WAP Systems and the Space of Labels

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Ellis Semigroups and Ellis Actions

We write for a map $\Phi : S \times X \to X$

$$\begin{aligned} px &= \Phi(p,x) = \Phi^p(x) = \Phi_x(p) && \text{for } (p,x) \in S \times X. \\ AB &= \{px : p \in A \text{ and } x \in B\} && \text{for } A \times B \subset S \times X. \\ \Phi^{\#} &: S \to X^X && \text{is defined by } p \mapsto \Phi^p. \\ \Phi_{\#} &: X \to X^S && \text{is defined by } x \mapsto \Phi_x. \end{aligned} \tag{1}$$

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A semigroup S is a set equipped with $M : S \times S \to S$ which is an associative multiplication, i.e.

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An action of a semigroup S on a nonempty set X is a map $\Phi : S \times X \to X$ which is an action, i.e.

$$\Phi^p \circ \Phi^q = \Phi^{pq} \quad \text{for all } p, q \in S. \tag{3}$$

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An element $u \in S$ is *idempotent* when $uu = u$, i.e. $\{u\}$ is a subsemigroup.

S is an *Ellis semigroup* and Φ is an *Ellis action* when S and X are compact spaces with $M^{\#}$ and $\Phi^{\#}$ continuous, i.e. M_q and Φ_x are continuous for each $q \in M, x \in X$.

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With $S = X^X$ with the product topology, define composition and evaluation:

$$\begin{aligned} Comp : S \times S \to S \qquad & \text{by} \quad (p,q) \mapsto p \circ q, \\ Eval : S \times X \to X \qquad & \text{by} \quad (p,x) \mapsto p(x). \end{aligned} \tag{4}$$

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If S_0 is a discrete semigroup and $\phi : S_0 \times X \to X$ is a continuous action on X, the *enveloping semigroup* of ϕ, denoted $E(\phi)$, is the closure in X^X of the image $\phi^{\#}(S_0)$, with M and Φ the restrictions of $Comp$ and $Eval$.

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If T is a homeomorphism on X, then for the *cascade* (X, T), the *enveloping semigroup* for the $\mathbb{Z}$ action is denoted $E(X, T)$.

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Let $Iso(X)$ be the -possibly empty- set of isolated points of X. If (X, T) is topologically transitive and $Iso(X) \neq \emptyset$ then $Iso(X)$ is a single dense orbit and so equals $Trans(X, T)$.

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If X is countable then $Iso(X)$ is dense in X.

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The system ϕ is *weakly almost periodic* ($=$ WAP) when every $p \in E(\phi)$ is continuous on X and so M, Φ are separately continuous, i.e. continuous in each variable separately. If (X, T) is WAP , then $E(\phi)$ is abelian.

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The condition that M and Φ are jointly continuous is *almost periodicity*. $E(X, T)$ is a compact topological group and Φ is an equicontinuous topological action.

The Space of Labels

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We will call $\mathbf{m} \in \mathbb{Z}_+^{\mathbb{N}}$ an $\mathbb{N}$-vector when it has finite support, i.e. $supp\ \mathbf{m} = \{\ell : \mathbf{m}_\ell > 0\}$ is finite. We call $\#supp\ \mathbf{m}$ the *size* of $\mathbf{m}$ and call $|\mathbf{m}| = \Sigma_\ell\ \mathbf{m}_\ell$ the *norm* of $\mathbf{m}$. The countable set $FIN(\mathbb{N})$ of $\mathbb{N}$ vectors is a monoid under addition. We think of the domain $\mathbb{N}$ as an infinite set of colors.

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For an $\mathbb{N}$-vector $\mathbf{m}$ and a positive integer ℓ^* we define $\mathbf{m} \wedge [1, \ell^*]$

$$(\mathbf{m} \wedge [1, \ell^*])_\ell \quad = \quad \begin{cases} \mathbf{m}_\ell & \text{for} \quad \ell \ \leq \ \ell^*, \\ 0 & \text{for} \quad \ell \ > \ \ell^*. \end{cases} \tag{5}$$

Definition A -possibly empty- set $\mathcal{M}$ of $\mathbb{N}$-vectors is called a *label* when

- **Heredity Condition** $0 \leq \mathbf{m}^1 \leq \mathbf{m}$ and $\mathbf{m} \in \mathcal{M}$ imply $\mathbf{m}^1 \in \mathcal{M}$.

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For example, $\emptyset$ and $0 = \{\mathbf{0}\}$ are finite labels. $\mathcal{M} \neq \emptyset$ iff $\mathbf{0} \in \mathcal{M}$. $\mathcal{M}$ is a *positive label* when it is neither empty nor 0.

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For a label $\mathcal{M}$ and $\ell^* \in \mathbb{N}$

$$\mathcal{M} \wedge [1, \ell^*] \quad = \quad \begin{cases} \emptyset & \text{when } \ell^* = 0, \\ \{ \, \mathbf{m} \wedge [1, \ell^*] \; : \; \mathbf{m} \in \mathcal{M} \, \} & \text{when } \ell^* > 0. \end{cases} \tag{6}$$

Thus, $\mathcal{M} \wedge [1, \ell^*] \; = \; \{ \, \mathbf{m} \in \mathcal{M} \; : \; supp \, \mathbf{m} \subset [1, \ell^*] \, \}$.

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For a label $\mathcal{M}$ and an $\mathbb{N}$-vector $\mathbf{r}$

$$\mathcal{M} - \mathbf{r} \quad = \quad \{\ \mathbf{w} \in FIN(\mathbb{N})\ :\ \ \mathbf{w} + \mathbf{r} \in \mathcal{M}\}. \tag{7}$$

Thus, $\mathcal{M} - \mathbf{r}\ =\ \{(\mathbf{m} - \mathbf{r}) \vee \mathbf{0} : \mathbf{m} \in \mathcal{M}\}$.

Given $N \in \mathbb{Z}_+$, define the finite label

$$\mathcal{B}_N \quad = \quad \{\mathbf{m} \in \mathbb{Z}_+^{\mathbb{N}} : \mathbf{m} < N \text{ and } \mathit{supp}\, \mathbf{m} \subset [1, N]\}. \qquad (8)$$

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On $\mathcal{LAB}$ we define an ultrametric by

$$d(\mathcal{M}_1, \mathcal{M}_2) = inf\ \{\ 2^{-N}\ :\ N \in Z_+ \text{ and } \mathcal{M}_1 \cap \mathcal{B}_N = \mathcal{M}_2 \cap \mathcal{B}_N\ \}. \quad (9)$$

Notice that since $\mathcal{B}_0 = \emptyset$, $\mathcal{M}_1 \cap \mathcal{B}_0 = \mathcal{M}_2 \cap \mathcal{B}_0$ is always true.

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Let $\mathcal{M}^i$ be a sequence of labels. Define the labels

$$\begin{aligned} LIMSUP\ \mathcal{M}^i \quad &= \quad \bigcap_i [\ \bigcup_{j \geq i} \{\mathcal{M}^j\}\ ], \\ LIMINF\ \mathcal{M}^i \quad &= \quad \bigcup_i [\ \bigcap_{j \geq i} \{\mathcal{M}^j\}\ ]. \end{aligned} \quad (10)$$

Clearly, $\mathbf{m} \in LIMSUP$ iff frequently $\mathbf{m} \in \mathcal{M}^i$ and $\mathbf{m} \in LIMINF$ iff eventually $\mathbf{m} \in \mathcal{M}^i$ and so $LIMINF \subset LIMSUP$.

A sequence $\mathcal{M}^i$ is Cauchy iff $\mathcal{M}^i \cap \mathcal{N}$ is eventually constant for any finite label $\mathcal{N}$. In that case, $LIMSUP = LIMINF$ is the limit, which we denote $LIM\ \mathcal{M}^i$.

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Theorem Let $\mathcal{L} \subset \mathcal{LAB}$ be either the set of bounded labels or the set of labels of finite type. A subset $\Phi \subset \mathcal{L}$ is compact iff Φ is closed in the relative topology of $\mathcal{L}$ and $\bigcup\ \Phi \in \mathcal{L}$.

Theorem For $\mathbf{r} \in FIN(\mathbb{N})$ the map $P_{\mathbf{r}}$ on $\mathcal{LAB}$, given by $P_{\mathbf{r}}(\mathcal{M}) = \mathcal{M} - \mathbf{r}$ is continuous. The map $P : FIN(\mathbb{N}) \times \mathcal{LAB} \ \rightarrow \ \mathcal{LAB}$ given by $P(\mathbf{r}, \mathcal{M}) = \mathcal{M} - \mathbf{r} = P_{\mathbf{r}}(\mathcal{M})$ is a continuous monoid action of $FIN(\mathbb{N})$ on $\mathcal{LAB}$.

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Let $\Theta(\mathcal{M})$ be the closure in the space of labels of the set $\{ \ \mathcal{M} - \mathbf{r} \ : \ \mathbf{r} \in FIN(\mathbb{N})\}$. That is, $\Theta(\mathcal{M})$ is the orbit closure of $\mathcal{M}$ with respect to the $FIN(\mathbb{N})$ action.

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Let $\Theta'(\mathcal{M})$ be the closure in the space of labels of the set $\{ \ \mathcal{M} - \mathbf{r} \ : \ \mathbf{r} \in FIN(\mathbb{N}) \ \text{ with } \mathbf{r} > \mathbf{0}\}$. Thus, $\Theta(\mathcal{M}) = \Theta'(\mathcal{M}) \cup \{\mathcal{M}\}$. A label $\mathcal{M}$ is called *recurrent* when $\mathcal{M} \in \Theta'(\mathcal{M})$. The set of recurrent labels is a dense G_δ subset of $\mathcal{LAB}$.

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$\Theta(\mathcal{M})$ and $\Theta'(\mathcal{M})$ are compact, $FIN(\mathbb{N})$ invariant subsets.

Definition A label $\mathcal{M}$ is *finitary* if it is bounded and

- **Finitary condition** Whenever $\{S_i\}$ is a sequence of finite subsets of $\mathbb{N}$ with $\bigcup_i S_i$ infinite, there are only finitely many subsets S of $\mathbb{N}$ such that eventually $S \cup S_i \in Supp\ \mathcal{M}$.

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A label $\mathcal{M}$ is finitary iff whenever $\mathbf{r}^i$ is a sequence in $FIN(\mathbb{N})$ with $\bigcup_i \ supp\ \mathbf{r}^i$ infinite and $\{\mathcal{M} - \mathbf{r}^i\}$ convergent then then $LIM\ \{\mathcal{M} - \mathbf{r}^i\}$ is a finite label.

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For a finitary $\mathcal{M}$ the compact set $\Theta(\mathcal{M})$ consists of $\{\mathcal{M} - \mathbf{r} : \mathbf{r} \in FIN(\mathbb{N})\}$ and certain additional finite labels. Furthermore, the action of $FIN(\mathbb{N})$ on $\Theta(\mathcal{M})$ is WAP.

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Technical as the condition seems it is not hard to construct a rich supply of examples.

Labeled subshifts

If B is an odd positive integer and $k : \mathbb{Z} \to \mathbb{Z}$ is an odd function with $k(n) = B^{n-1}$ for all $n \in \mathbb{N}$ then every integer t has a unique *symmetric base* B *expansion* $t = \Sigma_{i=1}^{\infty} \epsilon_i k(i)$ with $\epsilon_i \in \mathbb{Z}$ such that $|\epsilon_i| < B/2$ and $\epsilon_i \neq 0$ for finitely many i. Fix $B \geq 5$.

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Definition An *expansion of length* $r \geq 0$ for $t \in \mathbb{Z}$ is a finite sequence $j_1, \ldots, j_r \in \mathbb{Z}$ such that $|j_i| > |j_{i+1}| > 0$ for $i = 1, \ldots, r-1$ and with

$$t \ = \ k(j_1) + k(j_2) + \cdots + k(j_r).$$

$0 \in \mathbb{Z}$ has the empty expansion with length 0.
When there is an expansion for t we will say that t is *expanding* or that t is an *expanding time*. We let $IP(k)$ denote the set of expanding times.

Thus, the expanding times are the integers with a base B expansion such that $|\epsilon_i| \leq 1$ for all i. So $IP(k)$ is a rather sparse subset of $\mathbb{Z}$. It can be shown to have Banach density zero.

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We now fix an infinite coloring of $\mathbb{N}$, a partition of $\mathbb{N}$ by an infinite sequence

$$\mathcal{D} \quad = \quad \{D_\ell : \ell \in \mathbb{N}\}$$

of infinite sets, numbered so that $minD_\ell < minD_{\ell+1}$. Hence, $minD_1 = 1$ and $minD_\ell \geq \ell$.

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The *support map* $n \mapsto \ell(n)$ associates to each $n \in \mathbb{N}$ the member of $\mathcal{D}$ which contains it, so that $n \in D_{\ell(n)}$. That is, $\ell(n)$ is the color of n.

Definition If $j_1, \ldots, j_r \in \mathbb{Z}$ is the expansion of $t \in IP(k)$,

$$t \;=\; k(j_1) + k(j_2) + \cdots + k(j_r),$$

then the *length vector* for the expansion is the $\mathbb{N}$-vector

$$\mathbf{r} \;=\; \mathbf{r}(t) \;=\; \chi(\ell(|j_1|)) + \chi(\ell(|j_2|)) + \cdots + \chi(\ell(|j_r|)),$$

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$0 \in IP(k)$ has the empty expansion with length vector $\mathbf{0}$.

Thus, $\mathbf{r}(t)$ counts the number of colors which occur at places where there is a nonzero term in the expansion of t.

Definition For a label $\mathcal{M}$ define $A[\mathcal{M}] \subset \mathbb{Z}$ to consist of those $t \in IP(k)$ which have length vector $\mathbf{r}(t) \in \mathcal{M}$.

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Let $x[\mathcal{M}] \in \{0,1\}^{\mathbb{Z}}$ be the characteristic function of $A[\mathcal{M}]$. Thus,

$$x[\mathcal{M}]_t \;=\; 1 \qquad \Longleftrightarrow \qquad \mathbf{r}(t) \in \mathcal{M}.$$

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Theorem The map $x[\cdot]$ defined by $\mathcal{M} \mapsto x[\mathcal{M}]$ is a homeomorphism from $\mathcal{LAB}$ onto its image in $\{0,1\}^{\mathbb{Z}}$.

The key result which relates the T dynamics on $X(\mathcal{M})$ with the $FIN(\mathbb{N})$ dynamics on $\Theta(\mathcal{M})$ is the following.

Lemma Let $\{t^i\}$ be a sequence of expanding times with r^i the length of t^i. If $\{|j_{r^i}(t^i)|\} \to \infty$ then

$$Lim_{i\to\infty}\, sup_{\mathcal{M}\in\mathcal{LAB}}\, d(T^{t^i}(x[\mathcal{M}])\ ,\ x[\mathcal{M}-\mathbf{r}(t^i)]\ )\ =\ 0. \quad (11)$$

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That is, the pair of sequences $(\{T^{t^i}(x[\mathcal{M}])\}$ and $\{x[\mathcal{M}-\mathbf{r}(t^i)]\})$ are uniformly asymptotic in $\{0,1\}^{\mathbb{Z}}$.

The condition $\{|j_{r^i}(t^i)|\} \to \infty$ says that the place lowest nonzero digit of t^i tends to infinity with i.

From this one is able to prove:

Theorem Let $\mathcal{M}$ be a label of finite type. $X(\mathcal{M}) = \{T^k(x[\mathcal{N}]) : k \in \mathbb{Z},\ \mathcal{N} \in \Theta(\mathcal{M})\}$, and in $X(\mathcal{M})$ the fixed point $e = x[\emptyset]$ is the only recurrent point.

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By using constructions of finitary labels, one is thus able to get a large collection of interesting WAP examples.

For example, there are two transfinite approaches to the *Birkhoff Center*, the closure of the set of recurrent points.

For a system (X, T) let X' be the closed set of nonwandering points and $X'' \subset X'$ be the closure of union the sets of omega and alpha limit points.

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With $X_0 = X$ we can let $X_{\alpha+1} = X'_\alpha$ (the *high road*) or $X_{\alpha+1} = X''_\alpha$ (the *low road*) and in either case $X_\beta = \bigcap_{\alpha<\beta} X_\alpha$ for β a limit ordinal. Each stabilizes at the Birkhoff Center.

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In 2000, Shapovalov constructed countable WAP subshifts which a require arbitrarily large countable ordinals to stabilize via the high road, but his examples stabilize after two steps via the low road. Using finitary labels we construct examples of countable WAP subshifts which a require arbitrarily large countable ordinals to stabilize via the low road.

For a subshift (X, T) a subset $K \subset \mathbb{Z}$ is called an *independent set* if the restriction to X of the projection $\pi_K : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^K$ is surjective.

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In a private conversation Tomasz Downarowicz asked us whether it is the case that every WAP system is null. Using labels, we can obtain

(a) topologically transitive, WAP subshifts which are non-null;

(b) subshifts arising from finite type labels which are not tame.

Thank you.