Hankel determinants of a Toeplitz sequence

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joint work with Cor Kraaikamp and Jeffrey Shallit

Our result

The period doubling sequence is generated by

$$0 \to 01, \ 1 \to 00$$

A Hankel matrix of a symbolic sequence $abcdef\ldots$ is

$$\begin{pmatrix} a & b & c & d & e & f \\ b & c & d & e & f & g \\ c & d & e & f & g & h \\ d & e & f & g & h & i \\ e & f & g & h & i & j \\ f & g & h & i & j & k \end{pmatrix}$$

The Jacobsthal numbers $0, 1, 1, 3, 5, 11, 21, \cdots$ are generated by the recursion $J_{n+1} = J_n + 2J_{n-1}$.

Our result

Write $H(n)$ for the $n \times n$ Hankel matrix and $\Delta(n)$ for its determinant.

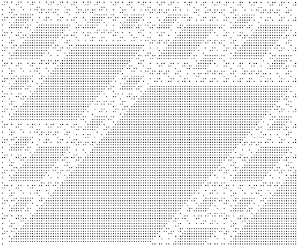
- $H(2^k)$ can be diagonalized by a Hadamard matrix.
- The Hankel determinants of the period doubling sequence are products of powers of Jacobsthal numbers.
- There does not seem to be a closed form formula for $\Delta(n)$, but there is an efficient algorithm to compute its Jacobsthals.
- There are many recursive formulas such as:

$$\Delta(2^k j) = \Delta(2^k)\Delta(j)^{2^k}$$

for odd j (observed by Jason Bell and Kevin Hare).

Why would we care?

The Hankel determinants for Thue-Morse sequence are not yet well understood. Modulo two, Allouche et al 1999 found:



Theorem Allouche, Peyrière, Wen, Wen (1998) For ThueMorse, the Hankel transformed sequences mod 2 are automatic.

Why would we care?

The proof depends on 16 recursions, including

Theorem 2.1. — *For $p \geq 0$ and $n \geq 1$, one has*

1) $$|\mathcal{E}^{2p}_{2n}| = |\mathcal{E}^{p}_{n}| \cdot |\Delta^{p}_{n}| - |\overline{\mathcal{E}^{p}_{n}}| \cdot |\overline{\Delta^{p}_{n}}| - 2|\mathcal{E}^{p}_{n}| \cdot |\overline{\Delta^{p}_{n}}| \equiv |\mathcal{E}^{p}_{n}| \cdot |\Delta^{p}_{n}| + |\overline{\mathcal{E}^{p}_{n}}| \cdot |\overline{\Delta^{p}_{n}}|,$$

2) $$|\overline{\mathcal{E}^{2p}_{2n}}| = 4|\mathcal{E}^{p}_{n}| \cdot |\overline{\Delta^{p}_{n}}| + |\overline{\mathcal{E}^{p}_{n}}| \cdot |\Delta^{p}_{n}| + 2|\overline{\mathcal{E}^{p}_{n}}| \cdot |\overline{\Delta^{p}_{n}}| \equiv |\overline{\mathcal{E}^{p}_{n}}| \cdot |\Delta^{p}_{n}|,$$

3) $$|\mathcal{E}^{2p}_{2n+1}| = |\mathcal{E}^{p}_{n+1}| \cdot |\Delta^{p}_{n}| - |\overline{\mathcal{E}^{p}_{n+1}}| \cdot |\overline{\Delta^{p}_{n}}| - 2|\mathcal{E}^{p}_{n+1}| \cdot |\overline{\Delta^{p}_{n}}| \equiv |\mathcal{E}^{p}_{n+1}| \cdot |\Delta^{p}_{n}| + |\overline{\mathcal{E}^{p}_{n+1}}| \cdot |\overline{\Delta^{p}_{n}}|,$$

4) $$|\overline{\mathcal{E}^{2p}_{2n+1}}| = 4|\mathcal{E}^{p}_{n+1}| \cdot |\overline{\Delta^{p}_{n}}| + |\overline{\mathcal{E}^{p}_{n+1}}| \cdot |\Delta^{p}_{n}| + 2|\overline{\mathcal{E}^{p}_{n+1}}| \cdot |\overline{\Delta^{p}_{n}}| \equiv |\overline{\mathcal{E}^{p}_{n+1}}| \cdot |\Delta^{p}_{n}|,$$

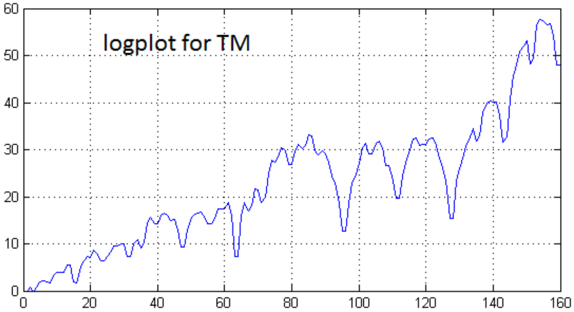
5) $$|\mathcal{E}^{2p+1}_{2n}| = (-1)^n \{ |\mathcal{E}^{p+1}_{n}| \cdot |\Delta^{p}_{n}| - 2|\mathcal{E}^{p+1}_{n}| \cdot |\overline{\Delta^{p}_{n}}| + |\overline{\mathcal{E}^{p+1}_{n}}| \cdot |\Delta^{p}_{n}| - |\overline{\mathcal{E}^{p+1}_{n}}| \cdot |\overline{\Delta^{p}_{n}}| \} \equiv |\mathcal{E}^{p+1}_{n}| \cdot |\Delta^{p}_{n}| + |\overline{\mathcal{E}^{p+1}_{n}}| \cdot |\Delta^{p}_{n}| + |\overline{\mathcal{E}^{p+1}_{n}}| \cdot |\overline{\Delta^{p}_{n}}|,$$

6) $$|\overline{\mathcal{E}^{2p+1}_{2n}}| = (-1)^n \{ 4|\mathcal{E}^{p+1}_{n}| \cdot |\overline{\Delta^{p}_{n}}| - |\overline{\mathcal{E}^{p+1}_{n}}| \cdot |\Delta^{p}_{n}| + 2|\overline{\mathcal{E}^{p+1}_{n}}| \cdot |\overline{\Delta^{p}_{n}}| \} \equiv |\overline{\mathcal{E}^{p+1}_{n}}| \cdot |\Delta^{p}_{n}|,$$

7) $$|\mathcal{E}^{2p+1}_{2n+1}| = (-1)^{n+1} \{ |\mathcal{E}^{p}_{n+1}| \cdot |\Delta^{p+1}_{n}| - 2|\mathcal{E}^{p}_{n+1}| \cdot |\overline{\Delta^{p+1}_{n}}|$$

Why would we care?

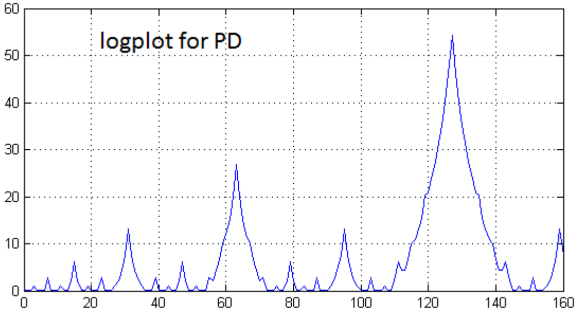
Computing parity Hankel determinants is not trivial (Allouche et al prove that $\Delta(n)$ is odd). Computing Hankel determinants of Thue Morse appears to be difficult.



Conjecture [Allouche and Shallit] liminf is 1 and limsup is ∞.

Why would we care?

Hankel determinants of the Period Doubling sequence are much more manageable:



It easily follows from our results that liminf is 1 and limsup is ∞.

Why would we care?

Hankel determinants play a role in transcendental number theory.

Automatic number: consider the generating function

$$f(X) = \sum a_n X^n$$

of an automatic sequence (a_n). Then $f(1/b)$ is called an automatic number for $b \in \mathbb{N}$ and $1/b$ within the domain of convergence.

Theorem [Adamczewski-Bugeaud, (2007)] Automatic numbers are either rational or transcendental.

This settled an old problem of Cobham and is a step on the way to settling that a "Mahlerian function" assumes either rational or transcendental values at algebraic numers.

Why would we care?

Typically, a real number should be transcendental and have irrationality measure two.

Question Is the irrationality measure of an automatic number equal to two?

A first step in this direction is
Theorem [Adamczewski-Rivoal, (2009)] Automatic numbers have finite irrationality measure.

Hankel determinants can be used to push this further, in specific cases. The Padé (n, m) approximation is

$$f(X) = \frac{p(X)}{q(X)} + O(X^{n+m+1})$$

for $deg(p) = n$ and $deg(Q) = m$.

Why would we care?

Hankel determinants can be used to compute c in

$$f(X) - \frac{p(X)}{q(X)} = cX^{n+m+1} + O(X^{n+m+2})$$

Apparently, first developed by Kamae et al (1999), who give explicit Padé approximants and Hankel determinants for the Fibonacci sequence.

In particular, $c = \frac{\Delta(k')}{\Delta(k)}$, a quotient of consecutive non-zero Hankel determinants for the (k, k') Padé approximation with $k < k'$. This idea was extended in:

Theorem [Bugeaud, Han, Wen, Yao (2015)] For a certain class of $f(X)$ the irrationality measure of $f(1/b)$ is equal to two if there exists a sequence $n_1 < n_2 < n_3 < \cdots$ such that $\limsup \frac{n_{i+1}}{n_i} = 1$ and none of the Hankel determinants $\Delta(n_i)$ are equal to zero.

Why would we care?

Very rough outline of proof: use (n_k, n_{k+1}) Padé approximants

$$f(X) - \frac{p_k(X)}{q_k(X)} = cX^{n_k+n_{k+1}+1} + O(X^{n_k+n_{k+1}+2})$$

Substitute $1/b$ and find a sequence rational approximations

$$r_k = \frac{p_k(1/b)}{q_k(1/b)}$$

that is good enough to be picked up by the continued fraction.

Because of the conditions on n_k, the denominators of r_k are bounded. Hence the irrationality measure of $f(1/b)$ is equal to two.

NOW YOU KNOW, HANKEL DETERMINANTS ARE USEFUL!

Idea of our proof

The first Hankel determinants of the PD sequence are

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\Delta_{\mathbf{d}}(k)$	1	1	-1	-3	1	1	-1	-15	1	1	-1	-3	1	1	-9	-495

Observe the powers of two!

Theorem 1 $\Delta(2^k) = -J_3^{2^{k-3}} J_4^{2^{k-4}} \cdots J_{k-2}^{2^2} J_{k-1}^2 J_k J_{k+1}$.

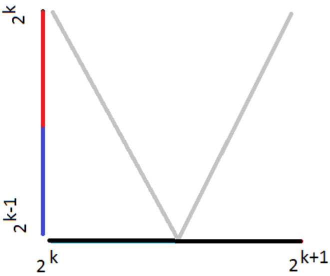
Proof is by diagonalization by the Hadamard matrix $M(2^k)$ of order 2^k:

$$M(2^k) \cdot H(2^k) \cdot M(2^k) = Diag$$

It is not hard to prove this by induction. Jacobsthal numbers appear on the diagonal.

Idea of our proof

Theorem 2 The Jacobsthal numbers in $\Delta(n)$ are generated by iterating the inverted tent map, starting with n:



Each iteration produces a power of a Jacobsthal number. If the iteration hits the blue half of the interval, the index in the Jacobsthal number is reset to 1. If the iteration hits the red half of the interval, the index of the Jacobsthal number increases by 1.

Idea of our proof

The iteration runs from $[2^k, 2^{k+1}]$ to $[2^{k-1}, 2^k]$ and eventually ends at a pure power 2^m, producing the final factor $\Delta(2^m)$

The size of the step is equal to the power of the Jacobsthal number. This explains the recursion $\Delta(2^k j) = \Delta(2^k)\Delta(j)^{2^k}$: for $2^k j$ the step size increases by 2^k, the iteration reaches 2^k and produces the final factor $\Delta(2^k)$.

We have found other recursive formulas, but we have not found a closed form formula for the Hankel determinants.

References

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Thank You!