Homeomorphisms of Tiling Spaces

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Joint work with Antoine Julien

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Outline

1. One question; 3 settings

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- Third setting: h isn't necessarily a homeomorphism, but is a general orbit-preserving continuous map.
- This talk mostly about FLC setting. Others are similar.

Main FLC result

Theorem

If $h : \Omega \to \Omega'$ is a homeomorphism of FLC tiling spaces, and if Ω is uniquely ergodic, then h is the composition of a "weak translation", a shape change, and an MLD equivalence.

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Corollary

The relation "is homeomorphic to" is generated by shape changes and MLD transformations.

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If $h : \Omega \to \Omega'$ is an orbit-preserving homeomorphism of general tiling spaces, and if Ω is uniquely ergodic, then h is the composition of a "weak translation", a shape change, and a topological conjugacy.

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If $h : \Omega \to \Omega'$ is an orbit-preserving homeomorphism of general tiling spaces, and if Ω is uniquely ergodic, then h is the composition of a "weak translation", a shape change, and a topological conjugacy.

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The relation "is orbit-equivalent to" is generated by shape changes and topological conjugacies.

Result for general maps

Theorem

Suppose $f : \Omega \to \Omega'$ is an orbit-preserving surjection of tiling spaces, and Ω is uniquely ergodic.

- *If Ω and Ω' are FLC, then h is homotopic to the composition of a shape change and a local derivation.*
- *In general, h is homotopic to the composition of a shape change and a factor map.*

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Corollary

To understand arbitrary orbit-preserving maps, study (a) shape changes, (b) factor maps, and (c) maps homotopic to the identity.

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- All MLD transformations are topological conjugacies, but not all conjugacies are MLD (Petersen, Radin-S).

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- "Continuous" means $F(T - x)$ is arbitrarily well-approximated by local pattern around x.
- If $F(T - x) - F(T - y)$ is determined exactly by local pattern around x and y, FLC is preserved, but shapes and sizes of tiles may change.

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Theorem (Kellendonk-S)

Every topological conjugacy is the composition of an MLD transformation and a shape conjugacy.

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- (Subject of Samuel Petite's talk?)

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Reminder:

Theorem

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- Theorem: $H^*_{PE}(\Omega, A) \simeq \check{H}^*(\Omega, A)$.

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- $[\mathcal{F}(\Omega)]$ gives class in $H^1_{PE}(\Omega, \mathbb{R}^d) \simeq \check{H}^1(\Omega, \mathbb{R}^d)$.

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- $[h]$ pulls geometric data from Ω' back to Ω.

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- 1) Work with Čech cohomology: $[\mathcal{F}(\Omega')] \in H^1_{PE}(\Omega', \mathbb{R}^d) \simeq \check{H}^1(\Omega', \mathbb{R}^d) \to \check{H}^1(\Omega, \mathbb{R}^d) \simeq H^1_{PE}(\Omega, \mathbb{R}^d)$.

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- $h_s = h \circ \tau_s$, where τ_s is a weak translation.

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Statement of the theorem

Theorem

Let $h_i : \Omega \to \Omega_i$ be two homeomorphisms ($i \in \{1,2\}$) of FLC tiling spaces. If $[h_1] = [h_2]$, then there exists a continuous $s : \Omega \to \mathbb{R}^d$ such that $\tau_s : T \mapsto T - s(T)$ is a homeomorphism, and there exists an MLD equivalence $\phi : \Omega_1 \to \Omega_2$ such that $h_2 \circ \tau_s = \phi \circ h_1$.

$$\begin{tikzcd} \Omega \arrow[r, "h_1"] \arrow[d, "\tau_s"'] & \Omega_1 \arrow[d, "\phi"] \\ \Omega \arrow[r, "h_2"] & \Omega_2 \end{tikzcd}$$

Sketch of proof

$$\begin{array}{ccc} \Omega & \xrightarrow{h_1} & \Omega_1 \\ {\scriptstyle \tau_s}\downarrow & & \downarrow{\scriptstyle \phi} \\ \Omega & \xrightarrow{h_2} & \Omega_2 \end{array}$$

- Since $[h_1] = [h_2]$, displacements in Ω_1 and Ω_2 agree, up to local correction.

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- But $\phi = h_2 \circ \tau_s \circ h_1^{-1}$ is homeomorphism, so ϕ is MLD map.

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- $C_\mu(\alpha) = \int_\Omega \alpha(0) d\mu$.
- Depends only on class of α: $C_\mu(d\beta) = 0$.
- Ergodic theorem: if μ ergodic, $C_\mu([\alpha]) =$ average value of $\alpha(x)$ for generic tiling.
- If Ω uniquely ergodic, $C_\mu([\alpha]) =$ average value of $\alpha(x)$ for every tiling.

$C_\mu(H^1(\Omega, \mathbb{R}^d))$

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- Example: $\mathcal{F}$ represented by constant 1-form $d\vec{x}$. $C_\mu(\mathcal{F}) = I$.
- $C_\mu([h])$ = linear transformation of $\mathbb{R}^d$ = large scale distortion induced on orbits by h.

Classes of homeomorphisms are non-singular

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- Restrict attention to how h acts on a single translational orbit.
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- For h to be onto, fluctuations must be large, but then h isn't 1:1.

Shape changes implement all non-singular classes

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If Ω is uniquely ergodic, $[\alpha] \in H^1(\Omega, \mathbb{R}^d)$ and $C_\mu([\alpha])$ is invertible, there is a shape change homeomorphism whose class is $[\alpha]$.

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- Deform one orbit by $x \to \int_0^x \alpha$. Extend by continuity.

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- $C_\mu([h])$ is invertible.
- There is a shape change $h_2 : \Omega \to \Omega_2$ with $[h_2] = [h]$.
- $$\begin{tikzcd} \Omega \arrow[r, "h"] \arrow[d, "\tau_s"'] & \Omega' \arrow[d, "\phi"] \\ \Omega \arrow[r, "h_2"] & \Omega_2 \end{tikzcd}$$
- $h = \phi^{-1} \circ h_2 \circ \tau_s$.

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- Theorems about Ruelle-Sullivan exactly same as before.

General maps

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- For structure theorem, must assume that h_1 is orbit equivalence. Weak translation τ_s is chosen to go up, but does not have to be invertible. Lack on injectivity of h_2 can come either from τ_s or from factor map ϕ.

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- For structure theorem, must assume that h_1 is orbit equivalence. Weak translation τ_s is chosen to go up, but does not have to be invertible. Lack on injectivity of h_2 can come either from τ_s or from factor map ϕ.

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- As long as h is surjective, $C_\mu[h]$ is still invertible.
- In applying structure theorem, $h_2 = h$ and h_1 is shape change with same class.

Recap

Theorem

If $h : \Omega \to \Omega'$ is a homeomorphism of FLC tiling spaces, and if Ω is uniquely ergodic, then h is the composition of a "weak translation", a shape change, and an MLD equivalence. There is a class $[h] \in H^1(\Omega, \mathbb{R}^d)$ that characterizes the shape change (up to MLD).

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If $h : \Omega \to \Omega'$ is a homeomorphism of FLC tiling spaces, and if Ω is uniquely ergodic, then h is the composition of a "weak translation", a shape change, and an MLD equivalence. There is a class $[h] \in H^1(\Omega, \mathbb{R}^d)$ that characterizes the shape change (up to MLD).

Theorem

If $h : \Omega \to \Omega'$ is an orbit-preserving homeomorphism of general tiling spaces, and if Ω is uniquely ergodic, then h is the composition of a "weak translation", a shape change, and a topological conjugacy. There is a class $[h] \in H^1_w(\Omega, \mathbb{R}^d)$ that characterizes the shape change (up to conjugacy).

More recap

Theorem

Suppose $f : \Omega \to \Omega'$ is an orbit-preserving surjection of tiling spaces, and $\Omega 1$ is uniquely ergodic.

- *If Ω and Ω' are FLC, then h is homotopic to the composition of a shape change and a local derivation.*
- *In general, h is homotopic to the composition of a shape change and a factor map.*
- *There is a distinguished class $[h]$in H^1 or H^1_w that parametrizes the shape change.*