Analysis and simulation of nonlinear and nonlocal transport equations

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Abstract This article is devoted to the analysis of some nonlinear conservative transport equations, includig the so-called aggregation equation with pointy potential, and numerical method devoted to its numerical simulation. Such a model describes the collective motion of individuals submitted to an attractive potential and can be written as a continuity transport equation with a velocity field computed through a self-consistent interaction potential. In the strongly attractive setting, L^p solutions may blow up in finite time, then a theory of existence of weak measure solutions has been defined. In this approach, we show the existence of Filippov characteristics allowing to define solutions of the aggregation initial value problem as a pushforward measure. Then numerical analysis of an upwind type scheme is proposed allowing to recover the dynamics of aggregates beyond the blowup time.

1 Introduction

This paper is devoted to existence and uniqueness, and numerical approximation of measure valued solutions to the following nonlinear and nonlocal transport equation in d space dimension,

$$\partial_t \rho + \operatorname{div}\big((V * \rho)\rho\big) = 0, \qquad t > 0, \quad x \in \mathbb{R}^d, \tag{1}$$

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complemented with the initial condition $\rho(0,x) = \rho^{ini}$. This equation governs the dynamics of a density of individuals, ρ at time $t > 0$, position $x \in \mathbb{R}^d$. The interaction between individuals is modelled by a given function $V : \mathbb{R}^+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$. One motivation is the so-called aggregation equation where $V = \nabla_x W$ derives from an interaction potential W whose gradient $\nabla_x W(x-y)$ measures the relative force exerted by a unit mass localized at a point y onto a unit mass located at a point x.

The aggregation equation appears in many applications in physics and population dynamics. One may cite for instance applications in granular media [3], crowd motion [13], collective migration of cells by swarming [29, 30, 33], bacterial chemotaxis [18, 19, 25]. In many of these examples, the potential W is usually mildly singular, i.e. W has a weak singularity at the origin. Due to this weak regularity, finite time blowup of (weak L^p) solutions has been observed for such systems and has gained the attention of several authors (see e.g. [28, 7, 5, 11]). Finite time concentration is sometimes considered as a very simple mathematical way to mimick aggregation of individuals, as opposed to diffusion.

Since finite time blowup of regular solutions occurs, a natural framework to study the existence of global in time solutions is to work in the space of probability measures. Two strategies have been proposed in the litterature. In [11], the aggregation equation is seen as a gradient flow minimizing the interaction energy in a Wasserstein space. In [25, 26, 12], this system is considered as a conservative transport equation with velocity field $\nabla_x W * \rho$. Then a flow Z can be constructed allowing to define the solution as a pushforward measure by the flow: $\rho = Z_\# \rho^{ini}$. See also [4] for a similar definition. To be able to define such a flow, some assumptions on the potential are needed that allows for mild singularity of the potential. The usual assumption consists in considering *pointy* potentials with singularity at the origin, such as the Morse potential $W(x) = e^{-|x|}$, or $W(x) = -|x|$. In this paper, we extend this assumption to a more general class of potentials.

Here we list the assumptions that will be used in the paper.

- We assume that there exists a function λ such that
$$\langle V(t,x) - V(t,y), x-y \rangle \leq \lambda(t)|x-y|^2, \quad \lambda \in L^1_{loc}(\mathbb{R}^+), \tag{2}$$
where $\langle \cdot,\cdot \rangle$ denotes the Euclidean inner product. We notice for the case of the aggregation equation, i.e. when $V(t,x) = \nabla_x W(x)$, this assumption is satisfied provided the interaction potential $W : \mathbb{R}^d \to \mathbb{R}$ is λ-concave, i.e. $x \mapsto W(x) - \frac{\lambda}{2}|x|^2$ is concave for some constant $\lambda \geq 0$.
- For the sake of simplicity of the presentation, we only consider bounded velocity fields, then we assume that there exists a nonnegative constant v_∞ such that for a.e. $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$,
$$|V(t,x)| \leq v_\infty. \tag{3}$$
- An interesting issue is raised when V is discontinuous, since, as already mentioned, it may imply blowup in finite time of weak L^p ($p > 1$) solutions. We may assume that there exists a finite set of discontinuity points. More precisely, there exists a finite set of points in $\mathbb{R}^d$, denoted $\xi_1, \ldots, \xi_L$, such that

$$V(t,x) = \sum_{\ell=1}^{L} \big(V_\ell(t,x-\xi_\ell) + V_\ell(t,x+\xi_\ell)\big) + V_r(t,x), \tag{4}$$

where for all $\ell = 1,\dots,L$, we assume

$$\begin{gathered} V_\ell \in L^\infty_{loc}(\mathbb{R}^+, C(\mathbb{R}^d \setminus \{0\}), \quad V_\ell(t,x) = -V_\ell(t,-x) \\ \text{and } V_r \in L^\infty_{loc}(\mathbb{R}^+, W^{1,\infty}(\mathbb{R}^d)). \end{gathered} \tag{5}$$

We notice also that since V_r is Lipschitz-continuous in space, then it satisfies (2) with a constant $\lambda(t) = \|DV_r(t)\|_\infty \in L^\infty_{loc}(\mathbb{R}^+)$. Thus, $V_s := V - V_r$ verifies also (2), and $V_s(t,x) = \sum_{\ell=1}^{L} \big(V_\ell(t,x-\xi_\ell) + V_\ell(t,x+\xi_\ell)\big)$ is odd.

Thus, to summarize, with respect to the preceding established results in [11, 12], we avoid three assumptions: we do not assume that V has only 1 singularity, we do not assume that V is odd, and we do not assume that V is the gradient of a potential. However, we still restrict to bounded velocity fields (see [4] for an approach in the radially symmetric case with non bounded velocity fields).

Although extremely accurate numerical schemes have been developed to study the blowup profile for smooth solutions, see e.g. [23, 24] for the aggregation equation, very few numerical schemes have been proposed to simulate the behaviour of solutions beyond blowup time. The so-called sticky particle method was shown to be convergent in [11] and used to obtain qualitative properties of the solutions such as the finite time total collapse. However, this method is not that practical to deal with the behavior of solutions after blowup in dimensions larger than one. In one dimension, this task has been performed in [25]. Recently, in higher dimensions, particle methods have been proposed and studied in [14, 9] but only the convergence for smooth solutions, before the blowup time, has been proved.

Finite volume schemes have been also developped, and the present paper stands in this frame. Note that the difficulty in this problem is twofold: first, the velocity is not smooth (and only one-sided Lipschitz-continuous), and second, it is a nonlinear problem.

In the linear case and when the given velocity field is only one-sided Lipschitz-continuous:

- in [21], the convergence of dissipative schemes is proven in dimension 1 (weak convergence in the sense of measures),
- in [16], the convergence of upwind-type or, more generally, some dissipative schemes, at order $1/2$ in Wasserstein distance, has been obtained (in any dimension).

In the fully nonlinear context, in [27], a finite volume scheme is proposed allowing to simulate the behaviour of the solution to the one dimensional aggregation equation (1) after blowup, and the authors prove its convergence. A finite volume method for a large class of PDEs including in particular (1) has been also proposed in [10] but no convergence result has been given. Finally, a finite volume scheme of Rusanov (or Lax-Friedrichs) type for general measures as initial data has been proposed and studied in [12]. Numerical simulations of solutions in dimension greater than one have been obtained, allowing to observe the behaviour after blowup. Its weak convergence in the sense of measure is proven. We propose in this paper to extend this result to the upwind scheme for a more general class of equations, that is system (1) with an interaction function V satisfying assumptions (2)–(5) only. This scheme is based on an idea developped in [25] and used later in [27, 12] which consists in using a careful discretization of the macroscopic velocity such that its product with the measure solution ρ is well-defined.

The outline of this paper is the following. In the next section, we recall briefly the theory of existence of solutions to the conservative transport equation with discontinous velocity field. In section 3, we establish the existence and uniqueness result. Section 4 is devoted to the numerical discretization. We show in particular the convergence of the numerical scheme towards the measure value solution. Finally, we conclude this paper with some numerical illustration in section 5.

2 Transport equation with discontinuous velocity field

2.1 Notations

All along the paper, we will make use of the following notations. We denote $C_0(\mathbb{R}^d)$ the space of continuous functions in $\mathbb{R}^d$ that tend to 0 at infinity. We denote $\mathscr{M}_b(\mathbb{R}^d)$ the space of Borel measures whose total variation is finite. For $\rho \in \mathscr{M}_b(\mathbb{R}^d)$, we denote by $|\rho|(\mathbb{R}^d)$ its total variation. ¿From now on, $\mathscr{M}_b(\mathbb{R}^d)$ is always endowed with the weak topology $\sigma(\mathscr{M}_b(\mathbb{R}^d), C_0(\mathbb{R}^d))$. For $T > 0$, we note $\mathscr{S}_{\mathscr{M}} := C([0,T]; \mathscr{M}_b(\mathbb{R}^d) - \sigma(\mathscr{M}_b(\mathbb{R}^d), C_0(\mathbb{R}^d)))$. For ρ a measure in $\mathscr{M}_b(\mathbb{R}^d)$ and Z a measurable map, we denote by $Z_\#\rho$ the pushforward measure of ρ by Z; it satisfies, for any continuous function ϕ,

$$\int_{\mathbb{R}^d} \phi(x)\, Z_\#\rho(dx) = \int_{\mathbb{R}^d} \phi(Z(x))\, \rho(dx).$$

We denote by $\mathscr{P}(\mathbb{R}^d)$ the subset of $\mathscr{M}_b(\mathbb{R}^d)$ of probability measures. We define the space of probability measures with finite second order moment by

$$\mathscr{P}_2(\mathbb{R}^d) = \left\{ \mu \in \mathscr{P}(\mathbb{R}^d),\ \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \right\}.$$

Here and in the following, $|\cdot|^2$ stands for the Euclidean norm that derives from the Euclidean inner product $\langle \cdot, \cdot \rangle$. This space is endowed with the Wasserstein distance d_W defined by (see e.g. [1, 34, 35])

$$d_W(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |y-x|^2\, \gamma(dx,dy) \right\}^{1/2} \qquad (6)$$

where $\Gamma(\mu,\nu)$ is the set of measures on $\mathbb{R}^d\times\mathbb{R}^d$ with marginals μ and ν. ¿From a minimization argument, we know that in the definition of d_W the infimum is actually a minimum (see [34, 32]). A map that realizes the minimum in the definition (6) of d_W is called an *optimal plan*, the set of which is denoted by $\Gamma_0(\mu,\nu)$. Then for all $\gamma_0\in\Gamma_0(\mu,\nu)$, we have

$$d_W(\mu,\nu)^2=\int_{\mathbb{R}^d\times\mathbb{R}^d}|y-x|^2\,\gamma_0(dx,dy).$$

2.2 Weak measure solutions for conservative transport equation

We recall in this section some useful results on weak measure solutions to the conservative linear transport equation with given velocity field b,

$$\partial_t u+\mathrm{div}(bu)=0;\qquad u(t=0)=u^0. \tag{7}$$

We start by the following definition of characteristics [20]:

Definition 1. Let us assume that $b=b(t,x)\in\mathbb{R}^d$ is a vector field defined on $[0,T]\times\mathbb{R}^d$ with $T>0$. A Filippov characteristic $X(t;s,x)$ that stems from $x\in\mathbb{R}^d$ at time s is a continuous function $X(\cdot;s,x)\in C([0,T],\mathbb{R}^d)$ such that $\frac{\partial}{\partial t}X(t;s,x)$ exists for a.e. $t\in[0,T]$ and satisfies

$$\frac{\partial}{\partial t}X(t;s,x)\in\big\{\mathrm{C}onvess(b)(t,\cdot)\big\}(X(t;s,x)),\quad \text{a.e. } t\in[0,T];\qquad X(s;s,x)=x.$$

¿From now on, we will use the notation $X(t,x)=X(t;0,x)$.

In this definition $Convess(E)$ denotes the essential convex hull of a set E. We recall its definition for the sake of completeness, see [20, 2] for more details. We denote by $Conv(E)$ the classical convex hull of E, i.e., the smallest closed convex set containing E. Given the vector field $b(t,\cdot):\mathbb{R}^d\longrightarrow\mathbb{R}^d$, the essential convex hull at point x is defined as

$$\{\mathrm{C}onvess(b)(t,\cdot)\}(x)=\bigcap_{r>0}\bigcap_{N\in\mathcal{N}_0}Conv\left[b\left(t,B(x,r)\setminus N\right)\right],$$

where $\mathcal{N}_0$ is the set of zero Lebesgue measure sets.

At this stage there is no smoothness assumption on b. Existence and uniqueness of a flow is classically ensured if b is smooth. A possible way to go beyond this, and use possibly discontinuous velocity fields, is to introduce the so-called one-sided Lipschitz continuity, see (8) below. The following existence and uniqueness result of Filippov characteristics ensures that the solution does not depend on the representative of b that is chosen.

Theorem 1 ([20]). *Let $T>0$. Let us assume that the vector field $b\in L^1_{loc}(\mathbb{R};L^\infty(\mathbb{R}^d))$ satisfies the OSL condition, that is for a.e. x and y in $\mathbb{R}^d$, a.e. $t\in[0,T]$,*

$$\langle b(t,x)-b(t,y),x-y\rangle \le \alpha(t)\|x-y\|^2, \quad \textit{for } \alpha \in L^1(0,T). \tag{8}$$

Then there exists a unique Filippov characteristic X associated to this vector field.

An important consequence of this result is the existence and uniqueness of weak measure solutions for the conservative linear transport equation. This result has been proved by Poupaud and Rascle [31, Theorem 3.3]:

Theorem 2 ([31]). *Let $T>0$. Let $b\in L^1([0,T],L^\infty(\mathbb{R}^d))$ be a vector field satisfying the OSL condition (8). Then for any $u_0\in \mathscr{M}_b(\mathbb{R}^d)$, there exists a unique measure solution u in $\mathscr{S}_{\mathscr{M}}$ to the conservative transport equation (7) such that $u(t)=X(t)_\# u_0$, where X is the unique Filippov characteristic, i.e. for any $\phi\in C_0(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d}\phi(x)u(t,dx)=\int_{\mathbb{R}^d}\phi(X(t,x))u_0(dx), \qquad \textit{for } t\in[0,T].$$

In one dimension, such solutions are equivalent to duality solutions defined by Bouchut and James in [8]. A pioneering numerical investigation of this equation in one dimension is provided in [21]. A numerical investigation of measure valued solutions defined in Theorem 2 with a convergence order proof is proposed in [16].

Finally, we recall the following stability result for the Filippov characteristics which has been established by Bianchini and Gloyer [6, Theorem 1.2]

Theorem 3 ([6]). *Let $T>0$. Assume that the sequence of vector fields b_n converges weakly to b in $L^1([0,T],L^1_{loc}(\mathbb{R}^d))$. Then the Filippov flow X_n generated by b_n converges locally in $C([0,T]\times\mathbb{R}^d)$ to the Filippov flow X generated by b.*

3 Filippov characteristic flow for the aggregation equation

This section is devoted to the existence of a Filippov flow for the aggregation equation (1) as it has been stated in [12] in a lightly less large context.

Under assumption (4), we define for $\rho\in C([0,T],\mathscr{P}_2(\mathbb{R}^d))$ the velocity field $\widehat{a}_\rho$ by

$$\begin{aligned}\widehat{a}_\rho(t,x)=\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\big(\widehat{V}_\ell(t,x-y-\xi_\ell)+\widehat{V}_\ell(t,x-y+\xi_\ell)\big)\rho(t,dy)\\ +\int_{\mathbb{R}^d}V_r(t,x-y)\rho(t,dy)\,,\end{aligned} \tag{9}$$

where $\widehat{V}_\ell$ is defined for $\ell=1,\dots,L$ by

$$\widehat{V}_\ell(t,x)=\begin{cases}V_\ell(t,x), & \text{when } x\neq 0;\\ 0, & \text{otherwise.}\end{cases}$$

¿From now on, we will use the notation

$$\widehat{V}(t,x) := \sum_{\ell=1}^{L} \left(\widehat{V}_\ell(t,x-\xi_\ell) + \widehat{V}_\ell(t,x+\xi_\ell)\right) + V_r(t,x). \tag{10}$$

The following theorem states existence and uniqueness of a solution defined by the Filippov characteristics. Its proof in the case where $V(t,x) = \nabla_x W(x)$ with $W \in C^1(\mathbb{R}^d \setminus \{0\})$ and λ-convex has been obtained in [12, Theorem 2.5 and 2.9]. In the present article we extend it to the case at hand.

Theorem 4. *Let V satisfy assumptions* (2)–(5) *and let ρ^{ini} be given in $\mathscr{P}_2(\mathbb{R}^d)$. There exists a unique solution $\rho \in C([0,+\infty), \mathscr{P}_2(\mathbb{R}^d))$ that satisfies in the sense of distributions the aggregation equation*

$$\partial_t \rho + \mathrm{div}(\widehat{a}_\rho \rho) = 0, \qquad \rho(0,\cdot) = \rho^{ini},$$

where $\widehat{a}_\rho$ is defined by (9)*, and is the pushforward measure $\rho := Z_\# \rho^{ini}$ where Z is the unique Filippov characteristic flow associated to the velocity field $\widehat{a}_\rho$.*

3.1 Sketch of the proof of Theorem 4

The proof of the existence and uniqueness result in Theorem 4 follows the ideas developped in [12]. For the sake of completeness, we recall the main steps of the proof and detail below the main changes to extend it to the case at hand.

Step 1: definition of the macroscopic velocity.
A difficulty when we want to deal with measure valued solutions to transport equation is that the velocity field should be defined carefully to be able to give a sense to the product in the divergence term in (1) when ρ is a measure. Here we use the definition (9) for the velocity field. This definition is motivated by Lemma 2 which is stated and proved below. Indeed, from assumptions (4)-(5), we have $V(t,x) = \sum_{\ell=1}^{L}(V_\ell(t,x-\xi_\ell) + V_\ell(t,x+\xi_\ell)) + V_r(t,x)$, with $V_\ell \in C^1(\mathbb{R}^d \setminus \{0\})$ for $\ell = 1,\dots,L$. Then, Lemma 2 implies that if we regularize V by a sequence V_n, for instance by taking the convolution of each V_ℓ with mollifiers, and if for all $t \geq 0$, $\rho_n(t)$ is a sequence of probability measures in $\mathscr{P}_2(\mathbb{R}^d)$ such that $\rho_n \rightharpoonup \rho$ in the sense of measures, then $V_n \star \rho_n \rightharpoonup \widehat{a}_\rho \rho$ in the sense of measures (see Lemma 2 in Section 3.2).

Moreover, we have the following one-sided Lipschitz estimate:

Lemma 1. *Let $\rho \in L^\infty(0,T,\mathscr{M}_b(\mathbb{R}^d))$ be nonnegative. Then under assumptions* (2)–(5)*, the function $(t,x) \mapsto \widehat{a}_\rho(t,x)$ defined in* (9) *satisfies the one-sided Lipschitz (OSL) estimate*

$$\langle \widehat{a}_\rho(t,x) - \widehat{a}_\rho(t,y), x-y \rangle \leq \lambda(t)|\rho(t)|(\mathbb{R}^d)\|x-y\|^2. \tag{11}$$

Proof. This result is an easy consequence of assumption (2) on V. Indeed, by definition (9), we have

$$\widehat{a}_\rho(t,x) - \widehat{a}_\rho(t,y) = \int_{\mathbb{R}^d} \big(\widehat{V}(t,x-z) - \widehat{V}(t,y-z)\big)\rho(dz),$$

where $\widehat{V}(t,x) = \sum_{\ell=1}^L (\widehat{V}_\ell(t,x-\xi_\ell) + \widehat{V}_\ell(t,x+\xi_\ell)) + V_r(t,x)$ is defined in (10). The conclusion follows directly from assumption (2) and the nonnegativity of ρ.

Step 2: approximation with Dirac masses.
We use the idea of atomization consisting in approximating the solution by a finite sum of Dirac masses: let us consider that for some integer $N > 0$,

$$\rho^{ini,N} = \sum_{i=1}^N m_i \delta(x - x_i^0), \qquad x_i^0 \neq x_j^0 \text{ for } i \neq j, \quad \sum_{i=1}^N m_i = 1, \quad \sum_{i=1}^N m_i |x_i^0|^2 < +\infty.$$

Then we look for a solution of the aggregation equation given by

$$\rho^N(t,x) = \sum_{i=1}^N m_i \delta(x - x_i(t)).$$

By definition (9), $\widehat{a}_{\rho^N}(t,x) = \sum_{i=1}^N m_i \widehat{V}(t, x - x_i(t))$, with $\widehat{V}$ defined in (10). ¿From Lemma 1, $\widehat{a}_{\rho^N}$ satisfies the OSL condition. Applying Theorem 1, it allows to define uniquely a Filippov characteristic, denoted $\widehat{X}^N$, associated to the velocity field $\widehat{a}_{\rho^N}$. By construction, from Theorem 2, the Poupaud-Rascle pushforward measure $\rho_{PR} := \widehat{X}^N{}_\# \rho^{ini,N}$ is the unique measure valued solution to the conservative linear transport equation

$$\partial_t \rho_{PR}^N + \mathrm{div}(\widehat{a}_{\rho^N} \rho_{PR}^N) = 0, \qquad \rho_{PR}^N(t=0) = \rho^{ini,N}.$$

Moreover, by definition of the pushforward measure,

$$\begin{aligned} \widehat{a}_{\rho_{PR}^N} &= \int_{\mathbb{R}^d} \widehat{V}(t,x-y)\rho_{PR}^N(t,dy) = \int_{\mathbb{R}^d} \widehat{V}(t, x - \widehat{X}^N(t,y))\rho^{ini,N}(dy) \\ &= \sum_{i=1}^N m_i \widehat{V}(t, x - \widehat{X}^N(t, x_i^0)) = \widehat{a}_{\rho^N}(t,x). \end{aligned}$$

Thus $\rho_{PR}^N = \rho^N$. It gives the existence result for initial data given by a finite sum of Dirac masses.

Step 3: passing to the limit $N \to +\infty$.
Making use of stability results, we may pass to the limit $N \to +\infty$ in the above construction. This step is the same as in [12]; for the sake of completeness, we recall the ideas but omit details. We assume that $\rho^{ini,N} \rightharpoonup \rho^{ini}$ as $N \to +\infty$. Then, since the velocity field $\widehat{a}_{\rho^N}$ is uniformly bounded, thanks to (3), we may extract a subsequence that converges in L^∞ weak-$\star$. Using the stability result of Theorem 3, we deduce that $\widehat{X}^N \to \widehat{X}$. As a consequence, we get the weak convergence $\rho^N \rightharpoonup \rho := \widehat{X}_\# \rho^0$. Finally, we apply the stability result of Lemma 2 to conclude the proof of existence.

Step 4: uniqueness.
Uniqueness is deduced from the contraction estimate in Wasserstein distance d_W in Proposition 1 below. Indeed, if we take $\rho^0 = \widetilde{\rho}^0 = \rho^{ini}$ in the estimate of Proposition 1, then we deduce that $\rho = \widetilde{\rho}$.

3.2 The macroscopic velocity

In the first step above, we have defined a macroscopic velocity for which the product in the divergence term in (1) has a sense. This definition relies on the following stability result.

Lemma 2. *Let V be a velocity field satisfying $V \in L^\infty_{loc}(\mathbb{R}^+, C(\mathbb{R}^d \setminus \{0\}))$, (2), (3) and $V(-x) = -V(x)$. Let $(V_n)_{n\in\mathbb{N}^*}$ be a sequence of odd functions in $C^1(\mathbb{R}^+ \times \mathbb{R}^d)$, uniformly bounded by v_∞ and such that for all $t \in \mathbb{R}^+$,*

$$sup_{x\in\mathbb{R}^d\setminus B(0,\frac{1}{n})} \left| V_n(t,x) - V(t,x) \right| \leq \frac{1}{n}, \qquad \textit{for all } n \in \mathbb{N}^*. \tag{12}$$

Let $\rho(t)$ be a probability measure for all $t \geq 0$. Let $(\rho_n(t))_n$ be a sequence of probability measures such that $\rho_n \rightharpoonup \rho$ weakly as measures as $n \to +\infty$, then for any $T > 0$, for every $\phi \in C_0([0,T] \times \mathbb{R}^d)$ and any $\xi \in \mathbb{R}^d$, we have

$$\begin{aligned} \lim_{n\to+\infty} \int_0^T \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(t,x) V_n(t,x-y-\xi) \rho_n(t,dx) \rho_n(t,dy) dt \\ = \int_0^T \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(t,x) \widehat{V}(t,x-y-\xi) \rho(t,dx) \rho(t,dy) dt. \end{aligned}$$

Proof. We first introduce some notations that simplify the computations:

$$\begin{aligned} \mu_n(t) &:= \rho_n(t) \otimes \rho_n(t,\cdot-\xi) - \rho_n(t,\cdot-\xi) \otimes \rho_n(t), \\ \mu(t) &:= \rho(t) \otimes \rho(t,\cdot-\xi) - \rho(t,\cdot-\xi) \otimes \rho(t), \\ D_n &:= \left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, x \neq y, |x-y| < \frac{1}{n} \right\}. \end{aligned} \tag{13}$$

We recall that since $\rho_n \rightharpoonup \rho$ weakly as measures, we have that $\rho_n(t) \otimes \rho_n(t,\cdot-\xi) \rightharpoonup \rho(t) \otimes \rho(t,\cdot-\xi)$ and $\mu_n \rightharpoonup \mu$ weakly in the sense of measures.

Let us fixed $t \in \mathbb{R}^+$ and let $\varepsilon > 0$. By definition of μ and D_n in (13) there exists $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|\mu|(t,D_n) \leq \varepsilon. \tag{14}$$

For such N, we observe that, for all $n \geq N$, $D_n \subset D_N$, and

$$|\mu_n|(t,D_n) \leq |\mu_n|(t,D_N) \leq |\mu_n - \mu|(t,D_N) + |\mu|(t,D_N). \tag{15}$$

¿From the weak convergence $\mu_n \rightharpoonup \mu$, we deduce that for n large enough, we have $|\mu_n - \mu|(t, D_N) \leq \varepsilon$. Injecting in (15), and using also (14), we deduce that for N large enough and $\forall n \geq N$, we have

$$|\mu_n|(t, D_n) \leq \varepsilon, \quad \text{and} \quad |\mu_n - \mu|(t, D_n) \leq \varepsilon. \tag{16}$$

For $\phi \in C_0([0,T] \times \mathbb{R}^d)$, we note

$$\begin{aligned} A_n(t) := &\iint_{\mathbb{R}^d \times \mathbb{R}^d} V_n(t, x-y-\xi)\phi(t,x)\rho_n(t) \otimes \rho_n(t)(dx,dy) \\ &- \iint_{\mathbb{R}^d \times \mathbb{R}^d} \widehat{V}(t, x-y-\xi)\phi(t,x)\rho(t) \otimes \rho(t)(dx,dy). \end{aligned}$$

After a change of variable, we may write

$$\begin{aligned} A_n(t) &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(t,x)\Big(V_n(t,x-y)\rho_n(t,x)\rho_n(t,y-\xi) \\ &\qquad\qquad - \widehat{V}(t,x-y)\rho(t,x)\rho(t,y-\xi)\Big)\, dxdy \\ &= \frac{1}{2}\iint_{\mathbb{R}^d \times \mathbb{R}^d} \Big[V_n(t,x-y)\Big(\phi(t,x)\rho_n(t,x)\rho_n(t,y-\xi) \\ &\qquad\qquad - \phi(t,y)\rho_n(t,y)\rho_n(t,x-\xi)\Big) \\ &\qquad - \widehat{V}(t,x-y)\Big(\phi(t,x)\rho(t,x)\rho(t,y-\xi) \\ &\qquad\qquad - \phi(t,y)\rho(t,y)\rho(t,x-\xi)\Big)\Big] dxdy, \end{aligned}$$

where we have used the symmetry assumption $V_n(-x) = -V_n(x)$ and $V(-x) = -V(x)$ for the last equality. We may rewrite

$$\begin{aligned} A_n(t) =& \frac{1}{2}\iint_{\mathbb{R}^d \times \mathbb{R}^d} (\phi(t,x) - \phi(t,y))\Big[\big(V_n(t,x-y) - \widehat{V}(t,x-y)\big)\rho_n(t,x)\rho_n(t,y-\xi) \\ &\qquad + \widehat{V}(t,x-y)\big(\rho_n(t,x)\rho_n(t,y-\xi) - \rho(t,x)\rho(t,y-\xi)\big)\Big]\, dxdy + \\ &\frac{1}{2}\iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(t,y)\big((V_n(t,x-y) - \widehat{V}(t,x-y))\mu_n + \widehat{V}(t,x-y)(\mu_n - \mu)\big) dxdy \\ =& I_n + II_n + III_n + IV_n. \end{aligned} \tag{17}$$

We bound each term of the right and side separately.

Let us consider the first term

$$I_n := \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\phi(t,x) - \phi(t,y))\big(V_n(t,x-y) - \widehat{V}(t,x-y)\big)\rho_n(t,x)\rho_n(t,y-\xi)dxdy.$$

Using assumption (12) and the bound $\|V_n\|_\infty \leq v_\infty$, the latter integral on $\mathbb{R}^d \times \mathbb{R}^d \setminus D_n$ is bounded by a term of order $\frac{1}{n}$. We are left with the integral over D_n. Since $\phi \in C_0([0,T] \times \mathbb{R}^d)$, there exists a compact $K \subset \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d \setminus K$,

$|\phi(x)| \leq \varepsilon$. On the compact K, ϕ is uniformly continuous, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x, y \in D_n \cap K$, $|\phi(x) - \phi(y)| \leq \varepsilon$. Then, we deduce

$$|I_n(t)| \leq \frac{2}{n}\|\phi\|_\infty + 2v_\infty \varepsilon. \tag{18}$$

For the second term,

$$II_n := \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\phi(t,x) - \phi(t,y))\widehat{V}(t, x-y) \\ \big(\rho_n(t,x)\rho_n(t,y-\xi) - \rho(t,x)\rho(t,y-\xi)\big)\,dxdy.$$

We use the fact that the function $(x,y) \mapsto (\phi(t,x) - \phi(t,y))\widehat{V}(t,x-y)$ is continous and the weak convergence in the sense of measures of ρ_n to deduce that

$$\lim_{n \to +\infty} II_n(t) = 0. \tag{19}$$

Considering now the third term,

$$III_n := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(t,y)(V_n(t,x-y) - \widehat{V}(t,x-y))\mu_n(t,x,y)\,dxdy.$$

We split the integral between the one on $\mathbb{R}^d \times \mathbb{R}^d \setminus D_n$ and the one on D_n. We get

$$|III_n| \leq \frac{2}{n}\|\phi\|_\infty + 2v_\infty \|\phi\|_\infty |\mu_n|(D_n) \leq \left(\frac{2}{n} + 2v_\infty \varepsilon\right)\|\phi\|_\infty, \tag{20}$$

where we use (16) for the last inequality.

The fourth term reads

$$\begin{aligned} IV_n &:= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(t,y)\widehat{V}(t,x-y)(\mu_n - \mu)(t,x,y)\,dxdy \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(t,y)\big(\widehat{V}(t,x-y) - V_N(t,x-y)\big)(\mu_n - \mu)(t,x,y)\,dxdy \\ &\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(t,y)V_N(t,x-y)(\mu_n - \mu)(t,x,y)\,dxdy, \end{aligned}$$

where $N \in \mathbb{N}$ will be chosen large enough. The second term of the right hand side converges to 0 as n goes to $+\infty$ since $\mu_n \rightharpoonup \mu$ and $(x,y) \mapsto \phi(t,y)V_N(t,x-y)$ is continuous. Then it is bounded by ε for n large enough. We bound the first term of the right hand side as for the term III_n, we obtain, for n large enough

$$|IV_n| \leq \left(\frac{4}{N} + 2v_\infty \varepsilon\right)\|\phi\|_\infty + \varepsilon. \tag{21}$$

Finally, injecting (18),(19),(20),(21) into (17), we deduce the a.e. convergence $A_n(t) \to 0$ as $n \to +\infty$. Moreover, we have the uniform bound $|A_n(t)| \leq 2v_\infty \|\phi\|_\infty$.

Applying the Lebesgue's dominated convergence theorem, we deduce that $\int_0^T A_n(t)dt$ goes to 0 as $n \to +\infty$. It concludes the proof.

3.3 Contraction estimate

Proposition 1. *Under assumptions* (2),(3),(4),(5) *on* V*, let* ρ^0 *and* $\widetilde{\rho}^0$ *be given in* $\mathscr{P}_2(\mathbb{R}^d)$*. Then, there exists a nonnegative constant* C *such that the corresponding solutions* $\rho = Z_\# \rho^0$ *and* $\widetilde{\rho} = \widetilde{Z}_\# \widetilde{\rho}^0$ *verify*

$$d_W(\rho(t), \widetilde{\rho}(t)) \le e^{C(t+\int_0^t \lambda(s)ds)} d_W(\rho^0, \widetilde{\rho}^0).$$

Proof. Let us consider γ an optimal map with marginals ρ^0 and $\widetilde{\rho}^0$ such that,

$$d_W(\rho^0, \widetilde{\rho}^0)^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^2 \gamma(dx_1, dx_2).$$

We compute, formally,

$$\begin{aligned}
&\frac{d}{dt} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |Z(t,x_1) - \widetilde{Z}(t,x_2)|^2 \gamma(dx_1, dx_2) \\
&\quad = 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle \widehat{a}_\rho(t, Z(t,x_1)) - \widehat{a}_{\widetilde{\rho}}(t, \widetilde{Z}(t,x_2)), Z(t,x_1) - \widetilde{Z}(t,x_2) \rangle \gamma(dx_1, dx_2) \\
&\quad = 2 \iiiint \langle \widehat{V}(t, Z(t,x_1) - Z(t,y_1)) - \widehat{V}(t, \widetilde{Z}(t,x_2) - \widetilde{Z}(t,y_2)), \\
&\qquad\qquad Z(t,x_1) - \widetilde{Z}(t,x_2) \rangle \, \gamma(dx_1, dx_2) \gamma(dy_1, dy_2),
\end{aligned}$$

where $\widehat{V}$ is defined in (10). ¿From assumption (4), we decompose

$$\frac{d}{dt} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |Z(t,x_1) - \widetilde{Z}(t,x_2)|^2 \gamma(dx_1, dx_2) = I_s + I_r,$$

with, using the notation $V_s = \sum_{\ell=1}^L \big(\widehat{V}_\ell(\cdot - \xi_\ell) + \widehat{V}_\ell(\cdot + \xi_\ell) \big)$,

$$\begin{aligned}
I_s = 2 \iiiint &\langle V_s(t, Z(t,x_1) - Z(t,y_1)) - V_s(t, \widetilde{Z}(t,x_2) - \widetilde{Z}(t,y_2)), \\
&\quad Z(t,x_1) - \widetilde{Z}(t,x_2) \rangle \, \gamma(dx_1, dx_2) \gamma(dy_1, dy_2); \\
I_r = 2 \iiiint &\langle V_r(t, Z(t,x_1) - Z(t,y_1)) - V_r(t, \widetilde{Z}(t,x_2) - \widetilde{Z}(t,y_2)), \\
&\quad Z(t,x_1) - \widetilde{Z}(t,x_2) \rangle \, \gamma(dx_1, dx_2) \gamma(dy_1, dy_2).
\end{aligned}$$

We treat each term separately. Using the symmetry of V_s (exchanging the role of x and y) we obtain

$$
\begin{aligned}
I_s = \iiiint & \langle V_s(t, Z(t,x_1) - Z(t,y_1)) - V_s(t, \widetilde{Z}(t,x_2) - \widetilde{Z}(t,y_2)), \\
& Z(t,x_1) - Z(t,y_1) - \widetilde{Z}(t,x_2) + \widetilde{Z}(t,y_2) \rangle \, \gamma(dx_1,dx_2)\gamma(dy_1,dy_2) \\
\leq \lambda(t) & \iiiint |Z(t,x_1) - Z(t,y_1) - \widetilde{Z}(t,x_2) + \widetilde{Z}(t,y_2)|^2 \, \gamma(dx_1,dx_2)\gamma(dy_1,dy_2),
\end{aligned}
$$

where we use assumption (2) satisfied by V_s. Expanding the right hand side, we deduce straightforwardly

$$
I_s \leq 4\lambda(t) \iint_{\mathbb{R}^d \times \mathbb{R}^d} |Z(t,x_1) - \widetilde{Z}(t,x_2)|^2 \gamma(dx_1,dx_2). \tag{22}
$$

For the term I_r, we deduce from the Lipschitz continuity of V_r that there exists a nonnegative constant C such that

$$
\begin{aligned}
|I_r| \leq C \iiiint & \big|Z(t,x_1) - Z(t,y_1) - \widetilde{Z}(t,x_2) + \widetilde{Z}(t,y_2)\big| \\
& |Z(t,x_1) - \widetilde{Z}(t,x_2)| \, \gamma(dx_1,dx_2)\gamma(dy_1,dy_2), \\
\leq C \iint & |Z(t,x_1) - \widetilde{Z}(t,x_2)|^2 \, \gamma(dx_1,dx_2) + \\
C \iiiint & |Z(t,y_1) - \widetilde{Z}(t,y_2)||Z(t,x_1) - \widetilde{Z}(t,x_2)| \, \gamma(dx_1,dx_2)\gamma(dy_1,dy_2), \\
\leq C \iint & |Z(t,x_1) - \widetilde{Z}(t,x_2)|^2 \, \gamma(dx_1,dx_2) + C \left(\iint |Z(t,x_1) - \widetilde{Z}(t,x_2)| \, \gamma(dx_1,dx_2) \right)^2.
\end{aligned}
$$

Using a Cauchy-Schwarz inequality, we deduce

$$
|I_r| \leq 2C \iint_{\mathbb{R}^d \times \mathbb{R}^d} |Z(t,x_1) - \widetilde{Z}(t,x_2)|^2 \, \gamma(dx_1,dx_2). \tag{23}
$$

Combining (22) and (23), we deduce

$$
\begin{aligned}
\frac{d}{dt} & \iint_{\mathbb{R}^d \times \mathbb{R}^d} |Z(t,x_1) - \widetilde{Z}(t,x_2)|^2 \gamma(dx_1,dx_2) \\
& \leq (4\lambda(t) + 2C) \iint_{\mathbb{R}^d \times \mathbb{R}^d} |Z(t,x_1) - \widetilde{Z}(t,x_2)|^2 \gamma(dx_1,dx_2).
\end{aligned}
$$

We conclude by a Gronwall argument and using the fact that

$$
d_W(\rho(t), \widetilde{\rho}(t))^2 \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |Z(t,x_1) - \widetilde{Z}(t,x_2)|^2 \gamma(dx_1,dx_2).
$$

The above formal computation can be made rigorous by using a regularization of the potential and passing to the limit in the regularization (we refer the interested reader to Proposition 3.4 in [12]).

4 Numerical analysis

4.1 Definition of the scheme

Let us first introduce an upwind type numerical scheme for the discretization of the aggregation equation. We denote by Δt the time step and consider a Cartesian grid with step Δx_i in the ith direction, $i = 1,\dots,d$, and $\Delta x = \max_i \Delta x_i$. We use standard notations for vectors $e_i = (0,\dots,1,\dots,0)$, $a = (a_1,\dots,a_d)$. We define the multi-indices

$$J = (J_1,\dots,J_d) \in \mathbb{Z}^d, \quad x_J = (J_1 \Delta x_1,\dots,J_d \Delta x_d).$$

We denote by $C_J = [(J_1 - \frac{1}{2})\Delta x_1,(J_1 + \frac{1}{2})\Delta x_1) \times \dots [(J_d - \frac{1}{2})\Delta x_d,(J_d + \frac{1}{2})\Delta x_d)$ the elementary cell.

For a given nonnegative measure $\rho^{ini} \in \mathscr{P}_2(\mathbb{R}^d)$, we define, for $J \in \mathbb{Z}^d$,

$$\rho_J^0 = \int_{C_J} \rho^{ini}(dx) \geq 0. \tag{24}$$

Since ρ^{ini} is a probability measure, the total mass of the system is $\sum_{J\in\mathbb{Z}^d} \rho_J^0 = 1$. We denote by ρ_J^n an approximation of the value $\rho(t^n,x_J)$, for $J \in \mathbb{Z}^d$. Assuming that an approximating sequence $(\rho_J^n)_{J\in\mathbb{Z}^d}$ is known at time t^n, then we compute the approximation at time t^{n+1} by the following scheme,

$$\rho_J^{n+1} = \rho_J^n - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i}\big((a_{iJ}^n)^+\rho_J^n - (a_{iJ+e_i}^n)^-\rho_{J+e_i}^n - (a_{iJ-e_i}^n)^+\rho_{J-e_i}^n + (a_{iJ}^n)^-\rho_J^n\big). \tag{25}$$

The notation $(a)^+ = \max\{0,a\}$ stands for the positive part of the real number a and respectively $(a)^- = \max\{0,-a\}$ for the negative part. The discrete macroscopic velocity is computed thanks to the following discretization of equation (9),

$$a_{iJ}^n = \sum_{K\in\mathbb{Z}^d} \rho_K^n V_{iJ,K}^n, \quad \text{where} \quad V_{iJ,K}^n := \int_{t^n}^{t^{n+1}} \widehat{V}_i(s,x_J - x_K)ds, \tag{26}$$

where $\widehat{V}_i$ is the ith components of the velocity field $\widehat{V}$ defined in (10).

Example 1. In one dimension, the scheme (25) reads

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\Big((a_i^n)^+\rho_i^n - (a_{i+1}^n)^-\rho_{i+1}^n - (a_{i-1}^n)^+\rho_{i-1}^n + (a_i^n)^-\rho_i^n\Big).$$

This scheme has the following interpretation. Defining $\rho_{\Delta x}^n = \sum_{i\in\mathbb{Z}} \rho_i^n \delta_{x_i}$, we construct the approximation at time t^{n+1} with the two following steps:

- The delta mass ρ_i^n located at position x_i, moves with velocity a_i^n to the position $x_i + a_i^n \Delta t$. Assuming a CFL condition $v_\infty \Delta t \leq \Delta x$, the point $x_i + a_i^n \Delta t$ belongs to the interval $[x_i,x_{i+1}]$ if $a_i^n \geq 0$, and to the interval $[x_{i-1},x_i]$ if $a_i^n \leq 0$.

- Then we make a linear interpolation of the mass ρ_i^n between x_i and x_{i+1} if $a_i^n \geq 0$ and between x_{i-1} and x_i if $a_i^n \leq 0$.

Finally, we emphasize that this scheme is not the standard finite volume upwind scheme for which the numerical velocity is computed at the interface $a_{i+1/2}^n$. This is due to the particular structure of the equation for which the product $\widehat{a}_\rho \rho$ should be defined properly. If in the discretization we choose the velocity in a different grid point that the density, it creates a shift in the definition of the product and the numerical solution does not converge to the solution of Theorem 4. This point has been already noted in [27, 22] where numerical simulations emphasized the wrong behaviour of numerical solutions computed with the classical upwind scheme.

4.2 Convergence analysis

In the following theorem, we establish the convergence of scheme (25) towards the unique solution of Theorem 4. More precisely the statement reads:

Theorem 5. *Let $\rho^{ini} \in \mathscr{P}_2(\mathbb{R}^d)$. Let us assume that V satisfies assumptions* (2)–(3)–(4)–(5). *Let $T > 0$ and $\rho = Z_\# \rho^{ini}$ be the unique measure solution on $[0,T]$ to the interaction equation* (1) *with initial data ρ^{ini} given by Theorem 4. Let us assume that the CFL condition holds:*

$$v_\infty \sum_{i=1}^{d} \frac{\Delta t}{\Delta x_i} \leq 1. \tag{27}$$

Let us define $(\rho_J^0)_{J\in\mathbb{Z}^d}$ by (24) *and define the reconstruction*

$$\rho_\Delta(t,x) = \sum_{J\in\mathbb{Z}^d} \rho_J^n \delta_{x_J}(x) \mathbf{1}_{[t^n,t^{n+1})}(t), \tag{28}$$

where the approximation sequence (ρ_J^n) is computed thanks to the scheme (25)–(26). *Then we have the weak convergence in the sense of measures $\rho_\Delta \rightharpoonup \rho$ in $\mathscr{M}_b([0,T]\times \mathbb{R}^d)$ as Δt and Δx_i go to 0 under the condition* (27).

Before going into the proof of this Theorem, we mention that this result extends to the upwind scheme and to the general system of equation (1) at hand the convergence result stated in [12]. We mention also that an estimate of the order of convergence in the same spirit as [15, 16] is under progress [17].

We first recall the following well-known properties for the upwind scheme, whose proof is left to the reader.

Proposition 2. *Let us assume that V satisfies assumptions* (2)–(3)–(4)–(5) *and consider $\rho^{ini} \in \mathscr{P}_2(\mathbb{R}^2)$. Let us assume that the CFL condition* (27) *holds. We define $(\rho_J^0)_{J\in\mathbb{Z}^d}$ by* (24) *and the reconstruction ρ_Δ by* (28), *where the approximation sequence (ρ_J^n) and (a_{iJ}^n) are computed thanks to the scheme* (25)–(26). *Then, we have*

(i) Positivity: for all $J \in \mathbb{Z}^d$, $n \in \mathbb{N}$, $i = 1,\dots,d$, $\rho_J^n \geq 0$, $|a_{iJ}^n| \leq v_\infty$.

(ii) *Mass conservation: for all* $n \in \mathbb{N}^*$, *we have* $\sum_{J\in\mathbb{Z}^d} \rho_J^n = \sum_{J\in\mathbb{Z}^d} \rho_{ij}^0 = 1.$

(iii) *Bound on the second moment: there exists a constant* $C > 0$ *such that for all* $n \in \mathbb{N}^*$, *we have*

$$M_2^n := \sum_{J\in\mathbb{Z}^d} |x_J|^2 \rho_J^n \le e^{Cn\Delta t}\big(M_2^0 + C\big).$$

Proof of Theorem 5

Step 1: weak convergence. Under the CFL condition (27), we deduce from Proposition (2) that the sequence $(\rho_\Delta)_\Delta$ is a sequence of nonnegative bounded measures which satisfies for all $t \in [0,T]$, $|\rho_\Delta(t)|(\mathbb{R}^2) = 1$. Therefore, we can extract a subsequence, still denoted $(\rho_\Delta)_\Delta$, converging for the weak topology towards ρ as Δt, and Δx_i go to 0, satisfying (27).

Step 2: identification of the limit. We choose $\Delta t > 0$ and $N_T \in \mathbb{N}^*$ such that $T = \Delta t N_T$ and condition (27) holds. Let $\phi \in \mathscr{D}([0,T)\times\mathbb{R}^d)$, we multiply (25) by ϕ and integrate on $[t^n,t^{n+1})\times\mathbb{R}^d$, using a discrete integration by parts, we get

$$\sum_{J\in\mathbb{Z}^d} (\rho_J^{n+1} - \rho_J^n)\phi_J^n = \sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} \frac{\Delta t}{\Delta x_i}\Big[(a_{iJ}^n)^+ \rho_J^n(\phi_{J+e_i}^n - \phi_J^n) + (a_{iJ}^n)^- \rho_J^n(\phi_J^n - \phi_{J-e_i}^n)\Big],$$

with the notation $\phi_J^n = \int_{t^n}^{t^{n+1}} \phi(t,x_J)dt$. ¿From a Taylor formula, we have

$$\phi_{J+e_i}^n = \phi_J^n + \partial_{x_i}\phi_J^n \Delta x_i + O(\Delta t \Delta x_i^2), \qquad \phi_{J-e_i}^n = \phi_J^n - \partial_{x_i}\phi_J^n \Delta x_i + O(\Delta t \Delta x_i^2).$$

Summing over n and using a discrete integration by parts, we deduce

$$\sum_{n=1}^{N_T} \sum_{J\in\mathbb{Z}^d} \rho_J^n(\phi_J^{n-1} - \phi_J^n) - \sum_{J\in\mathbb{Z}^d} \rho_J^0 \phi_J^0 = \sum_{n=0}^{N_T}\sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} \Delta t a_{iJ}^n \rho_J^n \partial_{x_i}\phi_J^n + O(\Delta t \Delta x).$$

Finally, using also a Taylor formula for the first term of the left hand side, we deduce

$$\sum_{n=1}^{N_T} \sum_{J\in\mathbb{Z}^d} \rho_J^n \int_{t^n}^{t^{n+1}} \partial_t \phi(t,x_J)\,dt + \sum_{J\in\mathbb{Z}^d} \rho_J^0 \frac{\phi_J^0}{\Delta t} + \sum_{n=0}^{N_T}\sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} a_{iJ}^n \rho_J^n \partial_{x_i}\phi_J^n = O(\Delta x + \Delta t). \tag{29}$$

Let us define the reconstruction for $i = 1,\dots,d$,

$$a_{i\Delta}(t,x) = \sum_{J\in\mathbb{Z}^d} a_{iJ}^n \mathbf{1}_{[t^n,t^{n+1})\times C_J}(t,x).$$

Using also the definition (28), we may rewrite (29) as

$$\int_0^T \int_{\mathbb{R}^d} \rho_\Delta(t,x)\partial_t\phi(t,x)\,dtdx + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}^d} a_{i\Delta}(t,x)\rho_\Delta(t,x)\partial_{x_i}\phi(t,x)\,dtdx$$
$$+ \int_{\mathbb{R}^d} \rho_\Delta^0(x)\phi(0,x)\,dx = O(\Delta x + \Delta t).$$

¿From the weak convergence of the sequence (ρ_Δ) (as a consequence of the first step), we may pass to the limit in the first term. Using also Lemma 2, we can pass to the limit in the second term. Then we deduce that the limit ρ is a solution in the sense of distributions of equation (1).

Step 3: conclusion. We have established that we can extract from the sequence $(\rho_\Delta)_\Delta$ a subsequence that converges weakly in the sense of measures towards a solution in the sense of distributions of the conservative transport equation (1). Moreover, we know from Theorem 4 that there exists a unique pushforward measure that solves equation (1).

We may invoke the superposition principle (see [1, Chapter 8]) to conclude that the limit ρ is the pushforward measure of Theorem 4. By uniqueness of such a solution, we deduce also that the whole sequence $(\rho_\Delta)_\Delta$ converges to ρ.

5 Numerical simulations

5.1 One dimensional examples

We consider an interval $[-2.5, 2.5]$ discretized with a Cartesian grid of size step $\Delta x = \frac{1}{80}$. As an initial data, we choose

$$\rho^{ini}(x) = e^{-10(x-1)^2} + e^{-10(x+1)^2}.$$

Then we implement the numerical scheme presented in section 4 for the function $V(t,x) = \partial_x W(x)$ where $W(x) = \frac{1}{2}|x| + \frac{1}{4}|x-\xi| + \frac{1}{4}|x+\xi|$ for $\xi = 0.5$. The times dynamics is plotted in Figure 1. For the matter of comparison, we display in Figure 2 the result obtained for the function $W(x) = |x|$, which corresponds to the case $\xi = 0$ in the previous example. We observe that in both case blowup in finite time occurs and that the solution concentrates in a Dirac delta in finite time. The only visible difference between the two graphs is the time of concentration which is smaller in the second case than in the first case.

5.2 Two dimensional examples

As an illustration, we propose now a numerical example in two dimensions. The spacial domain $[0,1] \times [0,1]$ is discretized with $N_x = 70$ nodes in the x-direction and $N_y = 70$ nodes in the y-direction and a time step $\Delta t = 10^{-3}$. We choose as an initial data:

$$\rho(t,x) = \mathbf{1}_{[0.2,0.8]\times[0.2,0.8]\setminus[0.3,0.7]\times[0.3,0.7]}.$$

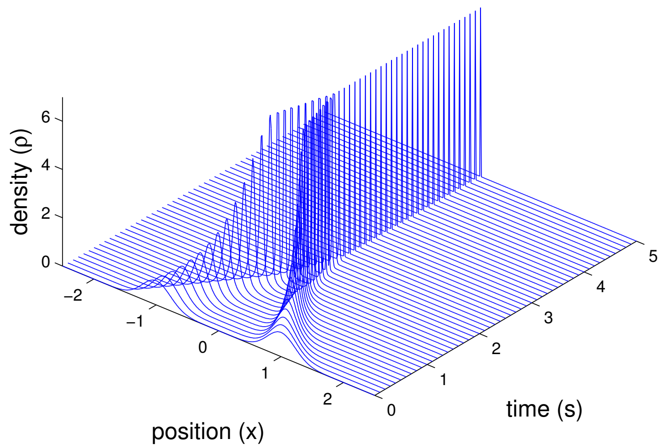

Fig. 1 Numerical simulation of equation (1) with $V(x) = \partial_x W(x)$ for $W(x) = \frac{1}{2}|x| + \frac{1}{4}|x-0.5| + \frac{1}{4}|x+0.5|$ and initial data compound of the sum of two bump functions.

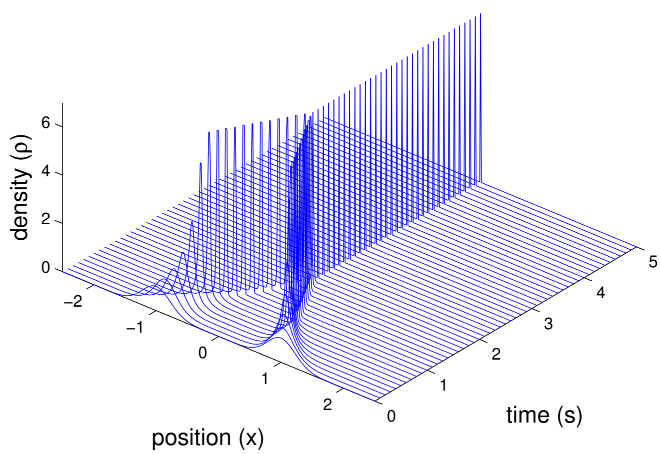

Fig. 2 Numerical simulation of equation (1) with $V(x) = \partial_x W(x)$ for $W(x) = |x|$ and initial data compound of the sum of two bump functions.

We consider $V(x) = \nabla_x W(x)$ with the following interaction potentials: $W_1(x) = e^{-5|x|}$ and $W_2(x) = -5|x|$. For $|x|$ close to 0, we have that $\nabla_x W_1 \sim \nabla_x W_2$. Then the

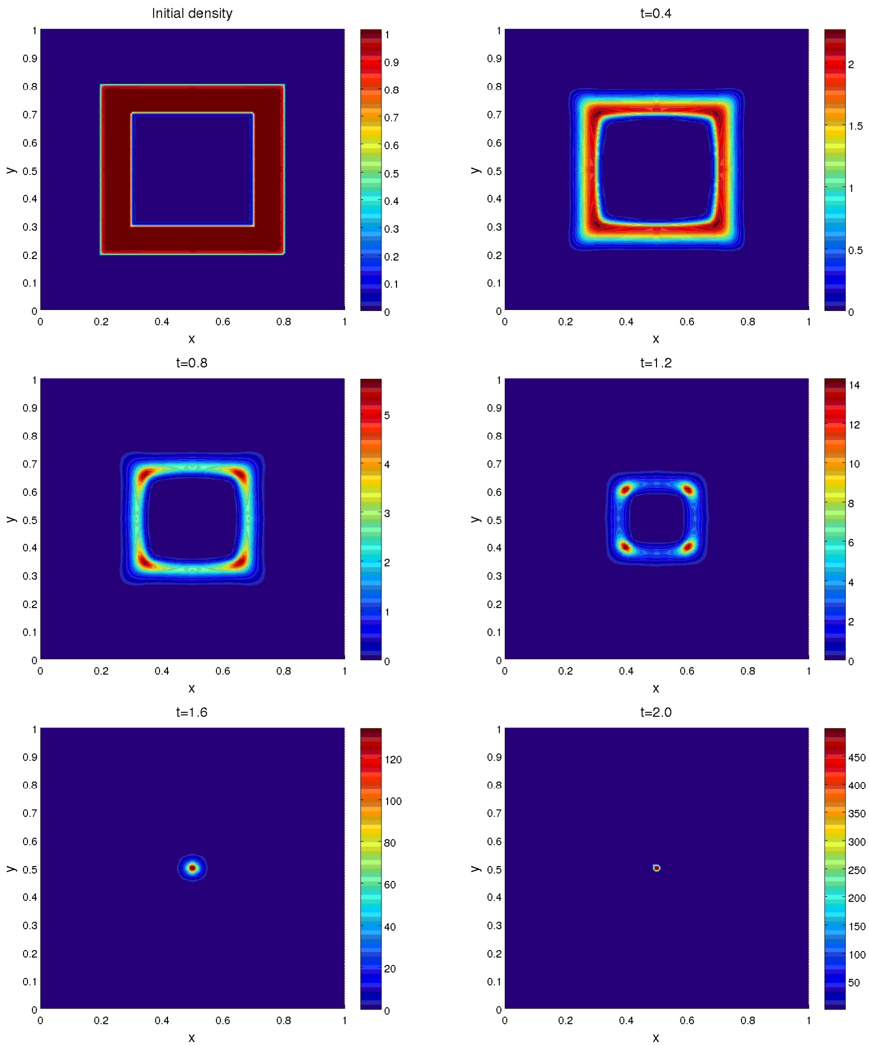

Fig. 3 Time dynamics of the numerical solution of the aggregation equation (1) with $V = \nabla_x W$, where $W(x) = e^{-5|x|}$ is a Morse potential. ¿From top left to bottom right, the times considered are $t = 0$ (initial data), $t = 0.4$, $t = 0.8$, $t = 1.2$, $t = 1.6$ and $t = 2$.

short range interaction is similar between both potential, but the long range interaction is different. The numerical results are displayed in Figure 3 for the potential W_1 and in Figure 4 for the potential W_2.

We observe as expected the aggregation in finite time of ρ towards a Dirac delta in the center of the domain. It is also interesting to observe that the time dynamics during this step of concentration is different between both potentials. In both case the density ρ keeps a shape similar to the inital square shape which tighten as time

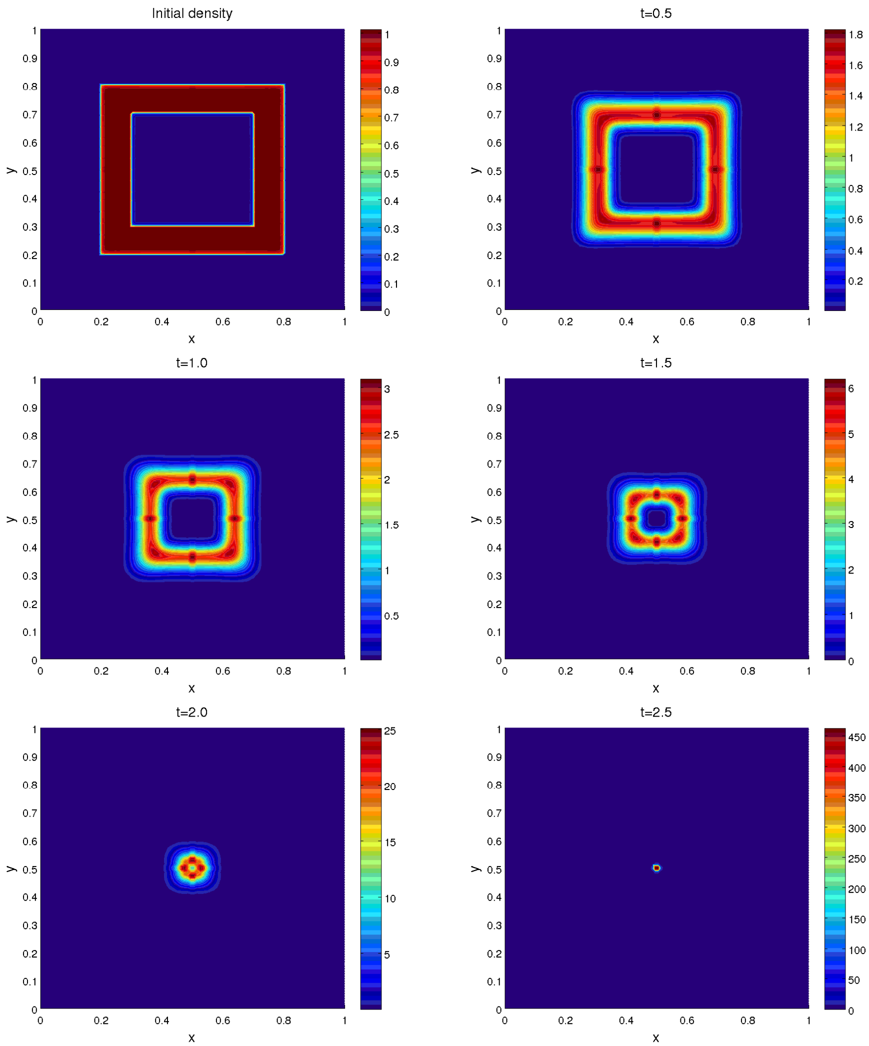

Fig. 4 Time dynamics of the numerical solution of the aggregation equation (1) with $V = \nabla_x W$, where $W(x) = -5|x|$ is a Newtonian potential. ¿From top left to bottom right, the times considered are $t = 0$ (initial data), $t = 0.5$, $t = 1$, $t = 1.5$, $t = 2$ and $t = 2.5$.

increases. However in the case of the Morse potential (Fig 3), we notice a strong concentration at the corners of the square, whereas in the case of the Newtonian potential (Fig 4) the density concentration is homogeneous along the edges of the square with a slight concentration in the middle of the edges.

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