Determinacy of model constructions

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Imagine that two people want to build a structure together, as a game. There are basically two ways of doing so : a first method consists in progressively defining the properties the final structure is to satisfy, and at the end, in seeing what kind of structures satisfy them all. In the dual approach, the players start with a structure, and extend it as the game goes by; it is only for the final structure that the problem of which properties are satisfied is addressed. Here we deal with games of both kinds : Ziegler's and Fraïssé's games.

Each player wants the final structure to satisfy some properties, but has to stick with what has already been chosen. Is it always possible for one of the player to achieve his/her ends, regardless of what the other plays?

We develop our analysis following the central line: the duality between two styles of playing games, equivalently of building models. In the first section, we introduce formally what a game is and go over the basic properties of our setting. The second section is devoted to construction of models as limits of conditions, namely the Ziegler game. In the third section, which is our last, the dual construction is presented. There, we construct models as limits of their substructures.

1 Games and determinacy

We must first define exactly what a game is:

Definition 1. A set of instructions for playing a game is called a **pre-game**. A **game** is a pre-game with a criterion for deciding which player wins.

We consider two-player pre-games played by Héloïse and Abélard, who we call respectively $\exists$(loise) and $\forall$(belard). The pre-games consist of a non-empty set C of **conditions**, ordered by inclusion. Player $\forall$ begins by choosing a condition p_0, then $\exists$ chooses a condition p_1 such that $p_0 \subseteq p_1$, and so on. Thus they form a chain $(p_n)_{n<\omega}$ of conditions. The player whose turn it is to choose a condition is allowed to know what has been played since the beginning of the game.

At the end of the play, $p_\omega = \bigcup_{n<\omega} p_n$ determines a structure A called the **compiled structure**. We will decide who the winner is depending on the properties of this compiled structure. Let us precise what kind of properties we study :

Definition 2. A **property** φ is a class of structures which is closed under isomorphism. A structure is thus said to **satisfy** or to **have** φ if it lies in the class φ. By extension, we will consider sentences as properties too.

We are now ready to play a full game :

Definition 3. Let G be a pre-game and φ a property. We define the game $\boldsymbol{G(\varphi)}$ as the pre-game G in which $\exists$ wins if the compiled structure satisfies φ and $\forall$ wins otherwise.

Now that we have all the rules, let us consider the different tactics the players can use. As most model theorists do, we will support Héloïse and wonder if she can play in such a way that whatever Abélard plays, she manages to win. We begin by introducing the notion of strategy :

Definition 4. An **∃-strategy** is a family $\{\sigma^n | n \ odd\ ,\ \sigma^n : C^n \to C\}$ of functions which tells ∃ how to play : more precisely, if $p_0, ..., p_{n-1}$ are the conditions chosen at the first n steps, ∃ will choose p_n to be $\sigma^n(p_0, ..., p_{n-1})$. Likewise, we define an **∀-strategy**.

Definition 5. An ∃-strategy in a game is said to be **winning** if, when ∃ uses it, she wins the game regardless of what ∀ plays. Likewise, we define a winning ∀-strategy.

Definition 6. Let φ be a property and G a game. We say that φ is **determined** with respect to G if one of the players has a winning strategy for the game $G(\varphi)$.

If every property is determined with respect to G, we say that G is **wholly determined**.

When ∃ has a winning strategy for the game $G(\varphi)$, we can deem the associated property φ nice, hence the following definition:

Definition 7. Let φ be a property. We say that φ is **enforceable** when ∃ has a winning strategy for the game $G(\varphi)$. It is called **coenforceable** when, almost as nice, ∀ has a winning strategy for the game $G(\neg\varphi)$. Finally, we say that a condition p **forces** φ if ∃ has a strategy which is winning whenever ∀ chooses a condition $p_0 \supseteq p$ as his first move.

Let us now prove an elementary lemma which establishes a link between those notions :

Lemma 1. *i) A property φ is enforceable if and only if every condition forces φ.*

ii) A property φ is coenforceable if and only if some condition forces φ.

iii) Every enforceable property is coenforceable.

Proof. *i*) It is immediate from the definition.

ii) Suppose that φ is coenforceable and let σ be a winning ∀-strategy for $G(\neg\varphi)$. Then the condition σ^0 forces φ. For suppose that, in the game $G(\varphi)$, ∀ plays $p_0 \supseteq \sigma^0$; ∃ can win by choosing p_{2n} to be $\sigma^{2n+1}(\sigma^0, p_0, p_1, ..., p_{2n-1})$ for each $n < \omega$.

Conversely, suppose that a condition p forces φ and let σ be an ∃-strategy which is winning whenever ∀ chooses a first condition containing p. Then ∀ has a winning strategy for $G(\neg\varphi)$ which consists in playing p as his first move and then using σ.

iii) It follows from the two previous points. □

In the next two sections, we will present two particular games and study their determinacy.

2 The Ziegler pre-game

Let us begin with an example to introduce the principle of the game:

Example. *We work in the language $\mathcal{L} = \{1, \cdot, ^{-1}\}$ of groups. We define a condition to be a finite set of equations and inequations of $\mathcal{L} \cup W$, where W is a countable set of new constants. The set of conditions is ordered by natural inclusion. At the end of a game, the compiled group is the group presented by W and the set of all equations in p_ω.*

The particular case of groups illustrates well this game and the construction it yields. Remarkably, the general discussion follows the same lines and shows that the setting is far more general than that of group theory.

The Ziegler's pre-game. *Let $\mathcal{L}$ be a countable language and T a consistent $\mathcal{L}$-theory. Let W be a countable set of constants we call* ***witnesses****.*

A ***condition*** *for the pre-game is a finite set of atomic and negated atomic sentences of $\mathcal{L}(W)$. The set of conditions is naturally ordered by inclusion.*

At the end of a play, let U be the set of all atomic sentences deducible from $T \cup p_\omega$. As U is $=$-closed, there is, up to isomorphism, a unique $\mathcal{L}(W)$-structure A' such that $Diag^+(A') = U$: the ***canonical model*** *of U (see [Hod93]). Then the* ***compiled structure*** *A is obtained by considering A' as an $\mathcal{L}$-structure.*

We now wish to establish a criterion for the Ziegler pre-game on a theory to be wholly determined. We will need the following theorem [Hod85]:

Dichotomy theorem. *In the Ziegler pre-game on any theory T, exactly one of the following holds:*

- *There is a countable set X of structures such that :*
  $i)$ the property "The compiled structure A is isomorphic to some structure in X" is enforceable
  $ii)$ for any structure B in X, the property "A is isomorphic to B" is coenforceable

- *For every enforceable property φ, there are continuum many non-isomorphic structures which have φ.*

Theorem 1. *The Ziegler pre-game on a theory T is wholly determined if and only if T is good.*

Proof. $\Leftarrow$] Suppose that T is a good theory. Let X be a set as in the dichotomy theorem and φ be a property.

- Suppose there is a structure B in X which fails to have property φ. Since it is coenforceable to have A isomorphic to B, it is also coenforceable to have $\neg\varphi$ (using the same strategy).

- Conversely, if every structure in X has property φ, then condition $i)$ gives $\exists$ a winning strategy for the property "A is isomorphic to some structure in X" thus for the property φ.

$\Rightarrow$] Suppose now that T is a bad theory. It follows that, since the property "A is a countable model of $Diag^+(T)$" is always true, and thus enforceable, there are 2^ω non-isomorphic countable models of $Diag^+(T)$, which we enumerate as $\{C_\alpha | \alpha < 2^\omega\}$.

Let us call an ordinal $\alpha < 2^\omega$ **constrainable** when some condition forces the property " A is isomorphic to C_α", that is when it is coenforceable. Since $\mathcal{L} \cup W$ is countable, there are only countably many conditions and thus countably many constrainable ordinals.

In order to show that the pre-game is not wholly determined, we will build a property φ which defeats every strategy the players could use. Hence let us list all the $\forall$- and $\exists$-strategies as $\{\sigma_\beta | \beta < 2^\omega\}$.

By induction on β, we will build chains $\{X_\beta | \beta < 2^\omega\}$ and $\{Y_\beta | \beta < 2^\omega\}$ of subsets of 2^ω to define φ as " A is isomorphic to some C_α with $\alpha \in \bigcup_{\beta<2^\omega} Y_\beta$".

We will ask that,$\forall \beta < 2^\omega$:

- $X_\beta \cap Y_\beta = \emptyset$

- $|X_\beta| < 2^\omega$ and $|Y_\beta| < 2^\omega$
- $|X_{\beta+1} \setminus X_\beta| \leqslant 1$ and $|Y_{\beta+1} \setminus Y_\beta| \leqslant 1$

Let us put $X_0 = \emptyset$ and take Y_0 to bet the set of all constrainable ordinals. We take unions at limit ordinals. Suppose that X_β and Y_β have been chosen. We consider the two following cases :

- Suppose σ_β is a $\exists$-strategy.

  Put $Z = \{\alpha < 2^\omega |\ \textit{when } \exists \textit{ uses } \sigma_\beta,\ \textit{it is possible for } A \textit{ to be isomorphic to } C_\alpha\}$.

  Since we want to prevent σ_β from being a winning strategy for $G(\varphi)$, we have to make sure that $\forall$ is able to play against σ_β, that is to ensure that $Z \not\subseteq \bigcup_{\beta<2^\omega} Y_\beta$. Since the X_γ and Y_γ are disjoint, it suffices to ensure that $Z \cap X_{\beta+1} \neq \emptyset$.

  But now, let us notice that σ_β is a winning strategy for the property " A is isomorphic to some C_α, with $\alpha \in Z$". No matter what the players do, A must indeed be isomorphic to some C_γ, with $\gamma < 2^\omega$, since the compiled structure is a model of $Diag^+(T)$. And, by the choice of Z, this γ is necessarily in Z. Thus, since T is bad, there are continuum many non-isomorphic countable structures which have this (enforceable) property. In particular, $|Z| = 2^\omega$.

  Hence, since the induction hypothesis guarantees that $|X_\beta \cup Y_\beta| < 2^\omega$, we can choose x_β in $Z \setminus (X_\beta \cup Y_\beta)$ and we put $X_{\beta+1} = X_\beta \cup \{x_\beta\}$ and $Y_{\beta+1} = Y_\beta$.

- Now suppose σ_β is a $\forall$-strategy.

  As before, we put $Z = \{\alpha < 2^\omega |\ \textit{when } \forall \textit{ uses } \sigma_\beta,\ \textit{it is possible for } A \textit{ to be isomorphic to } C_\alpha\}$. We shall ensure that $Z \cap \bigcup_{\beta<2^\omega} Y_\beta \neq \emptyset$, and more precisely, that Z meets $Y_{\beta+1}$.

  But in this case, Z is not necessarily of cardinality 2^ω. If it is the case, then we can simply choose y_β in $Z \setminus (X_\beta \cup Y_\beta)$ and put $X_{\beta+1} = X_\beta$ and $Y_{\beta+1} = Y_\beta \cup \{y_\beta\}$.

  If not, we will see that Z already meets Y_β, by exhibiting a good theory which extends T: let $T' = T \cup \sigma_\beta^0$. We will show that T' fails to satisfy the second part of the dichotomy theorem and is thus the good theory we are looking for.

  By definition of Z, the condition σ_β^0 forces the property "A is isomorphic to some C_α, with $\alpha \in Z$". Hence it is enforceable with respect to T', for if σ is the strategy that enables $\exists$ to win whenever $\forall$ chooses a first condition which contains σ_β^0, and if, in the game on T', $\forall$ plays p^0 as his first move, $\exists$ can play $p^0 \cup \sigma_\beta^0$ which is still a condition for T'. And from that point on, $\exists$ can use σ to win.

  Since $|Z| < 2^\omega$, there are less than continuum many non-isomorphic structures which satisfy this (enforceable) property. This proves that T' is good. Then by the dichotomy theorem, there exists a set X of structures such that, for any $B \in X$, it is coenforceable, with respect to T', that "A is isomorphic to B".

  Since the compiled structure is necessarily isomorphic to some model of $Diag^+(T')$ which also happens to be a model of $Diag^+(T)$, it is necessarily isomorphic to some C_α. Thus, X must contain at least one structure B which is isomorphic to some C_γ. So there is a condition p which, with respect to T', forces the property ψ : "A is isomorphic to C_γ", by lemma 1. It follows that in the initial game on T, the condition $p \cup \sigma_\beta^0$ forces ψ. That means exactly that γ is constrainable so we have $\gamma \in Y_0 \subseteq Y_\beta$.

But γ is also in Z, for if $\forall$ plays σ^0_β, $\exists$ can choose the condition $p \cup \sigma^0_\beta$ and then use the previous strategy so that ψ holds.

Finally, we have proved that Z already meets Y_β, so in this last case, we can leave X_β and Y_β unchanged.

The construction of our two chains is now complete, and by the choice of φ, none of the σ_β is a winning strategy for $G(\varphi)$. Thus, φ is not determined and neither is the Ziegler pre-game on T. $\square$

Let us now switch to the other game, in which the compiled structure is obtained a completely different way since it takes actual structures as starting points.

3 The Fraïssé's pre-game

In this section we will analyze the dual game to that of Ziegler. The main idea is to construct structures as limits of their substructures. Indeed, the rules are based on a well-known method of in model theory: the Fraïssé's construction. As before, groups form a nice intuitive class although the discussion is not limited to them at all. It in fact was motivated by rational numbers with their usual order, which is nothing but the limit of finite ordered sets.

Example. *Let us call condition any finitely generated group. We order the set of conditions by the relation "is a subgroup of". At the end of a play, the compiled structure is simply the union p_ω of all the groups that had been chosen during the game, which is a group too.*

To define the pre-game, we need the following definition:

Definition 8. Let $\mathcal{L}$ be language and $\mathbf{K}$ a class of $\mathcal{L}$-structures. Let $\mathbf{K}_{fg}$ be the class of all finitely generated structures of $\mathbf{K}$. We say that $\mathbf{K}$ is a **Fraïssé class** if it satisfies the following properties:

- $\mathbf{K}$ is closed under unions of countable chains.
- If $B \in \mathbf{K}$ and C is embeddable in B, then $C \in \mathbf{K}$.
- If C_1 and C_2 are in $\mathbf{K}_{fg}$, then there exists $D \in \mathbf{K}_{fg}$ such that both C_1 and C_2 are embeddable in D.
- If B, C_1 and C_2 are in $\mathbf{K}_{fg}$ and B is embeddable in both C_1 and C_2, then there exists $D \in \mathbf{K}_{fg}$ and embeddings of C_1 and C_2 in D which agree on B (when B is seen as a substructure of C_1 and C_2).

The Fraïssé's pre-game. *Let $\mathcal{L}$ be a countable language and $\mathbf{K}$ be a Fraïssé class of $\mathcal{L}$-structures. A **condition** for the pre-game is a structure in $\mathbf{K}_{fg}$. We order the set of conditions by the relation " is a substructure of". The **compiled structure** is then the union of the chain of structures of $\mathbf{K}_{fg}$ that have been chosen by the players, and is thus in $\mathbf{K}$ by the first requirement in the definition of a Fraïssé class.*

As in the Ziegler's pre-game, we want to know when the Fraïssé's pre-game is wholly determined. We first see how Fraïssé's construction gives us a sufficient condition; we will see in a second section that the latter is in fact a necessary condition too.

3.1 Fraïssé's construction

We will need the following definitions:

Definition 9.
- An equivalence class of structures under the relation of isomorphism is called an **isomorphism type**. Let $\mathbf{F}$ be the set of all the isomorphism types of structures of $\mathbf{K}_{fg}$.
- Let B be a structure. The **age** of B is the set of all elements of F which are the isomorphism types of substructures of B. We write it $age(B)$.
- Let B be a structure. We say that B is **weakly homogeneous** if for every pair of structures C and D such that $C \subseteq B$, C is embeddable in D and whose isomorphism types are in $age(B)$, there is an embedding from D to B which is the identity on the image of C in D.

Remarks. - *The cardinality of* $\mathbf{F}$ *is at most* 2^ω*.*

An isomorphism type in $\mathbf{K}_{fg}$ *indeed corresponds to the* $\mathcal{L}$*-type of a finite tuple : if two tuples have the same type, the structures they generate are isomorphic. Hence, as the language is countable, there are at most* 2^ω *types and* $|\mathbf{F}| \leqslant 2^\omega$ *too.*

- If a structure is countable, then its age is countable too.

- If two countable weakly homogeneous structures have the same age, they are isomorphic.

Fraïssé's theorem. *Let* $\mathcal{L}$ *be a countable language and* $\mathbf{J}$ *be a non-empty countable set of finitely generated* $\mathcal{L}$*-structures. Suppose that* $\mathbf{J}$ *satisfies the following properties :*

(HP) If $B \in \mathbf{J}$ *and* C *is a finitely generated substructure of* B*, then* C *is isomorphic to some structure in* $\mathbf{J}$*.*

(JEP) If B *and* C *are in* $\mathbf{J}$*, then there is* $D \in \mathbf{J}$ *such that both* B *and* C *are embeddable in* D*.*

(AP) If B, C_1 *and* C_2 *are in* $\mathbf{J}$*, and* B *is embeddable in both* C_1 *and* C_2*, then there is a structure* $D \in \mathbf{J}$ *and embeddings of* C_1 *and* C_2 *in* D *which agree on* B*.*

Then there exists a countable weakly homogeneous $\mathcal{L}$*-structure* D *such that* $age(D) = \mathbf{J}$*. Moreover,* D *is unique up to isomorphism. It is called the* ***Fraïssé limit*** *of the class* $\mathbf{J}$*.*

Example. *The Fraïssé limit of finite linear orderings is the rationals, with their usual order.*

Proof. The last remark guarantees the uniqueness of the Fraïssé limit.

Without loss of generality, we can suppose that $\mathbf{J}$ is closed under isomorphism.

We shall build a chain $(D_i)_{i<\omega}$ of structures of $\mathbf{J}$ so that for every B and C in $\mathbf{J}$ such that $B \subseteq C$ and f embeds B in some D_i , then there are $j > i$ and an embedding g of C in D_j which extends f.

Suppose that we have built such a chain. Then the union D of the D_i is the structure we are looking for. It is indeed weakly homogeneous by the previous property. Besides, the age of D is included in $\mathbf{J}$ since the age of each D_i is (by the hereditary property (HP)). To show that $age(D)$ is in fact equal to the whole $\mathbf{J}$, let us take B in $\mathbf{J}$. Then (JEP) gives us a structure $C \in \mathbf{J}$ such that both B and D_0 are embeddable in C. By choice of the D_is, the identity map on D_0 extends to an embedding of C in some D_j. Hence, B is a isomorphic to a substructure of D_j and of D, and lies in the age of D.

Let us now construct that chain. We first choose any structure D_0 in $\mathbf{J}$ and build the rest of the sequence by induction: suppose that D_k has been chosen. Let us enumerate as $((f_{ij}, B_{ij}, C_{ij}))_{i\leqslant k, j<\omega}$ all the triples (f, B, C) where $B \subseteq C$ and f is an embedding from B to C. Let us also choose a bijection π between $\mathbb{N}^2$ and $\mathbb{N}$ so that for every $i, j \in \mathbb{N}$, $\pi(i, j) \geqslant i$. Thus there exists a unique couple (i, j) such that $k = \pi(i, j)$, and we have $i \leqslant k$. We then build D_{k+1} by the amalgamation property (AP) such that there is an embedding g_{ij} of C_{ij} into D_{k+1} which extends f_{ij}, and the chain satisfies the wanted property. $\square$

3.2 Fraïssé's pre-game determined, a sufficient condition

The previous construction allows us to state the first determinacy result about Fraïssé's pre-game:

Theorem 2. *If* $\mathbf{F}$ *is countable, then the Fraïssé's pre-game on the class* $\mathbf{K}$ *is wholly determined.*

Proof. Since $\mathbf{K}$ is a Fraïssé class, $\mathbf{K}_{fg}$ satisfies the condition of Fraïssé's theorem. Thus, there is a countable weakly homogeneous structure D of age $\mathbf{K}_{fg}$. Besides, D is unique up to isomorphism, so, if $\exists$ follows the method of construction given in the proof of Fraïssé's theorem, she can make the compiled structure A isomorphic to D. So, if φ is a property satisfied by D, $\exists$ has a winning strategy for $G(\varphi)$. If D does not have the property φ, then $\neg\varphi$ is enforceable, and also coenforceable by lemma 1, so that $\forall$ has a winning strategy for $G(\varphi)$. $\square$

Let us now study the converse:

3.3 The uncountable case

Suppose that $\mathbf{F}$ is uncountable.

Definition 10. Let X be a set. By $\mathcal{P}_\omega(X)$, we denote the set of all countable subsets of X. Let W be a subset of $\mathcal{P}_\omega(X)$.

- W is **closed unbounded** in $\mathcal{P}_\omega(X)$ if :

  i) W is closed under unions of countable chains.

  ii) For every $Y \in \mathcal{P}_\omega(X)$, there exists $Z \in W$ such that $Y \subseteq Z$.

- We say that W is **fat** if it contains a closed unbounded set.

Lemma 2. *If* X *is uncountable, the intersection of two closed unbounded sets in* $\mathcal{P}_\omega(X)$ *is itself closed unbounded.*

Let now S be the set of all subsets W of $\mathbf{F}$ for which there exists a countable weakly homogeneous structure B in $\mathbf{K}$ such that $W = age(B)$. By the remark in the previous section, such W are countable and thus $S \subseteq \mathcal{P}_\omega(\mathbf{F})$.

Lemma 3. S *is closed unbounded in* $\mathcal{P}_\omega(\mathbf{F})$.

Let us now define a collection of properties whose enforceability we will establish a criterion for : for every $W \subseteq \mathcal{P}_\omega(\mathbf{F})$, let φ_W be the property " $age(A) \in W$".

Proposition 1. *The property* φ_W *is enforceable if and only if* W *is fat.*

Proof. $\Leftarrow$] It suffices to show that if W is closed unbounded in $\mathcal{P}_\omega(\mathbf{F})$, then φ_W is enforceable. Thus, let W be closed unbounded.

Let $(X_i)_{i<\omega}$ be a partition of ω into strictly countable subsets.

Let $i \in \omega$, and suppose that $\forall$ chooses a condition p_{2i}. Since W is unbounded, there is an element s_{2i} of W such that the isomorphism type of p_{2i} is in s_{2i}. If such an element has been found for p_{2i-2}, it is possible to choose s_{2i} such that $s_{2i-2} \subseteq s_{2i}$.

$\exists$ has to ensure that $age(A) \in W$. By definition of the s_{2i}, she already knows that $age(A) \subseteq \bigcup_{i<\omega} s_{2i}$. But $\bigcup_{i<\omega} s_{2i} \in W$ since W is closed under unions of countable chains. So she shall try to have $age(A) = \bigcup_{i<\omega} s_{2i}$ to win the game $G(\varphi_W)$.

For every t in s_{2i}, $\exists$ has to make sure that t is the isomorphism type of some substructure of A. Since s_{2i} is countable, we can enumerate s_{2i} by $\{t_n | n \in \omega\}$. Thus, if $n \in X_i$, then $\exists$ can choose p_{2n+1} to be $p_{2n} \cup T_n$, where T_n is a structure of type t_n. Thus, T_n will be a substructure of A and we will have $t_n \in age(A)$.

$\Rightarrow$] Suppose now that φ_W is enforceable and let σ be a winning $\exists$-strategy for $G(\varphi_W)$. We shall build a closed unbounded set which is contained in W.

Let Y be the set of all countable subsets of $\mathbf{F}$ which are closed under every function σ^n : $P \in Y$ if and only if $P \in \mathcal{P}_\omega(\mathbf{F})$ and for all $n \in \omega$, if the isomorphism types of $p_0, ..., p_{n-1}$ are in P, then the isomorphism type of $\sigma^n(p_0, ..., p_{n-1})$ is in P too.

Then Y is closed unbounded in $\mathbf{F}$. Closure is clear from the definition, and if $P \in \mathcal{P}_\omega(\mathbf{F})$, then it is included in its closure by the functions of σ (which is in Y because there are countably many functions in σ). Since $\mathbf{F}$ is uncountable, $Y \cap S$ is closed unbounded. Let us show that $Y \cap S \subseteq W$. Suppose that $s \in Y \cap S$.

Let $\mathbf{J}$ be the class of structures in $\mathbf{K}$ with age included in s.

$\mathbf{J}$ is a Fraïssé class: if $C \in \mathbf{J}$ and D is embeddable in C, then $age(D) \subseteq age(C) \subseteq s$ and $D \in \mathbf{J}$. $\mathbf{J}$ is clearly closed under unions of countable chains. Since $s \in S$, there is a countable weakly homogeneous structure $B \in \mathbf{K}$ such that $s = age(B)$. If C, D_1 and D_2 are in $\mathbf{J}_{fg}$, then their isomorphism types are in $age(B)$ (their age is included in s and they are finitely generated so their types is in $\mathbf{F}$ too). Suppose now that e_1 and e_2 are embeddings of C in D_1 and D_2 respectively. The weak homogeneity of B gives us two embeddings f_1 and f_2 of D_1 and D_2 in B such that $f_1 \circ e_1 = f_2 \circ e_2$. Then the union $f_1(D_1) \cup f_2(D_2)$ is in $\mathbf{J}_{fg}$ and satisfies the joint embedding property.

Hence, we can consider the Fraïssé's pre-game G' on the class $\mathbf{J}$. Since s is countable, by theorem 2 and Fraïssé's construction, the property " A is weakly homogeneous and $age(A) = s$" is enforceable (in G'). By lemma 1, it is also coenforceable.

Besides, as $s \in Y$, s is closed under the functions of σ, which allows $\exists$ to use σ in G'. Let us consider a particular play of G' where $\exists$ plays by σ and $\forall$ plays so that A is weakly homogeneous with age s.

But $\mathbf{J} \subseteq \mathbf{K}$, so G' is only a restriction of the game G. Thus we can play the same game on $\mathbf{K}$. Then σ is a winning strategy for $G(\varphi_W)$ so $age(A) \in W$, that is, $s \in W$, and W is fat.

□

We are now ready to state the converse of theorem 2:

Theorem 3. *If $\mathbf{F}$ is uncountable, then the Fraïssé's pre-game on $\mathbf{K}$ is not wholly determined.*

Proof. Let us simply sketch out the proof. Let $\mathcal{F}$ be the set of fat subsets of $\mathbf{F}$. $\mathcal{F}$ is then a filter by lemma 2. It suffices to show that $\mathcal{F}$ is not an ultrafilter on $\mathcal{P}_\omega(\mathbf{F})$, for then we would have some countable subset W of $\mathbf{F}$ such that neither W nor $\mathcal{P}_\omega(\mathbf{F}) \setminus W$ is fat. By proposition 1, it follows that neither φ_W nor $\varphi_{\mathcal{P}_\omega(\mathbf{F}) \setminus W}$ is enforceable, that is nor $\exists$ nor $\forall$ has a winning strategy for $G(\varphi_W)$ (because in a Fraïssé's pre-game, the joint embedding property of the class guarantees that enforceability and coenforceability are equivalent notions), and thus φ_W is not determined.

$\mathcal{F}$ is non-principal filter on $\mathbf{F}$, with the property that any countable intersection of members of $\mathcal{F}$ is still in $\mathcal{F}$. Besides, $|\mathcal{P}_\omega(\mathbf{F})| \leqslant 2^\omega$. But we can show ([JH99]) that if there exists a non-principal ultrafilter with the previous property on a set X, then $|X|$ must be strongly inaccessible, that is, for every $\beta < |X|$ then $2^\beta < |X|$ too. So $\mathcal{F}$ cannot be an ultrafilter. □

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