Dynamics on the graph of the torus parametrisation

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Randomness of primes and the Möbius function

- $\mu : \mathbb{N} \to \{-1, 0, 1\}$ defined by $\mu(1) = 1$ and
  - $\mu(n) = (-1)^t$ if n is a product of t distinct primes
  - $\mu(n) = 0$ if n is not squarefree

  e.g., $\mu(15) = \mu(3 \cdot 5) = (-1)^2 = 1$ and $\mu(20) = \mu(2^2 \cdot 5) = 0$
- consider the Mertens function $M(n) = \sum_{k=1}^{n} \mu(k)$
- μ "orthogonal" to constant sequence:

$$M(n) = o(n) \iff \text{PNT}$$

(LLN, Landau 1906, compare Tao 2009)

Möbius randomness principle

- sequence $\xi = (\xi(n))_{n\in\mathbb{N}}$ with values in finite $A \subset \mathbb{R}$
- for $\xi \in A^{\mathbb{N}}$ consider shift $T\xi(n) = \xi(n+1)$ and define compact flow

$$\mathcal{F}(\xi) = \overline{\{T^k\xi : k \in \mathbb{N}\}} \subseteq A^{\mathbb{N}}$$

- $\mathcal{F}(\xi)$ called *deterministic* if ξ has zero topological entropy

Conjecture

μ is orthogonal to any sequence in any deterministic flow.

- trivial flows, i.e., $\mathrm{card}(\mathcal{F}(\xi)) = 1$: prime number theorem
- Kronecker flows, i.e., $\mathcal{F}(\xi) = G$ cpct abelian group with T_g group translations: Vinogradov–Davenport (1937)
- distal flows, i.e., $\inf_{k\in\mathbb{N}} d(T^k x, T^k y) > 0$ for distinct $x, y \in \mathcal{F}(\xi)$ for certain "homogeneous nilflows" Green–Tao (2008)

Möbius flow and squarefree flow

Sarnak (2011): interpret μ itself dynamically

- Möbius flow $\mathcal{F}(\mu) \subset \{-1, 0, 1\}^{\mathbb{N}}$
- squarefree flow $\mathcal{F}(\sigma) \subset \{0, 1\}^{\mathbb{N}}$ where $\sigma = \mu^2 = (\mu(n)^2)_{n \in \mathbb{N}}$
- squaring yields factor map from $\mathcal{F}(\mu)$ to $\mathcal{F}(\sigma)$

Lemanczyk et al (2013): generalise σ to so-called $\mathcal{B}$-free integers

- $\mathcal{B} = \{b_1 < b_2 < \ldots\} \subset \{2, 3, \ldots\}$ pairwise coprime such that
$$\sum_{k \in \mathbb{N}} \frac{1}{b_k} < \infty.$$
- $\mathcal{B}$-free integers $V_{\mathcal{B}} \subset \mathbb{Z}$ consist of all integers having no factor in $\mathcal{B}$
- squarefree integers $V_{\mathcal{B}}$ obtained with $\mathcal{B} = \{p^2 : p \text{ prime}\}$

flows of weak model sets

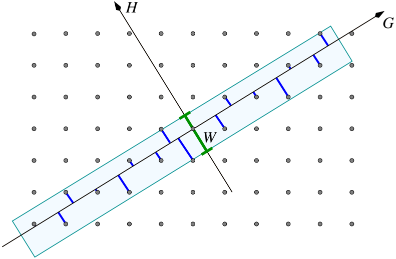
explicit calculations of $\mathcal{B}$-free flows $\mathcal{F}(V_{\mathcal{B}})$, e.g.:

- topological entropy of $(\mathcal{F}(V_{\mathcal{B}}), T)$
- pure point dynamical spectrum of $(\mathcal{F}(V_{\mathcal{B}}), T)$ with respect to pattern frequency measure
- obtained by comparison with a certain torus rotation
- simplex of invariant probability measures on $(\mathcal{F}(V_{\mathcal{B}}), T)$

analysis via weak model sets (Meyer 73, Baake–Moody–Pleasants 99, ...)

- structural insight into above results through underlying cp scheme
- extends theory of regular model sets

cut-and-project schemes and weak model sets



- G *physical space*, H *internal space*, LCA groups, $\mathcal{L}$ lattice in $G \times H$
- infinite strip parallel to G defined by compact *window* $W \subset H$
- *weak model set* $\curlywedge(W)$ by projecting lattice points inside strip to G
- assume wlog that projection of $\mathcal{L}$ is dense in H
- assume that distinct lattice points have distinct G-projection

cp scheme for squarefree integers

(cf. Meyer 73, Baake–Moody–Pleasants 99, Sing 05, ...)

- n squarefree $\iff n \mod p^2 \neq 0$ for all primes p
- consider the compact product group $H = \prod_p (\mathbb{Z}/p^2\mathbb{Z})$
- dense embedding of $\mathbb{Z}$ into H:

$$n \mapsto \iota(n) = (n \mod p^2)_p,$$

- take $G = \mathbb{Z}$ and $\mathcal{L} = \{(n, \iota(n)) : n \in \mathbb{Z}\} \subset G \times H$

Lemma

$(G, H, \mathcal{L})$ is a cp scheme, i.e., $\mathcal{L}$ is a lattice in $G \times H$, $\pi^G|_{\mathcal{L}}$ is 1-1 and $\pi^H(\mathcal{L})$ is dense in H.

squarefree integers as a weak model set

squarefree integers weak model set in $(G, H, \mathcal{L})$ with window

$$W = \prod_p (\mathbb{Z}/p^2\mathbb{Z}) \backslash \{0_p\}$$

- W closed as every component closed
- $\mathrm{int}(W) = \varnothing$, as no component of W is maximal
- hence $W = \partial W$

taking the canonical Haar measure m_H on H, we have

$$m_H(W) = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} \approx 0.6079\ldots$$

Density of squarefree integers is volume of window! (generally $\leqslant$)

diffraction of squarefree integers

diffraction theory:

- diffraction measure describes intensity of diffraction in a physical diffraction experiment.

explicit calculation for squarefree integers:

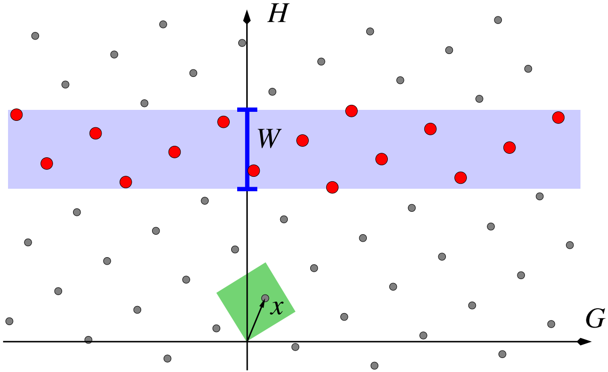
- pure point diffraction, i.e., no continuous component
- Bragg peak positions: rationals x with cubefree denominator q
- intensity of Bragg peak at x:

$$I(x) = \frac{1}{\zeta(2)^2} \prod_{p|q} \frac{1}{(p^2-1)^2}$$

(e.g., Baake–Moody–Pleasants 1999, Pleasants–Huck 2013)

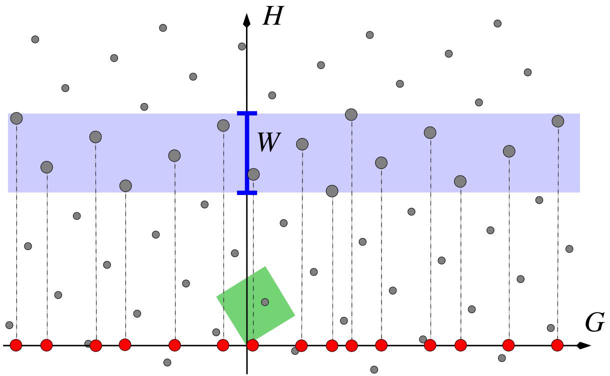
Squarefree integers are pure point diffractive!

torus parametrisation of weak model sets



green: FD of torus $\hat{X}$, red: $(\mathcal{L} + x) \cap (G \times W) = \mathrm{supp}(\nu_W(\hat{x}))$

torus parametrisation of weak model sets



green: FD of torus $\hat{X}$, red: $\pi^G((\mathcal{L}+x)\cap(G\times W)) = \mathrm{supp}(\nu_W^G(\hat{x}))$

description of weak model sets

torus parametrisation (Robinson 93, Baake, Pleasants et al 95, ...)

- fix cp scheme $(G, H, \mathcal{L})$, Haar measures m_G, m_H, window $W \subset H$
- weak model sets $\pi^G((\mathcal{L} + x) \cap (G \times W))$ for any $x \in G \times H$
- parametrisation by cpct torus $\hat{X} = (G \times H)/\mathcal{L}$ via $\hat{x} = x + \mathcal{L}$

torus dynamics (e.g. Moody 02)

- $\hat{T}_g \hat{x} = \hat{x} + (g, 0)$ minimal G-action on $\hat{X}$
- $(\hat{X}, \hat{T})$ uniquely ergodic, pure point dynamical spectrum
- induces dynamics on weak model sets
- Haar measure $m_{\hat{X}}$: any weak model set equally probable

We revisit and extend these approaches.

description of weak model sets

weak model sets as measures

- describe weak model set as measure via its Dirac comb
- measure $\nu_W(\hat{x})$ on $G \times H$, i.e.,

$$\nu_W(\hat{x}) = \sum_{y \in (\mathcal{L}+x) \cap (G \times W)} \delta_y$$

- projected measure $\nu_W^G(\hat{x})$ on G, i.e.,

$$\nu_W^G(\hat{x}) = \sum_{y \in (\mathcal{L}+x) \cap (G \times W)} \delta_{\pi^G(y)}$$

- induced dynamics on $G \times H$ easier than on projection to G

first analyse dynamics in $G \times H$ and then transfer results to G

flows from weak model sets

- flow of measure $\nu_W(\hat{x})$ on $G \times H$

$$\mathcal{M}_W(\hat{x}) = \overline{\{T_g \nu_W(\hat{x}) : g \in G\}}$$

- flow of projected measure $\nu_W^G(\hat{x})$ on G

$$\mathcal{M}_W^G(\hat{x}) = \overline{\{T_g \nu_W^G(\hat{x}) : g \in G\}}$$

- consider first the simpler flow spaces

$$\mathcal{M}_W := \mathcal{M}_W(\hat{X}), \quad \mathcal{M}_W^G := \mathcal{M}_W^G(\hat{X})$$

- as $\nu_W : \hat{X} \to \mathcal{M}_W$ and $\nu_W^G : \hat{X} \to \mathcal{M}_W^G$ are measurable maps, we may lift the torus Haar measure $m_{\hat{X}}$ to $\mathcal{M}_W$ and to $\mathcal{M}_W^G$ via

$$Q_{\mathcal{M}} = m_{\hat{X}} \circ \nu_W^{-1}, \qquad Q_{\mathcal{M}^G} = m_{\hat{X}} \circ (\nu_W^G)^{-1}$$

  Let us call these ergodic measures **Mirsky measures**.

measure-theoretic results: discrete spectrum

Theorem

Assume $m_H(W) > 0$. Then

(i) $(\hat{X}, m_{\hat{X}}, \hat{T})$ *and* $(\mathcal{M}_W, Q_{\mathcal{M}}, T)$ *are measure theoretically isomorphic.*

(ii) $(\mathcal{M}_W^G, Q_{\mathcal{M}^G}, T)$ *is a factor of* $(\hat{X}, m_{\hat{X}}, \hat{T})$.

(iii) *If* $m_H(\partial W) = 0$, *then* $(\mathcal{M}_W, T)$ *and* $(\mathcal{M}_W^G, T)$ *are uniquely ergodic.*

In particular, $(\mathcal{M}_W^G, Q_{\mathcal{M}^G}, T)$ *has pure point dynamical spectrum.*

remarks

- implies that diffraction measure of $Q_{\mathcal{M}^G}$ is pure point (Dworkin)
- extends results on $\mathcal{B}$-free systems (ii) and regular model sets (iii) (previous results explicit calculation and different measure definition)

pure point spectrum: arguments

measure-theoretic isomorphism from $\hat{X}$ to $\mathcal{M}_W$ via the map ν_W

- 1-1 except on $Z_W = \{\hat{x} \in \hat{X} : \nu_W(\hat{x}) = \underline{0}\}$
- $m_{\hat{X}}(Z_W) = 0$ if and only if $m_H(W) > 0$ as

$$\hat{x} = x + \mathcal{L} \in Z_W \iff x_H \in \bigcap_{\ell \in \mathcal{L}} (W^c - \ell_H) = \left(\bigcup_{\ell \in \mathcal{L}} (W - \ell_H) \right)^c$$

- hence also $Q_{\mathcal{M}}(\nu_W(Z_W)) = 0$ since

$$Q_{\mathcal{M}}(\nu_W(Z_W)) = m_{\hat{X}} \circ \nu_W^{-1}(\nu_W(Z_W)) = m_{\hat{X}}(Z_W)$$

measure-theoretic factor map from $\hat{X}$ to $\mathcal{M}_W^G$ via $\nu_W^G = \pi_*^G \circ \nu_W$

- need not be 1-1 as $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ need not be 1-1

dynamical properties of $(\mathcal{M}_W^G, T, Q_{\mathcal{M}^G})$ are inherited from $(\hat{X}, \hat{T}, m_{\hat{X}})$

injectivity properties of $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$

for $W \subset H$ recall

- W irredundant (aperiodic) if $h + W = W$ implies $h = 0$
- W topologically regular if $W = \overline{\mathrm{int}(W)}$

Lemma

Let W be topologically regular. Then the following are equivalent.

- *W is irredundant.*
- *$W \neq \varnothing$ and $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is a homeomorphism.*

for $\mathcal{B}$-free integers, there is a full measure subset where π_*^G is 1-1 (below)

flows of weak model sets $\nu_W(\hat{x})$

configurations of maximal density:

- for averaging in G, fix any tempered van Hove sequence $(A_n)_n$
- $\nu_W(\hat{x})$ has *maximal density* if

$$\lim_{n\to\infty} \frac{\nu_W(\hat{x})(A_n)}{m_G(A_n)} = \mathrm{dens}(\mathcal{L}) \cdot m_H(W)$$

(lim sup with $\leqslant$ always true)

Lemma (Moody 02, maximal density is generic)

Let $\hat{X}_{max} = \{\hat{x} \in \hat{X} : \nu_W(\hat{x})$ has maximal density$\}$. Then $\hat{X}_{max}$ is $\hat{T}$-invariant and has full Haar measure.

Moody also shows that pure point diffractivity is generic.

flows of weak model sets $\nu_W(\hat{x})$

Theorem

- *There is a full measure subset* $\hat{X}_0 \subseteq \hat{X}_{max}$ *such that* $\mathcal{M}_W(\hat{x}) = \mathrm{supp}(Q_{\mathcal{M}})$ *for all* $\hat{x} \in \hat{X}_0$*. We may choose*

$$\hat{X}_0 = \hat{X}_{max} \cap (\nu_W)^{-1}(\mathrm{supp}(Q_{\mathcal{M}}))$$

- $Q_{\mathcal{M}}$ *is the pattern frequency measure.*

$\mathcal{B}$-free integers:

- choose W such that $V_{\mathcal{B}} = \mathrm{supp}(\nu_W^G(\hat{0}))$
- $\hat{X}_0 = \hat{X}_{max}$ and $\mathcal{M}_W^G(\hat{x}) = \mathcal{M}_W^G$ for all $\hat{x} \in \hat{X}_{max}$
- $(\hat{X}, m_{\hat{X}}, \hat{T})$ isomorphic to $(\mathcal{M}_W^G(\hat{x}), Q_{\mathcal{M}^G}, T)$ for any $\hat{x} \in \hat{X}_{max}$.
- As $\nu_W^G(\hat{0})$ has maximal density, it has pure point spectrum.

$\mathcal{B}$-free integers: injectivity properties of $\pi_*^{\mathcal{G}} : \mathcal{M}_W \to \mathcal{M}_W^{\mathcal{G}}$

For $A \subseteq \mathbb{Z}$ write $A_k = \{g_k : g \in A\}$ where $g_k = g \mod b_k$.

Definition

$\mathcal{Y} = \{\nu \in \mathcal{M}_W : \text{card supp}(\pi_*^{\mathcal{G}}\nu)_k = b_k - 1 \text{ for all } k \in \mathbb{N}_0\}$

- $\mathcal{Y} \subseteq \mathcal{M}_W$ measurable, consists of all measures such that "every b-reduction misses exactly one coset".
- $\pi_*^{\mathcal{G}}(\mathcal{Y})$ studied before:
    - X_1 for square-free integers (Peckner 12)
    - Y for $\mathcal{B}$-free systems (Kulaga-Przymus 14)
    - $\mathbb{A}_1$ for visible lattice points (Baake–Huck 14)

The next lemmas have close analogues in the above publications.

$\mathcal{B}$-free integers: injectivity properties of $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$

Lemma

π_*^G is 1-1 on $\mathcal{Y}$. Hence $\pi_*^G|_{\mathcal{Y}} : \mathcal{Y} \to \pi_*^G(\mathcal{Y})$ is a Borel isomorphism.

(Assume $\nu, \nu' \in \mathcal{Y}$ such that $\pi_*^G \nu = \pi_*^G \nu'$. By the coset condition, ν, ν' must be supported on the same shifted lattice. Hence $\nu = \nu'$.)

Lemma

We have $\nu_W(\hat{X}_{max}) \subseteq \mathcal{Y}$. Hence, $\mathcal{Y}$ has full Mirsky measure.

(Two missing cosets contradicts maximal density.)

the approach Baake–Huck–Strungaru (2015)

- consider weak model sets of maximal (or minimal) density
- different technique: approximate weak model set from above (below) by regular model sets, i.e., $m_H(\partial W) = 0$

results:

- expressions for pattern frequencies as in the regular case
- yield autocorrelation and diffraction measure
- ergodicity of measure on $\mathcal{M}_W^G$ defined by pattern frequencies

topological results: the map $\nu_w : \hat{X} \to \mathcal{M}_w$

consider the following $\hat{T}$-invariant subsets of $\hat{X}$:

- $C_w \subset \hat{X}$ set of continuity points of ν_w wrt vague convergence.
  - C_w residual in $\hat{X}$, in particular dense in $\hat{X}$
  - $\hat{x} = x + \mathcal{L}$ continuity point iff x_H generic, i.e.,

$$(\partial W - x_H) \cap \pi^H(\mathcal{L}) = \varnothing \iff x_H \in \bigcap_{\ell \in \mathcal{L}} (\partial W^c - \ell_H)$$

- $Z_w \subset \hat{X}$ set of zero points of ν_w, i.e., $\hat{x}$ for which $\nu_w(\hat{x}) = \underline{0}$.
  - $\nu_w : \hat{X} \to \mathcal{M}_w$ is 1-1 on $\hat{X} \backslash Z_w$
  - Z_w empty if $\mathrm{int}(W) \neq \varnothing$

the map $\nu_w : \hat{X} \to \mathcal{M}_w$

Proposition (Upper semicontinuity of ν_w)

i) *If* $\lim_{n\to\infty} \hat{x}^n = \hat{x}$, *then* $\nu \leqslant \nu_w(\hat{x})$ *for all vague limit points* ν *of* $(\nu_w(\hat{x}^n))_n$, *and* $d\nu/d\nu_w(\hat{x})$ *takes only values* 0 *and* 1.

ii) $\hat{x} \in C_w$ *if and only if* $\{\nu \in \mathcal{M}_w : \nu \leqslant \nu_w(\hat{x})\} = \{\nu_w(\hat{x})\}$.

iii) C_w *is residual in* $\hat{X}$, *i.e., the complement of* C_w *in* $\hat{X}$ *is meagre.*

iv) *If* $\mathrm{int}(W) \neq \varnothing$, *then* $Z_w = \varnothing$ *and, even more,* $\underline{0} \notin \mathcal{M}_w$.

v) *If* $\mathrm{int}(W) = \varnothing$, *then* $Z_w = C_w$ *is residual in* $\hat{X}$. *If in addition* $W \neq \varnothing$, *then* $\underline{0} \in \overline{\nu_w(\hat{X} \backslash Z_w)}$.

topological results

- we have the graph dynamical systems (with canonical group actions)

$$\mathcal{GM}_w = \overline{\{(\hat{x}, \nu_w(\hat{x})) : \hat{x} \in \hat{X}\}}, \qquad \mathcal{GM}_w^{\mathcal{G}} = \overline{\{(\hat{x}, \nu_w^{\mathcal{G}}(\hat{x})) : \hat{x} \in \hat{X}\}}$$

- working on graph advantageous to working on projection

Proposition

$\pi_*^{\hat{x}} : (\mathcal{GM}_w, S) \to (\hat{X}, \hat{T})$ *is a topological almost 1–1 extension of its maximal equicontinuous factor.*

argument:

- $(\pi_*^{\hat{x}})^{-1}\{\hat{x}\} = \{(\hat{x}, \nu_w(\hat{x}))\}$ for all $\hat{x} \in C_w$ and C_w dense in $\hat{X}$
- hence $\pi_*^{\hat{x}}$ almost 1–1 extension of an equicontinuous factor $(\hat{X}, \hat{T})$
- as factor map from maximal equicontinuous factor to $(\hat{X}, \hat{T})$ has one-point fibre, it must coincide with $(\hat{X}, \hat{T})$

topological results

Theorem (Topological factors and extensions)

i) *If* $\mathrm{int}(W) \neq \varnothing$, *then* $\pi_*^{\hat{X}} \circ (\pi_*^{G \times H})^{-1} : (\mathcal{M}_W, T) \to (\hat{X}, \hat{T})$ *is a topological almost 1–1 extension of its maximal equicontinuous factor.*

ii) *If* $\mathrm{int}(W) \neq \varnothing$, *then the restriction of* T *to the subsystem* $\overline{\nu_W(C_W)} \subseteq \mathcal{M}_W$ *is an almost automorphic extension of* $(\hat{X}, \hat{T})$, *and it is the only minimal subsystem of* $(\mathcal{M}_W, T)$.

iii) *If* $\mathrm{int}(W) = \varnothing$ *and if* $(\mathcal{M}_W, T)$ *is topologically transitive, then* $(\mathcal{M}_W, T)$ *has no nontrivial equicontinuous factor.*

results ii), iii) translate to $(\mathcal{M}_W^G, S)$ if $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is 1–1

topological properties of flows

Theorem

i) *If* $\mathrm{int}(W) \neq \varnothing$, *then* $\overline{\nu_W(C_W)} = \mathcal{M}_W(\hat{x})$ *for all* $\hat{x} \in C_W$.

ii) *If* $m_H(\partial W) = 0$, *then* $\overline{\nu_W(C_W)} = \mathrm{supp}(Q_{\mathcal{M}})$

- hence for $m_H(\partial W) = 0$ the relevant topological and measure-theoretic dynamical systems coincide.
- results transfer to G for topologically regular windows

graphs versus skew products

move all information about lattice shift into first component!

$$\mathcal{GM}_w = \overline{\{(\hat{x}, \nu_w(\hat{x})) : \hat{x} \in \hat{X}\}}$$

- fix fundamental domain X of $\mathcal{L}$ and define

$$\mathcal{G}\Omega_w = \overline{\{(x, S_{-x}\nu_w(\hat{x})) : x \in X\}} \subset X \times \{0,1\}^{\mathcal{L}}$$

- corresponding G-action on 2^{nd} factor of $\mathcal{G}\Omega_w$ piecewise continuous
- defines skew product dynamical system $\mathcal{G}\Omega_w$ over base X
- $\mathcal{G}\Omega_w$ is extension of $\mathcal{GM}_w$ with injective factor map

skew products with actions of countable amenable groups

- entropy theory (cf. Huck–R 15)
- thermodynamic formalism (Ward–Zhang 92, Bogenschütz 93)