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Pattern-equivariant functions and cohomology

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Abstract
The cohomology of a tiling or a point pattern has originally been defined via the construction of the hull or the groupoid associated with the tiling or the pattern. Here we present a construction which is more direct and, therefore, more easily accessible. It is based on generalizing the notion of equivariance from lattices to point patterns of finite local complexity.

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1. Introduction

When topological invariants of tilings were first defined and computed [Con90, Con94, Kel95, Kel97, AP98] they arose as K-groups of associated C^*-algebras but it soon became clear that, apart from the order on K_0, the K-groups are isomorphic to the integer valued Čech cohomology of the continuous hull of the tiling, or equivalently, to the integer valued cohomology of the discrete tiling groupoid [FH99]. Neither the continuous hull nor the discrete groupoid of a tiling are mathematical objects which are easily accessible by the non-expert. The purpose of this paper is to present a formulation of (real valued) tiling or point pattern cohomology which we believe is easier to understand on an intuitive level, because it involves more standard mathematical objects. This does not mean that it is easier to compute the tiling cohomology in this formulation. But we hope that it helps others to understand what the cohomology of a tiling actually means.

We formulate our results below for Delone sets of finite local complexity which we call for short here point patterns. This covers the case of tilings of finite local complexity since the topological invariants mentioned depend only on MLD classes and tilings are mutually locally derivable from Delone sets.

We will proceed along the following lines: let $\mathcal{P}$ be a tiling or a point pattern in the Euclidean space $\mathbb{R}^n$. We construct the (real valued) cohomology of $\mathcal{P}$ as the cohomology of the sub-complex of the de Rham complex over $\mathbb{R}^n$ given by the $\mathcal{P}$-equivariant forms. The main new ingredient is thus the notion of $\mathcal{P}$-equivariance.

2. $\mathcal{P}$-equivariant functions and cohomology

In a frequently used approach to describe particle motion in aperiodic media one investigates a Hamiltonian of the form $H = -\frac{\hbar^2}{2m}\Delta + V$ on $L^2(\mathbb{R}^n)$ where Δ is the Laplacian (or even the magnetic Laplacian in the presence of an external magnetic field) and V is a potential which describes the local interaction between the particle and the medium. This potential incorporates the aperiodic structure of the material in such a way that $V(x)$ depends only on the local configuration one finds around x. So if $\mathcal{P}$ is a point pattern or a tiling representing the aperiodic structure (e.g., $\mathcal{P}$ is the set of equilibrium positions of the atoms in the material) then V should be a $\mathcal{P}$-equivariant function in the following sense.

Recall that a discrete point set has finite local complexity if up to translation there are only finitely many r-patches for any r, an r-patch being $B_r(x) \cap \mathcal{P}$, the intersection of an r-ball at some point $x \in \mathbb{R}^n$ with $\mathcal{P}$. This implies that $\mathcal{P}$ is uniformly discrete, i.e. that any two distinct points have distance larger than a given $r > 0$. A Delone set is a uniformly discrete point set which is relatively dense, i.e. there exists an $R > 0$ such that all R-patches contain at least one point.

Definition 2.1. *Let $\mathcal{P}$ be a subset of $\mathbb{R}^n$ of finite local complexity. We call a function $f : \mathbb{R}^n \to X$ into some set X strongly $\mathcal{P}$-equivariant if there exists an $r > 0$ such that*

$$B_r \cap (\mathcal{P} - x) = B_r \cap (\mathcal{P} - y) \qquad \text{implies} \quad f(x) = f(y)$$

(B_r is the closed r-ball around 0).

Suppose that $\mathcal{D}$ is locally derivable from $\mathcal{P}$ in the sense of [BSJ91]. This means that for all $r > 0$ there exists $R > 0$ such that $B_R \cap (\mathcal{P} - x) = B_R \cap (\mathcal{P} - y)$ implies $B_r \cap (\mathcal{D} - x) = B_r \cap (\mathcal{D} - y)$. It is immediate from this definition that then any strongly $\mathcal{D}$-equivariant function is also strongly $\mathcal{P}$-equivariant. In particular, the concept of $\mathcal{P}$-equivariance is defined not only for a single point set but for MLD classes of point sets.

Recall that the de Rham complex over $\mathbb{R}^n$ is the complex

$$\Omega^0(\mathbb{R}^n) \xrightarrow{\mathrm{d}} \Omega^1(\mathbb{R}^n) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(\mathbb{R}^n)$$

where $\Omega^k(\mathbb{R}^n)$ are the differential k-forms over $\mathbb{R}^n$ and d is the exterior derivative. Using standard coordinates on $\mathbb{R}^n$ a differential k-form can be written as $\sum_{i_1,\ldots,i_k} f_{i_1,\ldots,i_k}\, \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_k}$ with smooth functions $f_{i_1,\ldots,i_k} : \mathbb{R}^n \to \mathbb{R}$. Setting $X = \mathbb{R}$ we may therefore consider strongly $\mathcal{P}$-equivariant differential forms over $\mathbb{R}^n$ as those for which the functions $f_{i_1,\ldots,i_k}$ are smooth and strongly $\mathcal{P}$-equivariant. We denote them by $\check{\Omega}_{\mathcal{P}}(\mathbb{R}^n)$. Clearly $\mathrm{d}(\check{\Omega}_{\mathcal{P}}(\mathbb{R}^n)) \subset \check{\Omega}_{\mathcal{P}}(\mathbb{R}^n)$ and so

$$\check{\Omega}^0_{\mathcal{P}}(\mathbb{R}^n) \xrightarrow{\mathrm{d}} \check{\Omega}^1_{\mathcal{P}}(\mathbb{R}^n) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \check{\Omega}^n_{\mathcal{P}}(\mathbb{R}^n)$$

is a differential sub-complex of the de Rham complex.

Definition 2.2. *The $\mathcal{P}$-equivariant de Rham cohomology of $\mathbb{R}^n$ is the cohomology of the sub-complex defined by $\check{\Omega}_{\mathcal{P}}(\mathbb{R}^n)$. We denote it by $\check{H}_{\mathcal{P}}(\mathbb{R}^n)$, i.e.*

$$\check{H}^k_{\mathcal{P}}(\mathbb{R}^n) = (\ker \mathrm{d} \cap \check{\Omega}^k_{\mathcal{P}}(\mathbb{R}^n))/(\mathrm{Im}\,\mathrm{d} \cap \check{\Omega}^k_{\mathcal{P}}(\mathbb{R}^n)).$$

The following theorem connects the above definition with the cohomology of $\mathcal{P}$ defined as Čech cohomology of the continuous hull of $\mathcal{P}$. Its proof will be given elsewhere [KP] although we give some explanation in section 4.

Theorem 2.3. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a Delone set of finite local complexity. The real valued Čech cohomology of the continuous hull of $\mathcal{P}$ is isomorphic to the $\mathcal{P}$-equivariant de Rham cohomology of $\mathbb{R}^n$.*

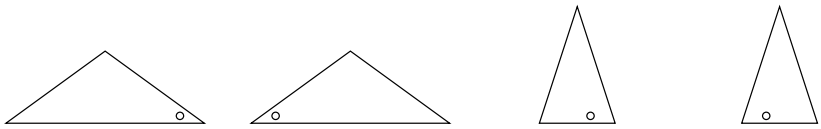

Figure 1.0 Figure 1.1 Figure 1.2 Figure 1.3

Figure 1. Tiles of a Penrose tiling (triangle version).

3. Example

Let us present some examples of $\mathcal{P}$-equivariant cohomology classes for $\mathcal{P} = \mathcal{T}_P$, a Penrose tiling, which may for our purposes be identified with the point set given by the centres of mass of its tiles. In the triangle version $\mathcal{T}_P$ has 40 different translational congruence classes of tiles which are the decorated triangles of figure 1 and their images under ten-fold rotation. Together with a reflection, e.g. the reflection in the x-axis, the ten-fold rotations form a symmetry group which is the dihedral group D_{10}. We call the translational congruence classes of tiles prototiles. We denote by $t_0, t_{10}, t_{20}, t_{30}$, respectively, the prototile depicted in figure 1.0, 1.1, 1.2, 1.3, respectively, and denote by t_{10k+l}, $0 \leqslant k \leqslant 3$, $0 \leqslant l \leqslant 9$, prototile t_{10k} rotated by $\frac{l\pi}{5}$ anticlockwise.

Since the tiling is two dimensional $\check{H}^k_{\mathcal{T}_P}(\mathbb{R}^2)$ is non-trivial only in $k = 0, 1, 2$. The case $k = 0$ is simple, in fact for any point pattern in any dimension $\check{H}^0_{\mathcal{P}}(\mathbb{R}^n) = \ker \mathrm{d} \cap \check{\Omega}^0_{\mathcal{P}}(\mathbb{R}^n) = \mathbb{R}$ holds.

3.1. $k = 2$

A closed strongly $\mathcal{T}_P$-equivariant 2-form is given by $f\,\mathrm{d}x_1\,\mathrm{d}x_2$ where f is a smooth strongly $\mathcal{T}_P$-equivariant function. Such functions can be constructed as follows: take a prototile t_i and let x_i be its centre of mass. Let δ_{t_i} be the Dirac comb associated with t_i by which we mean the sum of δ-functions at points $y \in \mathbb{R}^n$ such that $t_i - x_i$ occurs at $\mathcal{T}_P - y$. In other words δ_{t_i} is a Dirac comb supported on the centre of masses of all tiles which are translationally congruent to prototile t_i. Furthermore, let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function of compact support. Then the convolution product $f = \delta_{t_i} * \rho$ is a smooth strongly $\mathcal{T}_P$-equivariant function. In fact, any smooth strongly $\mathcal{T}_P$-equivariant function can be approximated in such a way if one allows for Dirac combs which are supported on arbitrary point sets which are locally derivable from $\mathcal{T}_P$. The difficult part is to determine when two 2-forms differ by a strongly $\mathcal{T}_P$-equivariant exact form. It follows from the Poincaré lemma for compactly supported de Rham cohomology that different choices of ρ yield the same cohomology class as long as the average height $\overline{\rho} := \int_{\mathbb{R}^2} \rho\,\mathrm{d}x_1\,\mathrm{d}x_2$ is kept fixed. We will therefore define $\alpha_i(\overline{\rho}) = \delta_{t_i} * \rho\,\mathrm{d}x_1\,\mathrm{d}x_2$ remaining ambiguous about the precise form of ρ.

To go further we have to combine theorem 2.3 with known results. Recall that the Penrose tiling is a substitution tiling and that its substitution can be used to compute its second cohomology as a quotient of the cohomology of the AF-groupoid $\mathcal{R}_\Sigma$ defined by the substitution in [Kel97][1]. More precisely, the so-called substitution matrix defines an endomorphism $\sigma : \mathbb{Z}^{40} \to \mathbb{Z}^{40}$ and one finds the integer second cohomology of the Penrose tiling as a quotient $\mathbb{Z}^{40}/\Gamma$. The real second cohomology of the Penrose tiling, which we

[1] The highest non-vanishing cohomology group is called the group of coinvariants in [Kel97].

denote here by $H^2(\mathcal{T}_P, \mathbb{R})$, can therefore be identified with a complement of the real span $\mathbb{R}\Gamma$ of the sub-group $\Gamma \subset \mathbb{Z}^{40}$ in $\mathbb{R}^{40}$. Such a complement has been computed [Kel97]

$$H^2(\mathcal{T}_P, \mathbb{R}) \cong \mathbb{R}^8 = E\big(\tau_+^2\big) \oplus E(\tau_-^2) \oplus E(-\tau_+) \oplus E(-\tau_-)$$

where

$$\tau_\pm = \xi_\pm + \xi_\pm^{-1} = \frac{1 \pm \sqrt{5}}{2} \qquad \xi_+ = \mathrm{e}^{\frac{\pi \mathrm{i}}{5}} \quad \xi_- = \mathrm{e}^{\frac{3\pi \mathrm{i}}{5}}$$

and $E(s)$ denotes the eigenspace of σ (extended to a linear map on $\mathbb{R}^{40}$) corresponding to the eigenvalue s.

We can use this to present specific strongly $\mathcal{T}_P$-equivariant 2-forms which form a linear base for $\check{H}_{\mathcal{T}_P}(\mathbb{R}^2)$. In order to do so we first note that we can identify the standard base $\{e_i\}_i$ of $\mathbb{Z}^{40}$ with indicator functions on certain cylinder sets of the discrete hull. These cylinder sets are in one-to-one correspondence with the prototiles and theorem 2.3 allows us to identify the class of e_i in $\mathbb{R}^{40}/\mathbb{R}\Gamma$ with the cohomology class of the 2-form $\alpha_i(1)$. All we have to do therefore is to determine a base for the above complement and this is best done by a symmetry analysis similar to that given in [ORS02].

In terms of the ten-fold rotation matrix ω the substitution matrix σ is given by

$$\sigma = \begin{pmatrix} \omega^4 & \omega^0 & 0 & \omega^6 \\ \omega^0 & \omega^6 & \omega^4 & 0 \\ \omega^3 & 0 & \omega^7 & 0 \\ 0 & \omega^7 & 0 & \omega^3 \end{pmatrix}.$$

The eigenspaces $E\big(\tau_\pm^2\big)$ are one dimensional and spanned by

$$\big(\tau_\pm^2 \mathbf{1}, \tau_\pm^2 \mathbf{1}, \mathbf{1}, \mathbf{1}\big)^T \tag{3.1}$$

where $\mathbf{1} = (1, \ldots, 1)$ is an eigenvector of ω to eigenvalue 1. Both eigenvectors are invariant under the D_{10} symmetry. The spaces $E(-\tau_\pm)$ are three dimensional. Each decomposes into a one-dimensional and a two-dimensional real irreducible representation of D_{10}. The one-dimensional irreducible real D_{10} module is spanned by

$$((1 - \tau_\pm)\mathbf{a}, (\tau_\pm - 1)\mathbf{a}, -\mathbf{a}, \mathbf{a})^T \tag{3.2}$$

where $\mathbf{a} = (1, -1, \ldots, 1)$ is an eigenvector of ω to eigenvalue -1. Using the matrix S which implements the reflection in the x-axis (see [Kel97]) one sees that the above two vectors are both invariant under it. The remaining two-dimensional parts of $E(-\tau_\pm)$ are easiest described if we complexify these spaces to observe that each splits into two one-dimensional complex irreducible D_{10} modules, one spanned by

$$b_\pm = \big(\tau_\pm \xi_\pm^2 \mathbf{z}_\pm, \tau_\pm \xi_\pm^{-1} \mathbf{z}_\pm, \xi_\pm \mathbf{z}_\pm, \mathbf{z}_\pm\big)^T \tag{3.3}$$

and the other by the complex conjugate $\bar{b}_\pm$. Here $\mathbf{z}_\pm = \big(1, \xi_\pm, \xi_\pm^2, \ldots, \xi_\pm^9\big)$, an eigenvector of ω to eigenvalue $\xi_\pm^{-1}$ and the reflection S acts as $Sb_\pm = -\xi_\pm \bar{b}_\pm$. We illustrate this result in figure 2 by marking the coefficients of $b_\pm$ and $\bar{b}_\pm$ into a figure of all prototiles using their identification with the base of $\mathbb{Z}^{40} \subset \mathbb{C}^{40}$.

From this analysis one can derive a set of generators for $\check{H}_{\mathcal{T}_P}(\mathbb{R}^2)$. From linear combinations of (3.1) (resp. (3.2)) one obtains B_1, B_2 (resp. B_3, B_4) where

$$B_1 = \sum_{i=0}^{19} \alpha_i(1) \qquad B_2 = \sum_{i=20}^{39} \alpha_i(1) \qquad B_3 = \sum_{i=0}^{19} (-1)^i \alpha_i(1) \qquad B_4 = \sum_{i=20}^{39} (-1)^i \alpha_i(1).$$

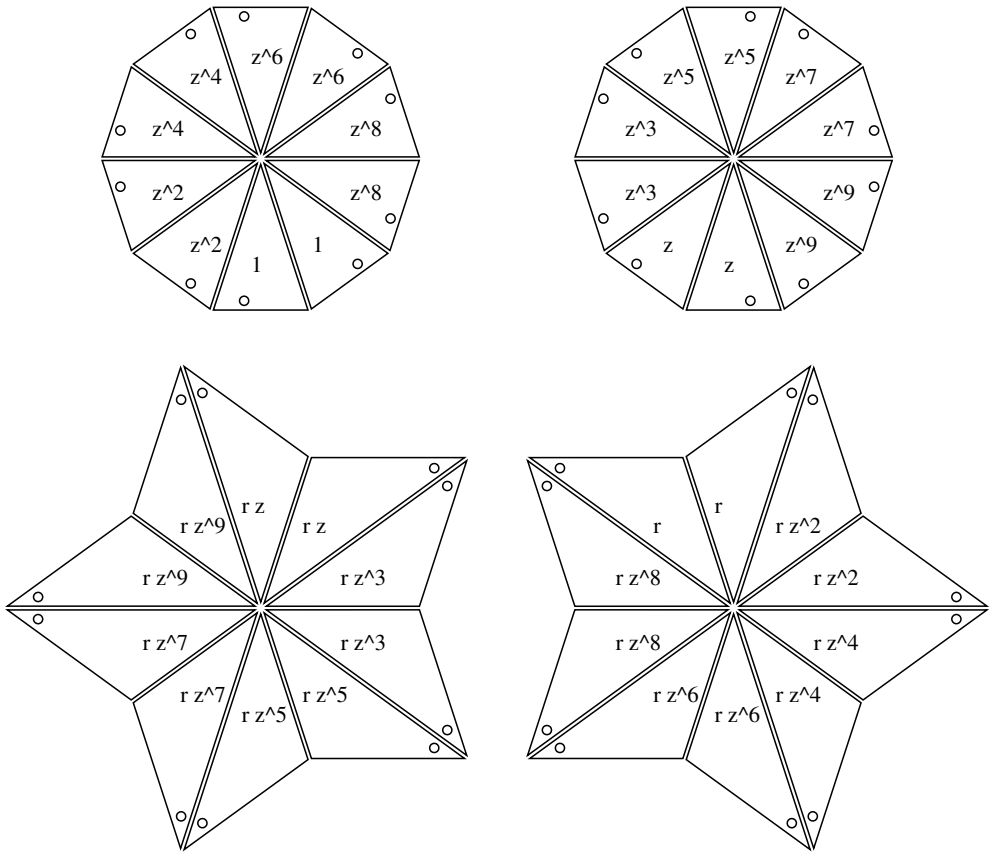

Figure 2. The coefficients of the four complex eigenvectors of (3.3) and its complex conjugate. The four possibilities are $(r, z) \in \{(\xi_+, \tau_+), (\xi_-, \tau_-), (\xi_+^{-1}, \tau_+), (\xi_-^{-1}, \tau_-)\}$.

The remaining four generators are more complicated expressions but straightforward to derive from (3.3). In particular, to obtain two two-dimensional real irreducible representations of D_{10} one has to take the real and the imaginary part of (3.3).

We mention that this method can also be used to determine a sub-group of finite index of the integer valued second cohomology. This amounts to taking linear combinations of the above vectors so that the result lies in the intersection of the above complement with $\mathbb{Z}^{40}$. In that case one obtains integer irreducible representations of D_{10} which are two, two and four dimensional, cf [ORS02].

3.2. $k = 1$

A closed strongly $\mathcal{T}_P$-equivariant 1-form is given by $f_1\,\mathrm{d}x_1 + f_2\,\mathrm{d}x_2$ where f_i are smooth strongly $\mathcal{T}_P$-equivariant functions satisfying $\frac{\partial f_2}{\partial x_1} = \frac{\partial f_1}{\partial x_2}$. A similar construction as the one presented for $k = 2$, namely one where one starts with a compactly supported closed 1-form and places it at all points of a uniformly discrete set which is locally derived from $\mathcal{T}_P$, produces only exact forms. This follows again from the Poincaré lemma.

To construct closed but non-exact 1-forms we use the well-known Ammann lines on the Penrose tilings. Ammann lines are five families of parallel lines which are locally derivable

from a Penrose tiling [GS87]. The distance between two consecutive parallel Ammann lines of one family is 1 or τ_+ (in suitable units) and the sequence of distances forms a Fibonacci sequence (called musical sequence in [GS87]). Fix one family $\mathcal{F}_1$ of Ammann lines and let (x, y) be orthonormal coordinates of $\mathbb{R}^2$ such that y is parallel and x normal to the Ammann lines of $\mathcal{F}_1$. In particular, a line N normal to the Ammann lines of $\mathcal{F}_1$ is parametrized by x and cut into a Fibonacci sequence by its intersection points with these Ammann lines. Let $\mathcal{P}_N$ be the set of intersection points in $N \cong \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a smooth strongly $\mathcal{P}_N$-equivariant function. Then $(x, y) \mapsto f(x)\,\mathrm{d}x$ is a closed strongly $\mathcal{T}_P$-equivariant 1-form over $\mathbb{R}^2$. In the same way as we reasoned for the case $k = 2$ above one can construct two strongly $\mathcal{P}_N$-equivariant functions each one associated with a prototile in N, namely by placing a Dirac comb on the centres of mass of either the unit intervals or the intervals of length τ_+. In the cohomology $\check{H}^1_{\mathcal{P}_N}(\mathbb{R})$ corresponding to the one-dimensional subsystem these two functions lead to two independent generators. Considering all families of Ammann lines this gives ten strongly $\mathcal{T}_P$-equivariant 1-forms. However, one may show that in $\check{H}^1_{\mathcal{T}_P}(\mathbb{R}^2)$ only four of them are independent. In fact, the appearance of these 1-forms is related to the higher dimensional lattice used to construct the Penrose tiling via the cut and projection method and somewhat independent of the details of the tiling which originate from the choice of windows [KP].

From [AP98, ORS02] we know (in combination with theorem 2.3) that $\check{H}^1_{\mathcal{T}_P}(\mathbb{R}^2) \cong \mathbb{R}^5$ and the latter splits into a sum of a four-dimensional irreducible C_{10} module with a one-dimensional irreducible C_{10} module. The four-dimensional irreducible C_{10} module is given by the above construction using the Ammann lines. The remaining one is related to a locally derivable orientation of the edges of the Penrose tiling [Gäh], it will not be discussed here.

4. $\mathcal{P}$-equivariant functions and the continuous hull of $\mathcal{P}$

For the definition of $\mathcal{P}$-equivariant cohomology we have considered smooth real valued strongly $\mathcal{P}$-equivariant functions over $\mathbb{R}^n$. To explain the relation with earlier definitions of cohomology groups for $\mathcal{P}$, in particular that involving the continuous hull, we now consider continuous complex valued functions.

Definition 4.1. *The algebra of complex continuous $\mathcal{P}$-equivariant functions $C_{\mathcal{P}}(\mathbb{R}^n)$ is the closure in the supremum norm of the space of complex continuous strongly $\mathcal{P}$-equivariant functions over $\mathbb{R}^n$ with pointwise multiplication.*

$C_{\mathcal{P}}(\mathbb{R}^n)$ is a commutative C^*-algebra with unit and by the Gelfand–Naimark theorem there exists a compact space $X_{\mathcal{P}}$ such that $C_{\mathcal{P}}(\mathbb{R}^n)$ is isomorphic to $C(X_{\mathcal{P}})$. The simplest example is certainly $\mathcal{P} = \emptyset$ in which $C_\emptyset(\mathbb{R}^n) \cong \mathbb{C}$ and so $X_\emptyset$ is a single point. Another simple example but with $\mathcal{P}$ being a Delone set is a regular lattice $\Gamma \subset \mathbb{R}^n$, then $C_\Gamma(\mathbb{R}^n) \cong C(\mathbb{T}^n)$, i.e. $X_\Gamma = \mathbb{T}^n = \mathbb{R}^n/\Gamma$, the n-torus. In general, $X_{\mathcal{P}}$ is a lot more complicated.

The set of subsets of $\mathbb{R}^n$ which have finite local complexity carries a metric

$$D(S, S') = \inf\left\{\frac{1}{r+1} | \exists x, x' \in \mathbb{R}^n, |x|, |x'| \leqslant \frac{1}{r} : B_r \cap (S - x) = B_r \cap (S' - x')\right\}.$$

The completion of the set of translates of $\mathcal{P}$ w.r.t. this metric,

$$\mathcal{M}_{\mathcal{P}} := \overline{\{\mathcal{P} - x | x \in \mathbb{R}^n\}}^D$$

is the continuous hull of $\mathcal{P}$, see e.g. [AP98]. By construction $D(S, S - x)$ is of order $|x|$ if the norm $|x|$ of x is small. In particular, $\mathcal{M}_{\mathcal{P}}$ is connected. If $\mathcal{P}$ is a Delone set then all elements of $\mathcal{M}_{\mathcal{P}}$ can be interpreted as Delone sets. (If $\mathcal{P}$ is not relatively dense then $\mathcal{M}_{\mathcal{P}}$ contains the empty set as element.)

Lemma 4.2. *Let $\mathcal{P}$ be a Delone subset of $\mathbb{R}^n$ which has finite local complexity. Then $X_{\mathcal{P}} \cong \mathcal{M}_{\mathcal{P}}$. More precisely, $\sigma : C(\mathcal{M}_{\mathcal{P}}) \to C_{\mathcal{P}}(\mathbb{R}^n)$,*

$$\sigma(f)(x) = f(\mathcal{P} - x)$$

is an algebra isomorphism.

Proof. The space $\mathcal{M}_{\mathcal{P}}$ is in general not a manifold but a foliated space in the sense of [MS88]. Its leaves are the sets of translates of points. They are locally homeomorphic to $\mathbb{R}^n$. Therefore, $\sigma(f)$ is continuous.

In the transverse direction $\mathcal{M}_{\mathcal{P}}$ is totally disconnected. In the (completely) periodic cases the transversals consist of finitely many points. For (completely) aperiodic Delone sets the transversals are Cantor sets. If we consider the canonical transversal $\Omega_{\mathcal{P}} = \{\omega \in \mathcal{M}_{\mathcal{P}} | 0 \in \mathcal{P}\}$ we can describe its topology as that being generated by the clopen sets $U_{P,p}$ where $p \in P \subset \mathcal{P}$, P a finite subset and $U_{P,p} = \{\omega \in \Omega_{\mathcal{P}} | P - p \subset \omega\}$. The topology of $\mathcal{M}_{\mathcal{P}}$ is then generated by $U_{P,p,\epsilon,y} = \{\omega \in \mathcal{M}_{\mathcal{P}} | \exists x \in B_\epsilon(y) : P - p - x \in \omega\}$ (cf [FHK02]). Moreover, we may restrict $\epsilon > 0$ to values smaller than the minimal distance r_0 of two points in $\mathcal{P}$. For such ϵ, $U_{P,p,\epsilon,y}$ is homeomorphic to $U_{P,p} \times B_\epsilon(y)$. It follows that $C(\mathcal{M}_{\mathcal{P}})$ is generated by continuous functions which are supported on sets $U_{P,p,\epsilon,y}$, $\epsilon < r_0$. Such functions are of the form $f_{P,p,\rho} = \delta_{P,p} * \rho$ where $\delta_{P,p}$ is a Dirac comb placed on the set $\{x \in \mathbb{R}^n | P - p \in \mathcal{P} - x\}$ and ρ is continuous and has support inside $B_{r_0}(y)$. But $\sigma(f_{P,p,\rho})$ is strongly $\mathcal{P}$-equivariant, the value for r in definition 2.1 being at most $r_0 + |y|$. This shows that σ maps $C(\mathcal{M}_{\mathcal{P}})$ into $C_{\mathcal{P}}(\mathbb{R}^n)$. From denseness of the leaf through $\mathcal{P}$ it follows that σ is injective.

If $g : \mathbb{R}^n \to \mathbb{C}$ is a strongly $\mathcal{P}$-equivariant continuous function define $\tilde{g} : \mathcal{M}_{\mathcal{P}} \to \mathbb{C}$ by $\tilde{g}(\omega) = g(\mathcal{P} - x)$ where x is such that $B_r \cap (\mathcal{P} - x) = B_r \cap \omega$ (the value for r from definition 2.1). Then $\tilde{g}$ is continuous and $\sigma(\tilde{g}) = g$. □

Let us indicate how the last lemma helps us to prove theorem 2.3, i.e. that the Čech cohomology of $\mathcal{M}_{\mathcal{P}}$ with coefficients in $\mathbb{R}$ is isomorphic to $\check{H}_{\mathcal{P}}(\mathbb{R}^n)$. Although $\mathcal{M}_{\mathcal{P}}$ may not be a manifold its leaves are and one can define functions which are smooth in the direction tangential to the leaves. This leads to the definition of tangential differential forms and consequently of tangential cohomology [MS88]. The inverse of the map σ above identifies the complex of strongly $\mathcal{P}$-equivariant differential forms with a sub-complex of the complex of tangential differential forms. $\mathcal{P}$-equivariant cohomology equates therefore with the cohomology of a sub-complex of tangential forms. On the other hand, there exists an analogue of the Čech–de Rham complex [BT82] providing us with a homomorphism from the Čech cohomology of $\mathcal{M}_{\mathcal{P}}$ (with real coefficients) to tangential cohomology of $\mathcal{M}_{\mathcal{P}}$. Its image is precisely $\mathcal{P}$-equivariant cohomology.

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References

[AP98] Anderson J E and Putnam I F 1998 Topological invariants for substitution tilings and their associated C^*-algebras *Ergod. Th. Dynam. Sys.* **18** 509–37

[BSJ91] Baake M, Schlottmann M and Jarvis P D 1991 Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability *J. Phys. A: Math. Gen.* **24** 4637–54

[BT82] Bott R and Tu L W 1982 *Differential forms in Algebraic Topology* (New York: Springer)

[Con90] Connes A 1990 *Géométrie Non Commutative* (Paris: InterEditions)

[Con94] Connes A 1994 *Non Commutative Geometry* (New York: Academic)

[FH99] Forrest A H and Hunton J 1999 The cohomology and K-theory of commuting homeomorphisms of the Cantor set *Ergod. Th. Dynam. Sys.* **19** 611–25

[FHK02] Forrest A H, Hunton J and Kellendonk J 2002 Cohomology of canonical projection tilings *Commun. Math. Phys.* **226** 289–322

[Gäh] Gähler F 2002 private communication

[GS87] Grünbaum B and Shephard G C 1987 *Tilings and Patterns* (New York: Freeman)

[Kel95] Kellendonk J 1995 Non commutative geometry of tilings and gap labelling *Rev. Math. Phys.* **7** 1133–80

[Kel97] Kellendonk J 1997 The local structure of tilings and their integer group of coinvariants *Commun. Math. Phys.* **187** 115–57

[KP] Kellendonk J and Putnam I F in preparation

[MS88] Moore C C and Schochet C 1988 *Global Analysis on Foliated Spaces (Mathematical Sciences Research Institute Publications vol 9)* (New York: Springer)

[ORS02] Ormes N, Radin C and Sadun L 2002 A homeomorphism invariant for substitution tiling spaces *Geom. Dedicata* **90** 153–82