Enveloping Semigroups and Tame Dynamical Systems

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Definitions

- Let G be a topological group, X a compact (usually metric) space. A **dynamical system (DS)** is a pair (X, G) with a jointly continuous action of G on X.
- A **factor map** $\pi : (X, G) \to (Y, G)$ is a continuous onto map intertwining the G actions. We also say that (X, G) is an **extension** of (Y, G).
- A DS (X, G) is **topologically transitive** if every invariant open set is dense (for metric systems this is equivalent to the existence of a dense orbit; we call the latter property **point transitivity**). The system is **minimal** if every orbit is dense.
- When the acting group $G = \mathbb{Z}$ is the group of integers we will write (X, G) as (X, T), where $T : X \to X$ is the homeomorphism which corresponds to $1 \in \mathbb{Z}$. We call such system a **cascade**.

Hereditary Almost Equicontinuity (HAE)

Definition

- A DS (X, G) is **equicontinuous at** $x_0 \in X$ if for every $\epsilon > 0$ there exists a neighborhood O of x_0 such that for every $x \in O$ and every $g \in G$ we have $d(gx, gx_0) < \epsilon$.
- A DS is **almost equicontinuous (AE)** if it is equicontinuous at a dense set of points.
- (X, G) is **hereditarily almost equicontinuous (HAE)** if every subsystem is AE.
- It is not hard to see that a minimal AE system is actually equicontinuous.

Weak mixing and AE

Definition

A DS (X, G) is **weakly mixing** if the product system $(X \times X, G)$ (with diagonal action $g(x, x') = (gx, gx'))$ is topologically transitive.

- A DS which is both weakly mixing and almost equicontinuous is necessarily trivial.
- To see this observe that, given $\epsilon > 0$ there is, by the AE property, a nonempty open set U such that for any $(x, x') \in \bigcup_{g \in G} gU \times gU$ we have $d(x, x') < \epsilon$. However, by weak mixing, this set is dense in $X \times X$. Thus the diameter of X is zero.

HAE and Rigidity for cascades

Definition

A cascade (X, T) is **uniformly rigid** if the there is a sequence $n_i \nearrow \infty$ such that the corresponding sequence of maps T^{n_i} tends uniformly to the identity.

Theorem

1. *An AE cascade with no isolated points is uniformly rigid.*
2. *Every topologically transitive, uniformly rigid cascade admits an AE extension.*
3. *There exist nontrivial weakly mixing, minimal, uniformly rigid cascades.*

Corollary

The AE property need not be inherited by either factors or subsystems.

HAE and WAP

On the other hand it is not hard to see that the HAE property is inherited by factors. It is also preserved by products and, by definition, by subsystems. This persistence of properties makes HAE a convenient class to work with. In particular this shows that there is a universal point transitive HAE dynamical system.

Definition

A DS (X, T) is **weakly almost periodic (WAP)** if the orbit $\{f \circ g : g \in G\}$ of each $f \in C(X)$ is weakly precompact in the Banach space $C(X)$.
Recall that (X, G) is equicontinuous iff in the definition above we require precompactness in the norm topology. Thus (X, G) is equicontinuous iff it is **Bohr almost periodic (AP)**.

Theorem

1. *Every WAP cascade is HAE [Akin-Auslaner-Berg].*
2. *There exist recurrent, topologically transitive, HAE cascades which are not WAP [Gl-Weiss].*

Enveloping semigroups

Definition

- The **enveloping semigroup** $E(X, G)$ is defined as the closure of the set of g-translations, $g \in G$, in the compact space X^X.
- It is not hard to check that $E(X, G)$ is a **right topological semigroup compactification** of G (i.e. $E(X, G)$ is a compact semigroup where right multiplication $R_p : q \mapsto qp$ is continuous.
- On the other hand, left multiplication $L_p : q \mapsto pq$ is continuous iff $p : X \to X$ is a continuous map.
- Note in particular that left multiplication by elements of G are therefore continuous and this makes $(E(X), G)$ a point transitive dynamical system.

Theorem

1. *A subset M of E is a minimal (left) ideal of the semigroup E iff it is a minimal subsystem of (E, G). Minimal ideals M in E exist and for each such ideal the set of idempotents in M, denoted by $J = J(M)$, is non-empty.*
2. *Let M be a minimal ideal and J its set of idempotents then:*
   (a) *For $v \in J$ and $p \in M$, $pv = p$.*
   (b) *For each $v \in J$, $vM = \{vp : p \in M\}$ is a subgroup of M with identity element v. For every $w \in J$ the map $p \mapsto wp$ is a group isomorphism of vM onto wM.*
   (c) *$\{vM : v \in J\}$ is a partition of M. Thus if $p \in M$ then there exists a unique $v \in J$ such that $p \in vM$; we denote by p^{-1} the inverse of p in vM.*
3. *Let K, L, and M be minimal ideals of E. Let v be an idempotent in M, then there exists a unique idempotent v' in L such that $vv' = v'$ and $v'v = v$. (We write $v \sim v'$ and say that v' is* equivalent *to v.) If $v'' \in K$ is equivalent to v', then $v'' \sim v$. The map $p \mapsto pv'$ of M to L is an isomorphism of G-systems.*

We have the following connections between the dynamical properties of the system (X, G) and the algebraic properties of $E(X, G)$.

- For every $x \in X$, $\overline{Gx} = Ex$
- $\overline{Gx}$ is minimal iff for every minimal ideal M in E, $\overline{Gx} = Mx$, iff in every minimal ideal there is an idempotent v such that $vx = x$.
- The pair (x, x') is proximal iff $px = px'$ for some $p \in E$ iff there exists a minimal ideal M in E with $px = px'$ for every $p \in M$.
- If (X, G) is minimal, then

$$P[x] = \{x' \in X : (x, x') \in P\} = \{vx : v \in \hat{J}\}.$$

  In particular $x \in X$ is a distal point iff $vx = x$ for every $v \in \hat{J}$.

- The relation P is an equivalence relation iff E contains a unique minimal ideal.
- (X, G) is distal iff E is a group.
- A distal system is pointwise minimal.
- (X, G) is distal iff $X \times X$ is pointwise minimal. Hence a factor of a distal system is distal.
- (X, G) is equicontinuous iff E is a compact topological group.

Examples

- For the Bernoulli scheme $\Omega = \{0, 1\}^{\mathbb{Z}}$ (a cascade under the shift) the enveloping semigroup is the Čech-Stone compactification $\beta\mathbb{Z}$. As a DS it is the universal point transitive $\mathbb{Z}$ system.
- For the transformation $T(x, y) = (x + \alpha, y + x)$ on the 2-torus, with irrational α, we have $E = \{S_{\beta,\gamma,\phi}\}$, $S(x, y) = (x + \beta, y + \phi(x) + \gamma)$, where β, γ are arbitrary points in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\phi : \mathbb{T} \to \mathbb{T}$ is **any** map with $\phi(x + x') = \phi(x) + \phi(x')$ and $\phi(\alpha) = \beta$.

Enveloping semigroup characterizations

Theorem

- *(X, G) is WAP $\iff$ every element of $E(X, G)$ is continuous [Ellis-Nerurkar].*
- *(X, G) is HAE $\iff$ $E(X, G)$ is metrizable [Gl-Meg-Usp].*

Note that in the latter case, as every element of $E(X, G)$ is the pointwise limit of a **sequence** of elements of G, it follows that each element p of E is a Baire-class-1 map; i.e. the inverse image $\{x \in X : px \in U\}$ of an open subset $U \subset X$, is F_σ subset of X. Alternatively the restriction of p to every closed subset has a point of continuity.

Rosenthal compacta

Definition

- Recall that a compact space K is called **a Rosenthal compactum** if it is homeomorphic to a compact subset of the space $B_1(X)$ of Baire-class-1 real-valued functions on a Polish space X, equipped with the pointwise convergence topology.
- A topological space K is a **Fréchet** space if for every $A \subset K$ and every $x \in \overline{A}$ there is a sequence $a_n \in A$ with $x = \lim_{n\to\infty} a_n$.

In a famous work Bourgain Fremlin and Talagrand have shown, among many other striking results, that every Rosenthal compactum is Fréchet.

A dynamical Bourgain-Fremlin-Talagrand dichotomy

We have the following dynamical version of the Bourgain-Fremlin-Talagrand dichotomy theorem.

Theorem (A dynamical BFT dichotomy, Köhler, Gl-Meg)

Let (X, G) be a metric dynamical system and let $E(X)$ be its enveloping semigroup. We have the following dichotomy. Either

1. *$E(X)$ is a separable Rosenthal compactum, hence with cardinality* $\mathrm{card}\ E(X) \leq 2^{\omega}$*; or*
2. *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence* $\mathrm{card}\ E(X) = 2^{2^{\omega}}$.

Tame dynamical systems

- Definition
  A DS is called **tame** if the first alternative occurs, i.e. $E(X)$ is Rosenthal compact.
  - One can measure the usefulness of a new mathematical notion by the number of seemingly unrelated ways by which it can be characterized. According to this principle the notion of tameness stands rather high. Tameness can be characterized by the lack of a certain “independence” property — where combinatorial Ramsey type arguments take a leading role, by the fact that the elements of the enveloping semigroup of a tame system are Baire class 1 maps. Finally a dynamical system is tame iff it can be represented on a Banach space which does not contain an isomorphic copy of ℓ_1.

Theorem

The following conditions on a metric dynamical system (X, T) are equivalent:

1. (X, T) *is tame.*
2. $\mathrm{card}(E(X, T)) = 2^{\aleph_0}$*.*
3. $E(X, T)$ *is a Fréchet space.*
4. *Every element of* $E(X, T)$ *is of Baire class 1.*
5. *Every element of* $E(X, T)$ *is Borel measurable.*

- Given a DS (X, G) each of the dynamical properties defined above defines a C^*-subalgebra of $C(X)$, and therefore also a **maximal factor** with the corresponding property:

$$AP(X) \subset WAP(X) \subset HAE(X) \subset Tame(X).$$

Globally, i.e. in the C^*-algebra $BRUC(G)$, we have:

$$AP(G) \subset WAP(G) \subset HAE(G) \subset Tame(G).$$

Entropy, tame and null systems

Recall the basic definitions of topological (sequence) entropy. Let (X, T) be a cascade, i.e., a $\mathbb{Z}$-dynamical system, and $A = \{a_0 < a_1 < \ldots\}$ a sequence of integers. Given an open cover $\mathcal{U}$ define

$$h^A_{top}(T, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} N(\bigvee_{i=0}^{n-1} T^{-a_i}(\mathcal{U}))$$

The **topological entropy along the sequence** A is then defined by

$$h^A_{top}(T) = \sup\{h^A_{top}(T, \mathcal{U}) : \mathcal{U} \text{ an open cover of } X\}.$$

When the phase space X is zero-dimensional, one can replace open covers by clopen partitions. We recall that a dynamical system (T, X) is called **null** if $h^A_{top}(T) = 0$ for every infinite $A \subset \mathbb{Z}$. Finally when $Y \subset \{0,1\}^{\mathbb{Z}}$, and $A \subset \mathbb{Z}$ is a given subset of $\mathbb{Z}$, we say that Y **is free on** A or that A **is an interpolation set for** Y, if

$$\{y|_A : y \in Y\} = \{0,1\}^A.$$

By a theorem of Huang and Kerr and Li every null system is tame. From results of Glasner-Weiss and Kerr-Li, the following results can be easily deduced.

Theorem

1. *A subshift $X \subset \{0,1\}^{\mathbb{Z}}$ has positive topological entropy iff there is a subset $A \subset \mathbb{Z}$ of positive density such that X is free on A.*
2. *A subshift $X \subset \{0,1\}^{\mathbb{Z}}$ is not tame iff there is an infinite subset $A \subset \mathbb{Z}$ such that X is free on A.*
3. *A subshift $X \subset \{0,1\}^{\mathbb{Z}}$ is not null iff for every $n \in \mathbb{N}$ there is a finite subset $A_n \subset \mathbb{Z}$ with $|A_n| \geq n$ such that X is free on A_n.*

Minimal tame systems

Theorem (A structure theorem for minimal tame DS, Huang, Kerr-Li, Gl)

Let (X, G) *be a tame minimal metrizable DS with G abelian. Then:*

- (X, G) *is an almost 1-1 extension of an equicontinuous system.*
- (X, G) *is uniquely ergodic and the corresponding measure preserving system is isomorphic to Haar measure on the equicontinuous factor.*

I will next describe some key points in the proof of this theorem. Unless I say otherwise Γ is assumed to be abelian.

Theorem

Let (X, Γ) be a metric tame dynamical system. Let $\mathfrak{M}(X)$ denote the compact convex set of probability measures on X (with the weak* topology). Then each element $p \in E(X, \Gamma)$ defines an element $p_* \in E(\mathfrak{M}(X), \Gamma)$ and the map $p \mapsto p_*$ is both a dynamical system and a semigroup isomorphism of $E(X, \Gamma)$ onto $E(\mathfrak{M}(X), \Gamma)$.

Proof.

Since $E(X, \Gamma)$ is Fréchet we have for every $p \in E$ a sequence $\gamma_i \to p$ of elements of Γ converging to p. Now for every $f \in C(X)$ and every probability measure $\nu \in \mathfrak{M}(X)$ we get, by the Riesz representation theorem and Lebesgue's dominated convergence theorem,

$$\gamma_i \nu(f) = \nu(f \circ \gamma_i) \to \nu(f \circ p) := p_* \nu(f).$$

This defines the map $p \mapsto p_*$. It is easy to see that this map is an isomorphism of dynamical systems, whence a semigroup isomorphism. Finally as Γ is dense in both enveloping semigroups, it follows that this isomorphism is onto. $\square$

As we have seen, when (X, Γ) is a metrizable tame system each element $p \in E$ is a limit of a sequence of elements of Γ, $p = \lim_{n\to\infty} \gamma_n$.

Note that under our assumptions (minimality and commutativity of Γ) the set pX is always a dense subset of X.

It follows that the subset $C(p)$ of continuity points of each $p \in E$ is a dense G_δ subset of X.

Lemma

Let (X, Γ) be a metrizable tame dynamical system, $E = E(X, \Gamma)$ its enveloping semigroup.

1. *For every $p \in E$ the set $C(p) \subset X$ is a dense G_δ subset of X.*
2. *For every idempotent $v \in E$, we have $C(v) \subset vX$.*

Proof.

1. As remarked above.

2. Given $x \in C(v)$ choose a sequence $x_n \in vX$ with $\lim_{n\to\infty} x_n = x$. We then have

$$vx = \lim_{n\to\infty} vx_n = \lim_{n\to\infty} x_n = x,$$

hence $C(v) \subset vX$.

$\square$

Point distality

A homomorphism $\pi : (X, \Gamma) \to (Y, \Gamma)$ is called **semiopen** if the interior of $\pi(U)$ is nonempty for every nonempty open subset U of X. When X is minimal every $\pi : (X, \Gamma) \to (Y, \Gamma)$ is semiopen. We will say that a subset $W \subset X \times X$ is a **S-set** if it is closed, invariant, topologically transitive, and the restriction to W of the projection maps are semiopen.

Theorem (van der Woude)

A metric minimal system (X, Γ) *is point distal iff every S-set in* $X \times X$ *is minimal.*

Proposition

Let (X, Γ) be a metric tame minimal system, then (X, Γ) is point distal.

Proof.
We will prove that the condition in the above theorem holds; i.e. that every S-set in $X \times X$ is minimal. So let $W \subset X \times X$ be an S-set.

Let v be a minimal idempotent in $E(X, \Gamma)$. By the above Lemma and Proposition the set $C(v)$ of continuity points of v is a dense G_δ subset of X and moreover $C(v) \subset vX$, so that vX is residual in X. Since by assumption the projection maps $\pi_i : W \to X$ are semiopen, it follows that the sets $\pi_i^{-1}(vX)$ are residual in W. Since W_{tr}, the set of transitive points in W, is a dense G_δ subset of W we conclude that the set $(\pi_1^{-1}(vX) \cap \pi_2^{-1}(vX)) \cap W_{tr} = (vX \times vX) \cap W_{tr}$ is residual in W and in particular it is nonempty. Now if (x, x') is any point in this intersection then $v(x, x') = (vx, vx') = (x, x')$ and (x, x') is a minimal point. Therefore $W = \overline{\mathcal{O}}_\Gamma(x, x')$ is minimal. $\square$

Theorem (Veech)

A metric minimal dynamical system is point distal iff it is an AI-system.

Once we know that (X, G) is point distal it is possible, using Veech's theorem, to reduce the proof to the case discussed in the following proposition:

Proposition

Let (X, Γ) be a minimal metric system and let $\phi : (X, \Gamma) \to (Y, \Gamma)$ be its maximal equicontinuous factor. Suppose further that Y is infinite and that we have the following diagram

$$(X, \Gamma) \xrightarrow{\pi} (Y^*, \Gamma) \xrightarrow{\theta} (Y, \Gamma),$$

where π is an isometric extension, θ is an almost 1-1 extension and $\phi = \theta \circ \pi$.

For every $y^* \in Y^*$ the fiber $\pi^{-1}(y^*)$ has the structure of a homogeneous space of a compact Hausdorff topological group and we let λ_{y^*} be the corresponding Haar measure on this fiber. Thus π is a RIM and open extension and $y^* \mapsto \lambda_{y^*}$, $Y^* \to \mathfrak{M}(X)$, is the corresponding section. Let $\Lambda : \mathfrak{M}(Y^*) \to \mathfrak{M}(X)$, defined by

$$\Lambda(\nu) = \int_{Y^*} \lambda_{y^*} \, d\nu(y^*),$$

be the associated affine injection. Set $\mathfrak{M}_m(X) = \{\Lambda(\nu) : \nu \in \mathfrak{M}(Y^*)\}$. Then the set

$$R = \{\nu \in \mathfrak{M}(X) : \text{the orbit closure of } \nu \text{ meets } \mathfrak{M}_m(X)\}$$

is a dense G_δ subset of $\mathfrak{M}(X)$.

A minimal dynamical system (X, Γ) is called **almost automorphic** if it is an almost 1-1 extensions of its largest equicontinuous factor.

Toeplitz systems and generalized Sturmian systems are almost automorphic.

Almost automorphy

Now recall that when (X, G) is tame we have $E(X, G) = E(\mathfrak{M}(X), G)$. In particular we have that

$$S := C_{\mathfrak{M}(X)}(u) \cap R$$

is a dense G_δ subset of $\mathfrak{M}(X)$.
Now if $\nu \in S$ then $u\nu = \nu$ and, u being a minimal idempotent, the closure of the Γ orbit of ν in $\mathfrak{M}(X)$ is a minimal set, whence this entire orbit closure is contained in $\mathfrak{M}_m(X)$. In particular $\nu \in \mathfrak{M}_m(X)$ and we conclude that

$$S = C_{\mathfrak{M}(X)}(u) \cap R \subset \mathfrak{M}_m(X).$$

Therefore S is dense in $\mathfrak{M}_m(X)$ and in turn, this implies the equality:

$$\mathfrak{M}_m(X) = \mathfrak{M}(X).$$

Finally, this means that π is an isomorphism; i.e. (X, G) is almost automorphic.

Unique ergodicity

Suppose that μ_1 and μ_2 are two invariant probability measures on X. Then, (X, Γ) being tame, $u_*\mu_i = \mu_i$ and we conclude that $\mu_i(uX) = 1$, for $i = 1, 2$.

Since θ is a proximal extension, for every $y \in Y = uY$ the fiber $\theta^{-1}(y)$ intersects uX at exactly one point: $\theta^{-1}(y) \cap uX = \{x\}$. Now by disintegrating each μ_i over η, inside the set uX, we conclude that $\mu_1 = \mu_2$. This proves the unique ergodicity of (X, Γ) .

It is also clear from the proof that the map $\theta : (X, \mu, \Gamma) \to (Y, \eta, \Gamma)$, where μ is the unique invariant measure on X, is an isomorphism of measure preserving systems.

$\square$

Examples

- As we have seen above the enveloping semigroup of an HAE system is metrizable, whence of course Rosenthal. Thus every HAE system is tame.
- Consider an irrational rotation $(\mathbb{T}, R_\alpha)$. Choose $x_0 \in \mathbb{T}$ and split each point of the orbit $x_n = x_0 + n\alpha$ into two points $x_n^\pm$. This procedure results in a **generalized Sturmian** dynamical system (X, T) which is a minimal almost 1-1 extension of $(\mathbb{T}, R_\alpha)$. The enveloping semigroup $E(X, T)$ is homeomorphic to the "two arrows" space, a basic example of a non-metrizble Rosenthal compactum. Thus (X, T) is tame but not HAE.

Examples

- Every **null** dynamical system is Tame.
- The actions of $G = SL_n(\mathbb{R})$ on both the sphere S^{n-1} and the projective space $\mathbb{P}^{n-1}$ ($n \geq 2$) are tame. In both cases the enveloping semigroup is Rosenthal but not metrizable. In the case of the projective space it is not even first countable. Again we conclude that these systems are tame but not HAE.

HNS and Tame subshifts

For a discrete group G and a finite alphabet $\mathcal{L} = \{1, \ldots, l\}$ let $\Omega = \mathcal{L}^G$. A closed G-invariant subset $X \subset \Omega$ is called a **subshift**.

Theorem

Let $X \subset \Omega$ be a subshift. The following conditions are equivalent:

1. (X, G) *is HAE.*
2. X *is countable.*

Theorem

Let $X \subset \Omega$ be a subshift. The following conditions are equivalent:

1. (X, G) *is tame.*
2. *For every infinite subset $L \subseteq G$ there exists an infinite subset $K \subseteq L$ such that $\pi_K(X)$ is a countable subset of $\mathcal{L}^K$.*

Asplund and Radon–Nikodým Banach spaces

Definition

- A Banach space V is **Asplund** if for every separable subspace $U \subset V$ its dual U^* is separable. Reflexive spaces and spaces of type $c_0(\Gamma)$ are Asplund.
- The Banach space V is **Radon–Nikodým** if it is of the form V^*, where V is an Asplund space.
- The Banach space V is **Rosenthal** if it does not contain an isomorphic copy of the space ℓ_1.

Banach representations

Definition

- A **representation** of a DS (X, G) on a Banach space V consists of a pair

$$h : G \to \mathrm{Iso}\,(V) \quad \text{and} \quad \alpha : X \to B_1^*,$$

where h is a group homomorphism, B_1^* is the unit ball in V^* and α is a factor map of DS:

$$\alpha(gx) = h(g)^* \alpha(x) \qquad \forall g \in G, x \in X.$$

- A representation is **proper** if α is an embedding.

The standard representation

Trivially every DS (X, G) has a proper representation on $C(X)$. Namely

$$\alpha(x) = \delta_x,$$

where δ_x is the point mass at x viewed as an element of $C(X)^*$.

Finding representations on geometrically “nicer” Banach spaces (Hilbert, Reflexive, Asplund, Rosenthal ...) is a more difficult task.

A theorem of Megrelishvili

Theorem

Let (X, G) be a compact metric DS. The following conditions are equivalent:

- (X, G) *is WAP.*
- (X, G) *admits a proper representation on a Reflexive Banach space.*
- *Every element of $E(X, G)$ is continuous.*

RN systems

Definition

A DS is **Radon–Nikodým (RN)** if it admits a proper representation on an Asplund Banach space. It is **weak Radon–Nikodým (WRN)** if it admits a proper representation on a Rosenthal Banach space.

Note that when $G = \{e\}$, i.e. there is no group action, we recover the definition of Radon–Nikodým compact spaces due to Namioka.

The equivalence of RN and HAE

Theorem (Gl-Meg-Usp)

Let (X, G) be a compact metric DS. The following conditions are equivalent:

- *(X, G) is HAE.*
- *(X, G) is RN; i.e. it admits a proper representation on an Asplund spce.*
- *$E(X, G)$ is metrizable.*

Tame dynamical systems

Theorem (Gl-Meg)

Let (X, G) be a compact metric DS. The following conditions are equivalent:

- *(X, G) is tame.*
- *(X, G) is WRN or Rosenthal representable; i.e. it admits a proper representation on a Banach space which does not contain ℓ_1.*
- *Every element of $E(X, G)$ is of Baire-class-1.*

The hierarchy of Banach representations

- Let X be a compact metrizable G-space,
- $E(X)$ the corresponding enveloping semigroup.
- f stands for an arbitrary function in $C(X)$.
- $fG = \{f \circ g : g \in G\}$ denotes its orbit.
- $\overline{fG}$ is the pointwise closure of fG in $\mathbb{R}^X$.

	Dynamical characterization	Enveloping semigroup	Banach representation
WAP	$\overline{fG}$ is a subset of $C(X)$	Every element is continuous	Reflexive
HNS	$\overline{fG}$ is metrizable	$E(X)$ is metrizable	Asplund
Tame	$\overline{fG}$ is Fréchet	Every element is Baire 1	Rosenthal

Table: The hierarchy of Banach representations

A dynamical approach to the examples of James and Lindenstrauss

- One of the important problems in Banach space theory in the mid 70's was how to construct a separable Rosenthal space (i.e. a Banach space not containing ℓ_1) which is not Asplund.
- The first counterexamples were given independently by James and Lindenstrauss.
- We will show next how counterexamples to the Banach space problem can be easily obtained from our dynamical results.

The Banach space examples

- We want to construct a Banach space which is Rosenthal (i.e. does not contain ℓ_1) but not Asplund.
- In view of the above dynamical results, all one needs to do is to construct a DS (X, G) which is tame but not HAE. This system will then be representable on a Rosenthal Banach space V which can not possibly be Asplund. For otherwise (X, G) would be HAE.
- Now recall that we have many examples of DS of this type; e.g. the generalized Sturmian cascades or $GL_n(\mathbb{R})$ acting on the sphere or the projective space.

Конец Фильма