{m+1}^k we can find a C^k-partition C so that f is C^k on each cell, so we may assume WLOG that f is C^k on a C^k-cell, C. Consider f'. By the C^1 case, we can find a C^1 partition of C such that f' is C^1 on each cell. Then f is C^{k+1} on each cell. It remains to show that each C^1-cell can be partitioned into C^{k+1}-cells. But the functions used to define each C^1 cell are on R^m, so by II_m^{k+1}, we can partition the cells so that the functions are C^{k+1} on C^{k+1}-cells, which finishes the proof.

7.4.3.1. Let $A \subseteq R^{m+1}$ be a definable set of dimension $\leq m$. Call a unit vector $u \in S^m \subseteq R^{m+1}$ an asymptotic direction for A if for each $\epsilon > 0$ and $r > 0$, there is a point $x \in X$ with $||x|| > r$ and $||(x/||x||) - u|| < \epsilon$. Show that the (definable) set of asymptotic directions for A is of dimension $< m$. (In particular, not every unit vector is an asymptotic direction for A.)

Assume the dimension of the set of asymptotic directions is m. Then there is some open set in R^m, U, containing only asymptotic directions. Let U be bounded. For each $x \in U$, if x is a good direction for A, then there is some upper bound r such that if $t > r$, $tx \notin A$. Let R be the maximum value of r on $\mathrm{cl}(U)$. The set $\{x/||x|| \mid x \in A, ||x|| > R\}$ is dense in U, so there is some open set, V, entirely contained in it. By the good directions lemma, there is some point, $v \in V$, which is a good direction for A, so $tv \notin A$ for $t > R$. But since $v \in V$, there is some $x \in A$, $||x|| > R$, such that $v = x/||x||$. Contradiction.

7.4.3.2. Let $A \subseteq R^m$ be definable and $\dim A \leq k < m$. Show that there is an $(m-k)$-dimensional linear subspace L of R^m all of whose translates $v + L$ ($v \in R^m$) meet A in only finitely many points. (Hint: proceed by induction on $m - k$.)

The case $m - k = 1$ is the theorem. In the case $m - k = n + 1$, first apply the theorem to get a line, L_1, all of whose translates meet A at finitely many points. Then consider $A + L_1$, that is $\{x \mid x = l + a,\ l \in L_1, a \in A\}$. This has dimension $\dim(A) + 1 \leq k + 1$, so by induction, there is an L', $m - k - 1$-dimensional linear subspace whose translates intersect $A + L_1$ in only finitely many points, and thus $L = L_1 + L'$ is an $m - k$-dimensional linear subspace whose translates intersect A in only finitely many points.

7.4.3.3. In this exercise, assume that R is an ordered field, not necessarily real closed.

Let $A \subseteq R^m$ be semilinear with $\dim(A) < m$. Show that A is contained in a finite union of affine subspaces of R^m of dimension $< m$, and derive that there is a good direction for A, that is, a nonzero vector $u \in R^m$ such that each line $p + Ru$ ($p \in R^m$) intersects A in only finitely many points.

Removing the inequalities from the definition of A, we have that A will be contained in a finite union of affine subspaces. Since inequalities correspond to open sets and therefore do not decrease the dimension, these affine subspaces will all have dimension $< m$. Any vector not in the span of the vectors giving the affine subspace will hit the subspace at only finitely many (in fact, 0 or 1) points for any translation. We can find a vector not in the span of any of the subspaces' vectors by induction. Let $A_1, \ldots, A_{n+1}$ be affine subspaces, with $A_i = V_i + w_i$, for V_i a vector subspace and w_i some vector. Suppose we have found v not in $V_1, \ldots, V_n$. Assume $v \in V_{n+1}$. Since $\dim(V_{n+1}) < m$, there is $u \notin V_{n+1}$. Then $v + tu \notin V_{n+1}$ for any $t \in R$. For each V_i, $i \leq n$, if $u \in V_i$, then $v + tu$ will not be in V_i. If $u \notin V_i$, then simple linear algebra shows that $v + tu \in V_i$ for only finitely many values of t. Thus, we can find t such that $v + tu \notin V_i$ for $i \leq n + 1$, and this vector will be a good direction for A.

4.1.17.2. Let $A \subseteq R^m$ be a definable set and $a \in R^m$. Show there is a number $d \in \{-\infty, 0, \ldots, \dim A\}$ such that $\dim(U \cap A) = d$ for all sufficiently small definable neighborhoods U of a in R^m, that is, for all definable neighborhoods of a in R^m that are contained in some fixed definable neighborhood of a in R^m.

Let $\mathcal{D}$ be a decomposition partitioning A. Let $E = \{D \in \mathcal{D} \mid a \in \mathrm{cl}(D)\}$. Then I claim that $e = \max(\{\dim(D) \mid D \in E\})$ is the required constant. Let U be any open set containing a. Then $U \cap D \neq \emptyset$ for any $D \in E$. It is easy to see that, since U contains a box, that $\dim(U \cap D) = \dim D$. Thus, for any open U, $\dim(U \cap A)$ is certainly at least e. As well, $\dim(U \cap A) = \max(\{\dim(U \cap D) \mid D \in \mathcal{D}\})$. Taking V open such that $V \cap D = \emptyset$ for $D \notin E$, we see that for $U \subseteq V$, $\dim U \cap A \leq e$, and thus $= e$.

4.1.17.3*. Show that if A is a d-dimensional cell, then $\dim_a(A) = d$ for all $a \in \mathrm{cl}(A)$.

This follows from the previous problem by taking $\mathcal{D}$ to be $\{A\}$.

4.1.17.4. Let $A \subseteq R^m$ be a definable set and $d \in \{0, \ldots, \dim A\}$. Show that the set $\{a \in R^m \mid \dim_a(A) \geq d\}$ is a definable closed subset of $\mathrm{cl}(A)$. Show also that if $A \neq \emptyset$, then $\dim\left(\{a \in \mathrm{cl}(A) \mid \dim_a(A) < d\}\right) < d$.

Let $\mathcal{D}$ be a cell decomposition of A (note $\mathcal{D}$ is definable). By the arguments of 17.2, an element x is in the set defined by $\dim_x(A) = d$ if and only if x is in the closure of D for some D in $\mathcal{D}$ with $\dim(D) = d$. Since there are only finitely many $D \in \mathcal{D}$, and each D is definable, along with its dimension, this set is definable.

For the second part, note that the desired set consists of $\bigcup\{\mathrm{cl}(D) \mid \dim(D) < d\}$, and since cl preserves dimension, and it is a finite union, the union has dimension less than d.

5.2.14.1. Finish the proof of (2.12) by showing that $V\,(\mathrm{pos}(\mathcal{L})) \geq m+1$.

Choose $f_1, \ldots, f_m$, and $a_1, \ldots, a_m \in X$, $f_i(a_j) = 0$ for $i \neq j$, $f_i(a_j) = 1$ for $i = j$. (These can be chosen by basic linear algebra.) Then for $u \subseteq \{1, \ldots, m\}$, let $f_u = \Sigma_i \delta_i f_i$, where $\delta_i = 1$ if $i \in u$, $\delta_i = 0$ if $i \in u$. Then $\mathrm{pos}(f_u) = \{a_i \mid i \in u\}$.

5.2.14.2. Show that $f^{\Phi \vee \Psi} \leq f^{\Phi} \cdot f^{\Psi}$, and derive that $f^{\Phi \wedge \Psi} \leq f^{\Phi} \cdot f^{\Psi}$.

Note that $(\Phi \vee \Psi)_x = \Phi_x \cup \Psi_x$. For any $x(1), \ldots, x(n) \in X$, $B((\Phi \vee \Psi)_{x(1)}, \ldots, (\Phi \vee \Psi)_{x(1)}) \subseteq B(\Phi_{x(1)}, \ldots, \Phi_{x(n)}, \Psi_{x(1)}, \ldots, \Psi_{x(n)})$. But the second boolean algebra is generated by the atoms in the boolean algebras for Φ and Ψ. There are at most $p_{f^{\Phi}}(n)$ and $p_{f^{\Psi}}(n)$ atoms in those, respectively. Thus, the number of atoms in the boolean algebra is the product of these two. The degrees of the polynomials are $f^{\Phi} - 1$ and $f^{\Psi} - 1$, so their product has degree $f^{\Phi} + f^{\Psi} - 2$. Thus, the number of atoms is given by a polynomial of degree $\leq f^{\Phi} + f^{\Psi} - 2$, which will be less than $p_{f^{\Phi}+f^{\Psi}-1}(n)$, so $f^{\Phi \vee \Psi} \leq f^{\Phi} + f^{\Psi} - 1 \leq f^{\Phi} \cdot f^{\Psi}$.

Using the fact that $f^{\neg \Phi} = f^{\Phi}$, it is easy to show that $f^{\Phi \wedge \Psi} \leq f^{\Phi} \cdot f^{\Psi}$.

5.2.14.3. Show that if Φ and Ψ are dependent, then $\Phi \vee \Psi$ and $\Phi \wedge \Psi$ are dependent.

Trivial from the previous problem.

5.2.14.4. Derive from the previous two exercises and proposition (2.12) that every semi-algebraic relation $\Phi \subseteq \mathbb{R}^M \times \mathbb{R}^N$ $(M, N > 0)$ is dependent.

Given any Φ, let $f_1(X_1, \ldots, X_M, Y_1, \ldots, Y_N), \ldots, f_n(X_1, \ldots, X_M, Y_1, \ldots, Y_N)$ be the polynomials used to define the set. Let f_i have degree m_i in the Y_j's. Then the set of positive values of polynomials of degree $\leq m_i$ has finite VC-index, by proposition (2.12). So $\{\mathrm{pos}(f_i(a, Y_1, \ldots, Y_N)) \mid a \in \mathbb{R}^M\}$ has finite VC-index. Φ is the result of a finite boolean combination of these sets, and so by the above problems, is dependent.

6.1.15.1. Show that the definable set of representatives indicated in the proof of (1.2) is definable in the model-theoretic structure $(R, <, 1, +, E)$. (Recall that this set of representatives is given by $T := \{e(A) \mid A \text{ is an equivalence class}\}$.)

Note that $e(X)$, for $X = \varphi(R, y)$, is given by a formula which uses $\varphi(x, y)$. (It is

$$
\begin{aligned}
&\varphi(x, y) \\
&\wedge (\forall z < x(\neg\varphi(z, y)) \\
&\quad \vee (\forall z(\varphi(z, y) \to \exists u < z(\varphi(u, y))) \\
&\qquad \wedge ((x = 0 \wedge \forall z \exists u < z, w > z(\varphi(u, y) \wedge \varphi(w, y))) \\
&\qquad\quad \vee (\exists z(\forall u < z(\varphi(u, y)) \wedge \neg\varphi(z, y) \wedge x = z - 1)) \\
&\qquad\quad \vee (\exists z(\forall u > z(\varphi(u, y)) \wedge \neg\varphi(z, y) \wedge x = z + 1)) \\
&\qquad\quad \vee (\exists z_1, z_2 > z_1(\forall w < z_1(\neg\varphi(w, y)) \wedge \forall z_1 < u < z_2(\varphi(u, y)) \wedge \neg\varphi(z_2, y) \wedge x + x = z_1 + z_2)))))
\end{aligned}
$$

.)

Let it be $e_\varphi(x, y)$. Let $\varphi(x, y) = E(x, y)$. Then T can be defined as $\exists y(e_\varphi(x, y))$.

6.1.15.2. Show that E has only finitely many equivalence classes of dimension $\dim(X)$, and that each of them is definable in the model-theoretic structure $(R, <, 1, +, E)$.

Consider the set $S = e_\varphi(x, y) \subset X \times X$, as defined above. By a result on dimension (proposition 4.1.5), the fibers S_x which have dimension $\dim(X)$ form a definable set, definable from S, and thus definable in this structure. This set has to have dimension 0, from another part of the same result. Thus, it is finite. Since each equivalence class with dimension $\dim(X)$ is mapped to a distinct point in this set, there can be only finitely many such classes.

6.1.15.3. Suppose all equivalence classes of E have the same Euler characteristic e. Show that then the Euler characteristic of X is a multiple of e. (In particular, this shows that for $e > 1$ there is no definable equivalence relation on R^m all of whose equivalence classes have exactly e elements.)

Consider the definable injective map $f : X \to X \times X$ given by $a \to (e_\varphi(x, a), a)$. Then the Euler characteristic of X is equal to the Euler characteristic of $f(X)$. The projection map has fibers with Euler characteristic e. and so by a previous result (corollary 4.2.11), the Euler characteristic of $f(X)$ is $eE(\pi(e_\varphi(x, y)))$, and so it is a multiple of e.

6.1.15.4. (Uniform continuity) Let $X \subseteq R^m$ be a closed and bounded definable set and $f : X \to R^n$ a continuous definable map. Show that there is for each $\epsilon > 0$ a $\delta > 0$ such that whenever $|x - y| < \delta$, $x, y \in X$, we have $|f(x) - f(y)| < \epsilon$.

Suppose not, for some ϵ. Then for each $t > 0$, there is some $x \in X$ such that $\exists y(|x - y| < t \wedge |f(x) - f(y)| > \epsilon)$. Through definable curve selection, we can choose a curve, $\gamma : (0, a) \to X$, which picks out such an x for each $t \in (0, a)$ and is continuous. $\lim_{t\to 0} \gamma(t)$ exists, since each component of γ is eventually monotonic, and thus has a limit, since X is bounded. Since X is closed, the limit is in X. Then at this point, f will not be continuous. Since this is impossible, our assumption is false, so f is uniformly continuous.

6.1.15.5. (Fixed point theorem) Let X be a nonempty closed bounded definable subset of R^m and $f : X \to X$ a definable map such that $|f(x) - f(y)| < |x - y|$ for all distinct points $x, y \in X$. Show that f has a unique fixed point.

Define $\gamma : (0, 1) \to X$ by $|f(\gamma(t)) - \gamma(t)| = t$ (using definable choice). Clearly the domain of γ cannot have a minimum element, since $|f(f(x)) - f(x)| < |f(x) - x|$. So assume $(0, \epsilon] \notin \mathrm{dom}(\gamma)$ for some $\epsilon > 0$. Then define $\gamma'(t) = \gamma(t + \epsilon)$, and we have a map which is defined in a neighborhood of 0. Thus, since X is closed and bounded, the limit of γ' is in X, and so there is an element of X such that $|f(x) - x| = \epsilon$, which contradicts our assumption about ϵ, so actually γ is defined on a neighborhood of 0, and so its limit, which is in X, is a fixed point.

6.1.15.6. (Uniform curve selection) Let $X \subseteq R^m$ be definable. Show there are definable maps $\epsilon : \partial X \to (0, \infty)$ and $\Gamma : \partial X \times (0, \epsilon) \to X$ such that for each $a \in \partial X$ the function $t \to \Gamma(a, t) : (0, \epsilon(a)) \to X$ is continuous, injective and satisfies $\lim_{t \to 0} \Gamma(a, t) = a$.

For each $a \in \partial X$ and $t \in (0, \infty)$, we have the set $|x - a| = t$, which can be written as $\varphi(x, a, t) = x - a = t \vee a - x = t$. Then, using notation from the first problem, we can define $e_\varphi(x, y, t)$. For a given choice of y and t, $e_\varphi(x, y, t)$ picks out a unique element such that $|x - y| = t$, if it exists. The set $e_\varphi(X, a, t)$ is nonempty for some interval near 0, so we can define $\psi(y) = \exists x e_\varphi(x, y, t)$, which defines a subset of R. Then let $\epsilon(y) = e_\psi(y)$, and $\Gamma(y, t)$ be the unique x such that $e_\varphi(x, y, t)$.

6.1.15.7. Given a map $f : A \to R^n$, $A \subseteq R^m$, we call f locally bounded if each point $a \in A$ has a neighborhood U in A such that $f(U)$ is bounded. Let $A \subseteq R^m$ be definable and $f : A \to R^n$ definable. Prove the following equivalence:

$$f \text{ is continuous} \leftrightarrow f \text{ is locally bounded and } \Gamma(f) \text{ is closed in } A \times R^n$$

.

It is clear that if f is continuous, it is locally bounded. Moreover, if f is continuous, then take $(a, b) \in \mathrm{cl}(\Gamma(f))$. We can find, for any $\epsilon > 0$, U containing a such that if $a' \in U$, $|(a, b) - (a', f(a'))| < \epsilon$. But we can also find V containing a such that if $a' \in V$, $|(a, f(a)) - (a', f(a'))| < \epsilon$ (since we are using the supnorm, V is just the usual open set guaranteed by continuity intersected with an ϵ-box around a). Thus, for any ϵ we can get that $|b - f(a)| < 2\epsilon$, and thus that $b = f(a)$, so $\Gamma(f)$ is closed.

For the converse, suppose f is not continuous at $a \in A$. Let $\epsilon > 0$ be a counterexample to continuity. Thus, for any open U with $a \in U$, we can find $a' \in U$ with $|f(a) - f(a')| > \epsilon$. Assume f is locally bounded, so fix V open and bounded, $a \in V$, such that $f(V)$ is bounded. By definable choice, for each $t > 0$, we can find such an a' in the box $B(t) \subset R^m$, centered at a with sides of length t. Let c be such that $B(c) \subseteq V$. Then we have a definable curve $\gamma : (0, c) \to B(c)$ with the properties $\gamma(t) \in B(t)$ and $|f(\gamma(t)) - f(a)| > \epsilon$. Now consider the curve $\gamma' : (0, c) \to B(c) \times f(B(c))$ defined by $\gamma'(t) = (\gamma(t), f(\gamma(t))$. Clearly $\gamma(t)$ goes to a, but clearly $f(\gamma(t))$ does not go to $f(a)$, so $\gamma'(t)$ cannot go to $(a, f(a))$, and so its limit is in fact not in $\Gamma(f) \cap \mathrm{cl}(V, f(V))$, if the limit exists. If $\Gamma(f)$ were closed, then γ' would have a limit in $\Gamma(f) \cap \mathrm{cl}(V, f(V))$ as $t \to 0$, since $\Gamma(f) \cap \mathrm{cl}(V, f(V))$ contains the image of γ' and would be closed and bounded. Since this is not true, $\Gamma(f)$ cannot be closed.

6.1.15.8. Consider the o-minimal model-theoretic structure $(\mathbb{R}, <)$ and the set

$$X := \left\{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\, 0 < y < 2\right\},$$

which is definable in $(\mathbb{R}, <)$ using the constants $0, 1, 2$. Note taht $(1, 2) \in \mathrm{cl}(X)$ and show that there is no subset Y of X such that Y is definable in $(\mathbb{R}, <)$ using constants, $\dim(Y) = 1$ and $(1, 2) \in \mathrm{cl}(Y)$.

By elimination of quantifiers for dense linear orders, we have that Y must be defined by some quantifier-free formula $\varphi(x, y, \bar{a})$, with $\bar{a}$ some constants from $\mathbb{R}$, and consists of a disjunction of orderings of $x, y, \bar{a}$, of the form $a_1 < x = y < a_5$ (or $x < a_2 < y = a_3$, etc.) Since Y is 1-dimensional, it is easy to see that every such ordering must have either $x = a_i$ or $y = a_i$ for some i. Thus, geometrically, Y is the finite union of points and horizontal and vertical lines. Since it is a finite union, we can take the closure of each of these components individually. The closures of the points are the points themselves, and since $Y \subseteq X$, they cannot include $(1, 2)$. The closure of a horizontal or vertical line is the line along with the endpoints, which necessarily have one coordinate the same as the points on the line. Since no horizontal or vertical line contained in X can have x-coordinate 1 or y-coordinate 2, the endpoints cannot either, and so $(1, 2)$ is not in the closure of any of these lines, and hence not in $\mathrm{cl}(Y)$.

6.2.5.1. Assume $(R, <, \mathcal{S})$ expands an ordered field. Show that then (2.4) holds even without the assumption that f is locally bounded. (2.4 states: Let $S \subset R^{m+n}$ be definable, $f : S \to R^k$ a locally bounded definable map, and $A \subseteq R^m$ a definable set such that for all $a \in A$ the map $f_a : S_a \to R^k$ is continuous, where $f_a(y) := f(a, y)$. Then there is a partition of A into definable subsets $A_1, \ldots, A_M$ such that each restriction

$$f|S \cap (A_i \times R^n) : S \cap (A_i \times R^n) \to R^k$$

is continuous.)

We show the result for $f : S \to R$, which suffices, because if we make each component of a map continuous, the map is continuous. We assume $A = \pi S$, since if not, redefine S. Go by induction on m.

Using the regular cell decomposition theorem, we can find a decomposition, $\mathcal{D}_1$, of R^{m+n} such that for each $C \in \mathcal{D}$, $f|C$ is continuous and monotonic or constant in every variable. If C is an $(i_1, \ldots, i_{m+n})$-cell, in van den Dries' notation, let $G_C = \{j \mid i_j = 1\}$. Define $a \sim_C b$ to be $a_i = b_i$ for $i \in G_C$ (the C will be dropped when it is clear from context). Define E_C to be $\{(x_1, \ldots, x_{m+n}) \mid \exists a \in C(a \sim x \wedge |a - x| = 1)\}$. Note that the a in the definition is uniquely defined by the first clause, and note that E_C is not in the closure of C or vice versa. Let X be the set of points at which f is not locally bounded (X is definable). Now take a decomposition $\mathcal{D}$ of R^{m+n} which partitions $\mathcal{D}_1$ and X and each E_C, and is also a stratification.

Let $C \in \mathcal{D}$ be a cell with $C \subseteq X$. Let $C' = \pi C \times R^n$. If πC is not open in R^m, we are done by the canonical projection and induction, so assume it is. We will derive a contradiction from this. Note that by this assumption, $\{1, \ldots, m\} \subseteq G_C$. Choose any $a \in C$. f is continuous on C, so there exists a box B_1 containing a such that for $b \in B_1 \cap C$, $|f(b) - f(a)| < 1$. Let B_2 be a box containing a such that if $j \in G_C$, and $(b_1, \ldots, b_j, \ldots, b_{m+n}) \in B_2 \cap C$, then there exists b'_j such that $(b_1, \ldots, b'_j, \ldots, b_{m+n}) \in C \setminus B_2$ (on those coordinates where C is open, B_2 is "inside" C). Let

$U = \{x \mid \exists b \in C(x \sim b \wedge |x - b|) < 1\}$. U is easily seen to be an open cell. Let V be a neighborhood of a containing only points from C or cells which are connected to C (this is possible by the definition of cells connecting to each other). Let B' be a box contained in $B_1 \cap B_2 \cap U \cap V$. $f|C'$ is not locally bounded, since C' is open in R^{m+n}, so for any $T > 0$, there exists $b_T \in B'$ with $|f(b_T)| > T$. Since there are only finitely many cells intersecting B', we can find some cell F with infinitely many such points. Let $B = B' \cap F$. Consider any $b = (b_1, \ldots, b_{m+n}) \in B$. Since $b \in U$, there is $b' \in C$ such that $b \sim b'$. Let $j \in G_F \setminus G_C$. We can find a least $t > 0$ such that $b_t = (b_1, \ldots, b_j + t, \ldots, b_{m+n}) \notin F$. This is so because F cannot be in E_C, since it borders C, and $|b - b'| < 1$, since $b \in U$, so by increasing t, we eventually make $|b_t - b'| = 1$, so $b_t \notin F$. Of course, the minimum necessary may occur before this. We can also find the greatest $s < 0$ such that $b_s \notin F$. By monotonicity of f on x_j, along with continuity, $f(b_s)$ and $f(b_t)$ bound $f(b)$. (This is the crucial place where πC must be open. If πC were not open, then j might be less than m, and then f might not be continuous on x_j, so we would not be able to bound $f(b)$.) Since $\mathcal{D}$ is a stratification, b_s and b_t lie in lower-dimension cells than F. Repeating this procedure with b_s and b_t, we can continue until we have $f(m(b)) < b < f(M(b))$, with $m(b)$ and $M(b)$ lying in cells $H(b)$ and $K(b)$, both in C', with $G_{H(b)} \setminus G_C = G_{K(b)} \setminus G_C = \emptyset$. It may be that $H(b) \neq H(c)$ for some $b, c \in B$, but if we consider an infinite sequence (not definable) in B, $(b^1, b^2, \ldots)$, approaching a such that $|f(b^i)| > i$, by passing to an infinite subsequence all b^i's must share the same H and K.

Fix such an H and K. Since $M(b) \sim b$ and $b \sim b'$, we have that for each $M(b)$, there is a unique $b' \in C$ with $M(b) \sim b'$. Since the b^i's are going to a, the ${b'}^i$'s must be going to a also. Since $G_H \setminus G_C = \emptyset$, the $M(b)$'s will go to a limit as well (each $M(b^i)_j$ for $j \in G_C$ is going to a_j, but this uniquely defines an element in $\bar{H}$). Let this limit be a_H. If $a_H \in H$, then construct B_3 for H the same way B_2 was constructed for C. On $\bar{B}_3 \cap H$, f is bounded, and since for large enough i every $M(b^i)$ is in B_3, $\{f(M(b^i)) \mid i < \omega\}$ is bounded. By a similar argument, if $a_K \in K$, $\{f(m(b^i)) \mid i < \omega\}$ is bounded. Since $f(m(b^i)) < f(b^i) < f(M(b^i))$ for all i, this is impossible.

Thus, $a_H \notin H$ or $a_K \notin K$. WLOG, $a_H \notin H$. Let N be the cell containing a_H. Since $a_H \in \mathrm{cl}(H)$ and $\mathcal{D}$ is a stratification, $\dim(N) < \dim(H) \leq \dim(C)$. This is true for every $x \in C$. For distinct $x, y \in C$, x_H and y_H must be distinct for any sequences approaching x and y that we choose, since $x \not\sim y$. There are only finitely many cells of dimension less than $\dim(C)$, and each point in C is associated with at least one point in their union, in an injective way. This is clearly impossible. Thus, our assumption was wrong and πC is not open.

6.2.5.2. Assume $(R, <, \mathcal{S})$ expands an ordered field. Let $S \subseteq R^{m+n}$ be definable, $f : S \to R^k$ a definable map, and $A \subseteq R^m$ a definable set such that $f|S \cap (A \times R^n)$ is injective and $f_a : S_a \to R^k$ is a homeomorphism from S_a onto $f_a(S_a)$ for all $a \in A$. Show that there is a partition of A into definable subsets $A_1, \ldots, A_M$ such that each restriction $f|S \cap (A_i \times R^n) : S \cap (A_i \times R^n) \to R^k$ is a homeomorphism from $S \cap (A_i \times R^n)$ onto $f\left(S \cap (A_i \times R^n)\right)$.

We have from the previous problem that we can make f continuous on cells. The question is whether we can make f^{-1} continuous. 6.3.9.1. Show that in the model-theoretic structure $(\mathbb{R}, <, 0, 1, 2)$ the definable set $\{(x, y) \mid 0 < x < 1, 0 < y < 2\} \cup \{(1, 2)\}$ is definably connected but not definably path connected.

Denote the set by X. As shown in problem 6.1.15.8, there is no definable path to $(1, 2)$ from any point in the rest of X, so X is not definably path connected. However, X is definably connected: suppose U and V are open sets partitioning X. Since the rectangle is certainly definably connected, either U or V must contain the whole rectangle. But then the other one must be just $(1, 2)$ if it is non-empty. But $\{(1, 2)\}$ is not open in X, since any open set containing $\{(1, 2)\}$ must also intersect the rectangle. So the other set is empty, and thus X is definably connected.

6.4.8.1. Let $f : X \to Y$ be a definably proper map. (This includes the assumption that X and Y are definable sets and f is definable and continuous.) Show that if $A \subseteq X$ is definable and closed in X, then $f(A)$ is closed in Y. Show that if $f' : X' \to Y'$ is a definably proper map, then $f \times f' : X \times X' \to Y \times Y'$ is definably proper. Show that if $g : Y \to Z$ is definably proper, then $g \circ f : X \to Z$ is definably proper.

For the first claim, let $y \in \mathrm{cl}(f(A))$. Then for each open box $B(t)$ with length t containing y, we can find $a \in A$ such that $f(a) \in B(t)$. Using definable choice, we can then make a map $\gamma : (0, c) \to A$ (for some $c > 0$), with $f(\gamma(t)) \in B(t)$. $f(\gamma)$ is completable in Y, with completion y. Thus it is completable in X, but since A is closed, its completion is in A, and it is clear that this completion's image is y, so $y \in f(A)$, and so $f(A)$ is closed.

For the second claim, let A be a closed bounded subset of $Y \times Y'$. Let $A_1 = \{y \in Y \mid \exists y'((y, y') \in A)\}$, and A_2 the corresponding set for Y'. Consider $B = (f \times f')^{-1}(A)$. Let B_1 and B_2 be the similar sets for X and X'. B is clearly bounded, since $B_1 = f^{-1}(A_1)$ and $B_2 = f'^{-1}(A_2)$ must be, since A_1 and A_2 are closed and bounded. Since f and f' are continuous, their cartesian product is as well, and thus the inverse image of a closed set is closed, so B is closed and bounded.

For the third claim, if $L \subseteq Z$ is closed and bounded, then $K = g^{-1}(L)$ is closed and bounded, so $f^{-1}(K)$ is closed and bounded, so $(f \circ g)^{-1}(L) = f^{-1}(g^{-1}(L)) = f^{-1}(K)$ is closed and bounded.

6.4.8.2*. Let $(R, <, \mathcal{S}')$ be an o-minimal structure with $\mathcal{S} \subseteq \mathcal{S}'$ and let $f : X \to Y$ be a definable continuous map between definable sets X and Y in R^m and R^n, where "definable" is taken in the sense of $\mathcal{S}$. Assume also that Y is locally closed in R^n. Show f is definably proper with respect to $(R, <, \mathcal{S})$ if and only if f is definably proper with respect to $(R, <, \mathcal{S}')$.

The reverse direction is trivial. In the forward direction, let $K \subseteq Y$ be closed and bounded, with $K \in \mathcal{S}' \setminus \mathcal{S}$. $f^{-1}(K)$ is certainly closed, so the question is whether it is bounded. But if K is bounded, we can take a box which contains it, intersect it with Y, and take the closure to get a set definable in $\mathcal{S}$, containing K, and closed and bounded. Then its inverse image will be bounded, and so K's will be as well.

6.4.8.3*. Let $f : X \to Y$ be a definable continuous map between definable sets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$. Show that f is proper if and only if f is definably proper.

First, the reverse direction. By the first problem $f(A)$ is closed for any definable closed A. Since any closed B is the intersection of definable closed B's, and the images are all closed, $f(B)$ will be closed. $\{y\}$ is closed and bounded, so $f^{-1}(\{y\})$ is closed and bounded, and thus compact.

In the forward direction, Bourbaki has apparently proved that if f is proper, the inverse image of a compact set is compact, which is more than we need.

7.2.12.1. (L'Hôpital's rule) Let I be an interval and $f, g : I \to R$ definable functions, and let a be one of the endpoints of the interval, possibly $a = +\infty$ or $a = -\infty$. Suppose that $g'(x) \neq 0$ for all $x \in I$ in some neighborhood of a, and that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, or $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = +\infty$. Then

$$\lim_{x\to a} \left(f(x)/g(x)\right) = \lim_{x\to a} \left(f'(x)/g'(x)\right).$$

(Note that both limits exist in R_∞, by Chapter 3, (1.6).)

(From Rudin) Assume a is the left endpoint (the right endpoint case is precisely analogous). Let the right-hand limit be A. If possible, choose a real q such that $q > A$, and choose r, $A < r < q$. We show the left-hand limit must be less than q. A similar argument will show that if $q' < A$, the left-hand limit is greater than q'. Thus, the limit exists and is equal to A (when $A = +\infty$, there is no q, but we can choose q' any real number, showing the limit is $+\infty$). Since the right-hand limit is A, we can find $c \in I$ such that $a < x < c$ implies $f'(x)/g'(x) < r$. If $a < x < y < c$, then by the generalized mean value theorem (Rudin 5.9), we can find $t \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r.$$

Now consider the different possibilities for $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)|$. If this limit is 0, then letting $x \to a$ in the above expression, we see that $f(y)/g(y) \leq r < q$, for $a < y < c$.

If the limits of $|f|$ and $|g|$ are ∞, then choose $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ for $x \in (a, c_1)$. Multiplying the above equation by $[g(x) - g(y)]/g(x)$, we have

$$\frac{f(x) - f(y)}{g(x)} < r\frac{g(x) - g(y)}{g(x)}$$

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}.$$

As we let $x \to a$, since the left-hand side goes to r, this shows that for some $c_2 \in (a, c_1)$, $f(x)/g(x) < q$ for $x \in (a, c_2)$.

Thus, in any case, we have that the left-hand limit is less than q. A precisely similar argument for $q' < A$ shows that it is greater than q', and so is A. The case where a is a right endpoint is analogous.

7.2.12.2. (Taylor's formula) Suppose the definable function $f : I \to R$ is $(n+1)$ times differentiable on the interval I, and let $a, b \in I$, $a < b$. Then

$$f(b) = f(a) + f'(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(b-a)^{n+1}$$

for some z with $a < z < b$.

(From Rudin) Let the polynomial above be $P(b)$. Let M be defined by $f(b) = P(b) + M(b-a)^{n+1}$, and let $g(t) = f(t) - P(t) - M(t-a)^{n+1}$. We must show that $(n+1)!M = f^{(n+1)}(z)$ for some $z \in (a, b)$. Differentiating this expression for g, we get $g^{(n+1)}(t) = f^{(n+1)}(t) - (n+1)!M$, for $t \in (a, b)$. So we will be done if we know that $g^{(n+1)}(z) = 0$ for some $z \in (a, b)$. Since $P^{(k)}(a) = f^{(k)}(a)$ for $k \leq n$, we have $g(a) = g'(a) = \ldots = g^{(n)}(a) = 0$. Our choice of M means that $g(b) = 0$, so there is some $z_1 \in (a, b)$ such that $g'(z_1) = 0$ by the mean value theorem, and then since $g'(a) = 0$, again there is some $z_2 \in (a, z_1)$ such that $g''(z_2) = 0$, and so on. This will give a z_{n+1} with $g^{(n+1)}(z_{n+1}) = 0$, which is the desired z.

7.3.3.1. Show that the remarks at the end of (3.1) go through with "C^1" replaced by "C^k." [The remarks note that inclusion maps are C^1, that compositions preserve C^1, and components are C^1 iff the function is too.]

The identity map is infinitely differentiable, the kth derivative of a composition involves only kth derivatives of each function, and the kth derivative of a function is defined componentwise.

7.3.3.2. State and prove the C^k-Cell Decomposition Theorem, $k \geq 1$.

The statement is the same as for the C^1 case (Theorem 7.3.2), with C^k replacing C^1. Note that, by first decomposing using the C^1 version of the theorem, we can consider only cells. So I_m^k can just say that, given C, there is a decomposition partitioning C such that each cell is C^k, and II_m^k can consider functions just on cells. For the proof, we go by induction on k and the dimension of the cells we are working with. We show I_{m+1}^{k+1} given I_m^{k+1}, I_{m+1}^k, and II_m^{k+1}. First, using I_{m+1}^k, decompose C into C^k-cells. Each function used to define these C^k cells is defined on (at most R^m). Thus, by II_m^{k+1}, there is a decomposition partitioning these cells such that all of these functions are C^{k+1} and the cells they are defined on are C^{k+1}, and hence this refinement of the original decomposition is C^{k+1}.

Now we show II_{m+1}^{k+1} given I. We have a function f on a cell, C. By II_{m+1}^k we can find a C^k-partition C so that f is C^k on each cell, so we may assume WLOG that f is C^k on a C^k-cell, C. Consider f'. By the C^1 case, we can find a C^1 partition of C such that f' is C^1 on each cell. Then f is C^{k+1} on each cell. It remains to show that each C^1-cell can be partitioned into C^{k+1}-cells. But the functions used to define each C^1 cell are on R^m, so by II_m^{k+1}, we can partition the cells so that the functions are C^{k+1} on C^{k+1}-cells, which finishes the proof.

7.4.3.1. Let $A \subseteq R^{m+1}$ be a definable set of dimension $\leq m$. Call a unit vector $u \in S^m \subseteq R^{m+1}$ an asymptotic direction for A if for each $\epsilon > 0$ and $r > 0$, there is a point $x \in X$ with $||x|| > r$ and

$||(x/||x||) - u|| < \epsilon$. Show that the (definable) set of asymptotic directions for A is of dimension $< m$. (In particular, not every unit vector is an asymptotic direction for A.)

Assume the dimension of the set of asymptotic directions is m. Then there is some open set in R^m, U, containing only asymptotic directions. Let U be bounded. For each $x \in U$, if x is a good direction for A, then there is some upper bound r such that if $t > r$, $tx \notin A$. Let R be the maximum value of r on $\mathrm{cl}(U)$. The set $\{x/||x|| \mid x \in A, ||x|| > R\}$ is dense in U, so there is some open set, V, entirely contained in it. By the good directions lemma, there is some point, $v \in V$, which is a good direction for A, so $tv \notin A$ for $t > R$. But since $v \in V$, there is some $x \in A$, $||x|| > R$, such that $v = x/||x||$. Contradiction.

7.4.3.2. Let $A \subseteq R^m$ be definable and $\dim A \leq k < m$. Show that there is an $(m-k)$-dimensional linear subspace L of R^m all of whose translates $v + L$ ($v \in R^m$) meet A in only finitely many points. (Hint: proceed by induction on $m - k$.)

The case $m - k = 1$ is the theorem. In the case $m - k = n + 1$, first apply the theorem to get a line, L_1, all of whose translates meet A at finitely many points. Then consider $A + L_1$, that is $\{x \mid x = l + a, l \in L_1, a \in A\}$. This has dimension $\dim(A) + 1 \leq k + 1$, so by induction, there is an L', $m - k - 1$-dimensional linear subspace whose translates intersect $A + L_1$ in only finitely many points, and thus $L = L_1 + L'$ is an $m - k$-dimensional linear subspace whose translates intersect A in only finitely many points.

7.4.3.3. In this exercise, assume that R is an ordered field, not necessarily real closed.

Let $A \subseteq R^m$ be semilinear with $\dim(A) < m$. Show that A is contained in a finite union of affine subspaces of R^m of dimension $< m$, and derive that there is a good direction for A, that is, a nonzero vector $u \in R^m$ such that each line $p + Ru$ ($p \in R^m$) intersects A in only finitely many points.

Removing the inequalities from the definition of A, we have that A will be contained in a finite union of affine subspaces. Since inequalities correspond to open sets and therefore do not decrease the dimension, these affine subspaces will all have dimension $< m$. Any vector not in the span of the vectors giving the affine subspace will hit the subspace at only finitely many (in fact, 0 or 1) points for any translation. We can find a vector not in the span of any of the subspaces' vectors by induction. Let $A_1, \ldots, A_{n+1}$ be affine subspaces, with $A_i = V_i + w_i$, for V_i a vector subspace and w_i some vector. Suppose we have found v not in $V_1, \ldots, V_n$. Assume $v \in V_{n+1}$. Since $\dim(V_{n+1}) < m$, there is $u \notin V_{n+1}$. Then $v + tu \notin V_{n+1}$ for any $t \in R$. For each V_i, $i \leq n$, if $u \in V_i$, then $v + tu$ will not be in V_i. If $u \notin V_i$, then simple linear algebra shows that $v + tu \in V_i$ for only finitely many values of t. Thus, we can find t such that $v + tu \notin V_i$ for $i \leq n + 1$, and this vector will be a good direction for A.