A simple 1D model of inviscid fluid-solid interaction

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Abstract. We analyze a one-dimensional fluid-particle interaction model, composed by the Burgers equation for the fluid velocity and an ordinary differential equation which governs the particle movement. The coupling is achieved through a friction term. One of the novelties is to consider *entropy weak solutions* involving *shock waves*. The difficulty is the interaction between these shock waves and the particle. We prove that the Riemann problem with arbitrary data always admits a solution, which is explicitly constructed. Besides, two asymptotic behaviors are described: the long-time behavior and the behavior for large friction coefficients.

Keywords. Burgers equation; fluid-particle interaction; Riemann problem; non-conservative product.

Contents

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1   Introduction

In this paper we consider a simple one-dimensional model of fluid-structure interaction. The fluid with velocity $u(t, x)$ is assumed inviscid and its motion is modeled by the inviscid Burgers equation. The structure is a particle localized at the point $h(t)$. The coupling between the fluid and the particle is achieved by a friction term between the fluid and the particle velocities, namely $\lambda (h'(t) - u(t, h(t)))$ where λ is the positive friction constant. The equations are the following.

$$\begin{cases} \partial_t u(t, x) + \partial_x (u^2/2)(t, x) = \lambda(h'(t) - u(t, h(t)))\delta_{h(t)}(x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ mh''(t) = -\lambda \left(h'(t) - u(t, h(t)) \right), & t \in \mathbb{R}_+, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ (h(0), h'(0)) = (0, v_0). \end{cases} \quad (1.1)$$

This simple model can be seen as a first step towards the understanding of the interaction between a structure and an inviscid fluid subject to shock waves.

In recent years, fluid-structure systems have been the subject of an active research and the theoretical analysis of this kind of systems have given rise to many publications. Several works have dealt with the study of the system composed by rigid bodies and a viscous incompressible fluid; see, for instance, [6], [7], [20], [13], [17], [18], [28], [26], [30], [32, 33] and [19]. Some authors have also considered structures of different type (see, for instance, [2], [4], [9]) or fluids of different type (see, for instance, [8], [12], [24]). However, up to now and up to our knowledge no results concerning the interaction between shocks and solids have been published. In this paper, we consider a simplified 1D model where shocks appear and where we have to deal with this interaction. Let us emphasize that our simple model is different from the systems studied in the above papers. More precisely, in (1.1), the velocity field of the fluid at the position of the particle is not equal to the particle velocity. This condition is relaxed through the friction term $\lambda(h'(t) - u(t, h(t)))$. However, when $\lambda \to \infty$, we recover formally the equality of the velocity fields of the fluid and of the particle (see [16] for a rigorous study in the viscous case). Another difference comes from the force which applies from the fluid to the particle. In the systems studied in the above papers, this force can be expressed through a stress tensor whereas here this force is due to the friction.

One of the main difficulties is the singular source term in the Burgers equation. Indeed, u can be discontinuous at the particle position, this leads to a difficulty in determining the non-conservative product $u(t, h(t))\delta_{h(t)}(x)$ which is not a distribution. This determination of the product is done in the following via a "regularization" of the particle, replacing the Dirac measure with a non-negative compactly supported density function.
Another important difficulty comes from the particle. Its motion is governed by an ordinary differential equation (ODE) with a discontinuous velocity field. It must be considered at least with the help of differential inclusions. Actually, using entropy inequalities, we are able to select the "good" value of the source term.

The first contribution of this paper is to provide a definition of solutions, based on "natural" approximations of system (1.1) (Definition 3.1). The second contribution is the construction of an explicit solution in this frame for specific initial data: we consider the case of an initial discontinuity superimposed with the particle, what will be referred to as a Riemann problem. We obtain a result of global existence for such initial data (Theorem 5.1). As already mentioned, one of the difficulties is due to the Dirac measure in the source term. Here, it corresponds to a linearly degenerate field (see Section 4) and may lead to resonance phenomenon, that is superimposition of non-linear waves with the particle. Isaacson and Temple studied such problems in [21]. Later, Goatin and LeFloch have extended their analysis to systems in [14]. They all consider non-conservative systems involving degenerate resonance, that is to say where the Jacobian matrix is no longer diagonalizable when its eigenvalues identify. In our case, the structure of the Jacobian matrix is somewhat different: when the eigenvalues identify, the Jacobian matrix becomes diagonal. In particular, this enables us to obtain a global (existence and) uniqueness result for a particle with a constant given velocity (instead of up to three solutions in [14] and [21]). The way to define the non-conservative product is similar to the one developed by Chinnayya, LeRoux and Seguin in [5] (which actually is equivalent to the definitions of [14] and [21]). This approach is based on a regularization of the Dirac measure and enables to define the non-conservative product, independently of the choice of regularization. It is worth noticing that the complete model does not admit self-similar

solutions in general (due to the source term). Therefore, the Riemann problem becomes much more difficult to solve and in particular, standard arguments for uniqueness cannot be invoked, due to the constructive character of the proof. Actually, the uniqueness for the Riemann problem is not tackled in this paper.

The outline of the paper is the following. In Section 2, some physical justifications for this model are given, with also formal estimates. Using the viscous regularization for deriving entropy inequalities and mollifying the particle to define the non-conservative product, we propose a definition of solution in Section 3. The following section is devoted to the resolution of the Riemann problem for the case of a particle with a given constant velocity. In Section 5, we prove the existence of a solution to the Riemann problem for the complete (fully coupled) system, and we describe some asymptotic behaviors.

2 Origin of the model

In system (1.1),

- $u(t,x)$ stands for the velocity of a one-dimensional "fluid" at time $t \in \mathbb{R}_+$ and position $x \in \mathbb{R}$ (for homogeneity considerations of the system, one can think of a fluid with constant density 1);

- $h(t)$ stands for the position of a punctual particle at time t with constant mass m.

The fluid velocity is assumed to be governed by the Burgers equation with a source term acting only at the position h,

$$\partial_t u + \partial_x u^2/2 = F\delta_h$$

where F represents a force term due to the presence of the particle, and the particle behaves according to Newton's law

$$mh''(t) = -F$$

where $-F$ is the force from the fluid to the particle, in conformity with the action-reaction principle. The drag force between the fluid and the particle is assumed to be local in $(t, h(t))$ and proportional (with a given positive coefficient λ) to the difference of the velocity of the particle, $h'(t)$, and the (local) fluid velocity $u(t, h(t))$: this leads to the right-hand side in (1.1).

From a slightly different point of view, this model can be formally derived from a classical fluid-particle interaction model. We here refer to [1] for sprays coupling a Eulerian gas and a Vlasov distribution of particles: one model studied by the authors writes in one space dimension

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) = -\mu \int_{\mathbb{R}} f(t,x,v)(u-v)\, dv, \\ \partial_t f + v\partial_x f + \partial_v (\mu(u-v)f) = 0, \end{cases}$$

Here $\rho(t,x)$ is the density of the surrounding fluid, $u(t,x)$ its velocity and p its pressure (a given function of the density). The spray (constituted of droplets) is described through its density in the phase space $f(t,x,v)$. The retained coupling force between the fluid and the (microscopic) particles is $\mu(u-v)f$ where μ is a given positive coefficient. This assumption turns out to be reasonable when the Reynolds number of the flow is small: see additionally [23] and [25] for the modeling of the drag force. Up to our knowledge, studies of such systems exist only for smooth solutions: see [1] and, in a rather similar context but with a diffusion term in the Vlasov equation, [3]. We here intend to study solutions involving shock waves, that is why we propose two drastic simplifications of the above system.

The first one consists in replacing the 2×2 Euler system with the scalar inviscid Burgers equation. This leads to

$$\partial_t u + \partial_x u^2/2 = -\mu n(t,x)\,(u(t,x) - W(t,x))$$

with $n(t,x) = \int_{\mathbb{R}} f(t,x,v)\, dv$ and $W(t,x) = \int_{\mathbb{R}} vf(t,x,v)\, dv/n(t,x)$.

The second one consists in assuming that the density of particles f is a Dirac measure in x and v: $f(t,x,v) = m\delta_{h(t)}(x)\delta_{V(t)}(v)$, with $m > 0$. On the one hand, this formally gives $n(t,x) = m\delta_{h(t)}(x)$, $W(t,x) = V(t)\delta_{h(t)}(x)$. The equation for the fluid then becomes

$$\partial_t u + \partial_x u^2/2 = -\mu m \left(u(t,x) - V(t)\right)\delta_{h(t)}(x).$$

On the other hand, this ansatz (formally) leads to

$$\partial_t f(t,x,v) = -m\left(h'(t)\delta'_{h(t)}(x)\delta_{V(t)}(v) + V'(t)\delta_{h(t)}(x)\delta'_{V(t)}(v)\right),$$
$$\partial_x f(t,x,v) = m\delta'_{h(t)}(x)\delta_{V(t)}(v),$$
$$\partial_v \left(\mu(u-v)f(t,x,v)\right) = \mu m(u-v)\delta_{h(t)}(x)\delta'_{V(t)}(v),$$

so that the Vlasov equation becomes

$$h'(t)\delta'_{h(t)}(x)\delta_{V(t)}(v) + V'(t)\delta_{h(t)}(x)\delta'_{V(t)}(v) - v\delta'_{h(t)}(x)\delta_{V(t)}(v) - \mu(u-v)\delta_{h(t)}(x)\delta'_{V(t)}(v) = 0,$$

i.e.

$$\delta'_{h(t)}(x)\left(h'(t)\delta_{V(t)}(v) - v\delta_{V(t)}(v)\right) + \delta'_{V(t)}(v)\left(V'(t)\delta_{h(t)}(x) - \mu(u-v)\delta_{h(t)}(x)\right) = 0,$$

which finally gives $V(t) = h'(t)$ for all t and

$$h''(t) = \mu\left(u(t,h(t)) - h'(t)\right).$$

Denoting $\lambda = \mu m$ completes the formal derivation of model (1.1).

Let us mention moreover that the diffusive system

$$\begin{cases} \partial_t u + \partial_x u^2/2 - \nu\partial^2_{xx} u = \lambda\left(V - u\right), \\ \partial_t f + v\partial_x f + \partial_v \left(\lambda(u-v)f\right) = 0, \\ V(t,x) = \int_{\mathbb{R}} vf(t,x,v)\,dv / \int_{\mathbb{R}} f(t,x,v)\,dv \end{cases} \tag{2.1}$$

(which could be derived replacing the compressible Navier-Stokes system (instead of the Euler system) with the viscous Burgers equation) has been extensively studied: see for example [11], [16] and [10], where it is shown to be well-posed, even for measure data.

Finally, let us mention that the model (1.1) satisfies several formal *a priori* estimates:

$$\frac{d}{dt}\left(\int_{\mathbb{R}} udx + mh'\right) = 0 \quad \text{(conservation of the total impulsion)}, \tag{2.2}$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}} u^2 dx + m(h')^2\right) + \lambda(h' - u(h))^2 = 0 \quad \text{(equation for the total kinetic energy)}, \tag{2.3}$$

$$\partial_t|u-\kappa| + \partial_x\left(\text{sgn}(u-\kappa)\frac{u^2-\kappa^2}{2}\right) + m\frac{d}{dt}|h'-\kappa|\delta_h \leqslant 0 \quad \forall\kappa \in \mathbb{R} \quad \text{(Kružkov entropy inequalities)}. \tag{2.4}$$

These relations will be justified in the next sections.

3 Definition of the solution

The problem (1.1) is not directly well defined since the generic solution are not smooth: thus the product $u\delta_h$ has to be defined if u is discontinuous at h. A natural idea is to add a vanishing regularizing term in the system, and, passing to the non-regularized limit, deduce the definition of the solution. Two different regularizations are investigated. The first one is a viscous regularization, which enables to derive entropy conditions. The second one is a thickening of the particle, for defining the non-conservative product.

Notation. In the sequel, $v(h)(t)$ stands for $v(t,h(t))$ for any function v defined in $\mathbb{R}_+ \times \mathbb{R}$. Besides, δ_h represents $\delta_{h(t)}(x)$.

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3.1 Entropy inequalities

We consider the following viscous regularization of problem (1.1), for $\varepsilon > 0$:

$$\begin{cases} \partial_t u - \varepsilon \partial_{xx}^2 u + \partial_x(u^2/2) = \lambda(h' - u(h))\delta_h, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ mh'' = -\lambda(h' - u(h)), & t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ (h(0), h'(0)) = (0, v_0). \end{cases} \tag{3.1}$$

This problem can be obtained from (2.1) with the ansatz $f(t,x,v) = m\delta_{h(t)}(x)\delta_{h'(t)}(v)$. Thus, it follows from [10] that it admits a unique solution (for the sake of completeness, we here propose another proof, see Theorem A.1 in the end of the paper), but the question whether the solutions of (3.1) converge towards the solution of (1.1) as ε goes to 0 remains an open problem. Nevertheless we propose to derive entropy inequalities from the viscous problem in order to select a "physical" solution of the inviscid one.

Let $\kappa \in \mathbb{R}$ and let us consider a smooth function G which approximates the function $x \mapsto |x - \kappa|$. By multiplying the first equation of (3.1) by $G'(u)\varphi$ with $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^* \times \mathbb{R})$ non-negative and integrating over $\mathbb{R}_+ \times \mathbb{R}$ we get

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \partial_t G(u) \, \varphi \, dx \, ds - \varepsilon \int_{\mathbb{R}_+} \int_{\mathbb{R}} \partial_{xx}^2 u \, G'(u) \, \varphi \, dx \, ds + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \partial_x \left(\frac{u^2}{2}\right) G'(u) \, \varphi \, dx \, ds$$
$$= \lambda \int_{\mathbb{R}_+} (h' - u(h)) \, G'(u(h)) \, \varphi(h) \, ds.$$

Then, multiplying the second equation of (3.1) by $G'(h')\varphi(h)$, we obtain

$$mh'' G'(h')\varphi(h) + \lambda(h' - u(h))G'(h')\varphi(h) = 0.$$

Summing both preceding equations allows to get

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \partial_t G(u) \, \varphi \, dx \, ds + m \int_{\mathbb{R}_+} \partial_t G(h') \, \varphi(h) \, ds + \varepsilon \int_{\mathbb{R}_+} \int_{\mathbb{R}} (\partial_x u)^2 G''(u) \, \varphi \, dx \, ds$$
$$- \varepsilon \int_{\mathbb{R}_+} \int_{\mathbb{R}} \partial_{xx}^2 G(u) \, \varphi \, dx \, ds + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \partial_x F(u)\varphi \, dx \, ds$$
$$+ \lambda \int_{\mathbb{R}_+} (h' - u(h))(G'(h') - G'(u(h))) \, \varphi(h) \, ds = 0,$$

where

$$F(x) = \int_\kappa^x G'(z)z \, dz.$$

We deduce from the above equation that

$$-\int_{\mathbb{R}_+} \int_{\mathbb{R}} G(u) \, \partial_t \varphi \, dx \, ds - m \int_{\mathbb{R}_+} G(h') \, \partial_t(\varphi(h)) \, ds - \int_{\mathbb{R}_+} \int_{\mathbb{R}} F(u)\partial_x \varphi \, dx \, ds$$
$$\leqslant \varepsilon \int_{\mathbb{R}_+} \int_{\mathbb{R}} G(u) \, \partial_{xx}^2 \varphi \, dx \, ds,$$

and therefore, passing to the limit $G \to |\cdot - \kappa|$ we get the entropy inequality for the viscous problem,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u - \kappa| \, \partial_t \varphi \, dx \, ds + m \int_{\mathbb{R}_+} |h' - \kappa| \, \partial_t(\varphi(h)) \, ds + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \mathrm{sgn}(u - \kappa)\frac{u^2 - \kappa^2}{2}\partial_x \varphi \, dx \, ds$$
$$\geqslant -\varepsilon \int_{\mathbb{R}_+} \int_{\mathbb{R}} |u - \kappa| \, \partial_{xx}^2 \varphi \, dx \, ds.$$

If we take *formally* the limit of the above inequality as $\varepsilon \to 0$, we obtain the following entropy inequalities for the inviscid problem

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u - \kappa| \, \partial_t \varphi \, dx \, ds + m \int_{\mathbb{R}_+} |h' - \kappa| \, \partial_t(\varphi(h)) \, ds + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(\mathrm{sgn}(u - \kappa)\frac{u^2 - \kappa^2}{2}\right) \partial_x \varphi \, dx \, ds \geqslant 0, \tag{3.2}$$

for all $\kappa \in \mathbb{R}$. In the sense of distributions, we have obtained (2.4).

3.2 How to handle the non-conservative product

We now focus on the non-conservative product in the first equation in (1.1).

By noting that $\delta_{h(t)}(x) = \partial_x H_{h(t)}(x)$ where $H_{h(t)}(x) = H(x - h(t)) = 0$ if $x \leqslant h(t)$, $H_{h(t)}(x) = H(x - h(t)) = 1$ if $x > h(t)$ (H is the classical Heaviside function), we can rewrite the problem (1.1) as

$$\begin{cases} \partial_t u + \partial_x(u^2/2) - \lambda(h' - u)\partial_x w = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ \partial_t w + h'\,\partial_x w = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ mh'' = -\lambda(h' - u(h)), & t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ w(0,x) = H(x), & x \in \mathbb{R}, \\ (h(0), h'(0)) = (0, v_0). \end{cases} \tag{3.3}$$

We study here the first two equations. The corresponding system is non-conservative, homogeneous, non-autonomous, and its quasi-linear form is

$$\partial_t \begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} u & -\lambda(h' - u) \\ 0 & h' \end{pmatrix} \partial_x \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

Lemma 3.1. *The system (3.4) has two eigenvalues u and h' and is hyperbolic:*

- *If $u \neq h'$, then it is strictly hyperbolic, the field associated with u is genuinely non-linear, the one associated with h' is linearly degenerate.*

- *If $u = h'$, then the corresponding matrix is $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$.*

This result is obtained by simple manipulations and we omit its proof.

The main difficulty is that the eigenvalues are not ordered. Therefore, when they identify, an accurate study of the double wave must be performed. As in [21, 14, 27, 5], we regularize w and we study the traveling wave corresponding to the velocity $h'(t)$ (depending on time t). More precisely, for $\varepsilon > 0$ we consider an approximation H^ε of H satisfying:

- $H^\varepsilon \in \mathcal{C}^0(\mathbb{R}) \cap \mathcal{C}^1([-\varepsilon/2, \varepsilon/2])$,

- H^ε is an increasing function on $[-\varepsilon/2, \varepsilon/2]$,

- $H^\varepsilon(x) = 0$ if $x \leqslant -\varepsilon/2$ and $H_\varepsilon(x) = 1$ if $x \geqslant \varepsilon/2$.

Using this approximation, we consider the following system

$$\begin{cases} \partial_t u^\varepsilon + \partial_x((u^\varepsilon)^2/2) - \lambda(h' - u^\varepsilon)\partial_x w^\varepsilon = 0, \\ w^\varepsilon(t,x) = H^\varepsilon(x - h(t)). \end{cases} \tag{3.5}$$

In the spirit of traveling waves studies, we look for solutions u^ε which follow the trajectory $h(t)$, of the form $u^\varepsilon(t,x) = U^\varepsilon(x - h(t))$. More precisely, we are interested in the restriction to $[-\varepsilon/2, \varepsilon/2]$ of the functions $U^\varepsilon(\xi)$ satisfying for all $t > 0$

$$-h'(t)(U^\varepsilon)'(\xi) + ((U^\varepsilon)^2/2)'(\xi) - \lambda(h'(t) - U^\varepsilon(\xi))(H^\varepsilon)'(\xi) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}), \tag{3.6}$$

$$U^\varepsilon(-\varepsilon/2) = U_L. \tag{3.7}$$

Since the flux in (3.5) is strictly convex, we naturally restrict the study to Lax solutions, that is to say solutions with bounded variation whose (possible) discontinuities are decreasing. The following lemma is an easy consequence of this hypothesis and of (3.6).

Lemma 3.2. *On every interval where U^ε is smooth, it satisfies*

$$(U^\varepsilon - h')(U^\varepsilon + \lambda H^\varepsilon)' = 0. \tag{3.8}$$

If U^ε is discontinuous in ξ_0, then one has

$$U^\varepsilon(\xi_0^+) + U^\varepsilon(\xi_0^-) = 2h'(t) \quad \text{and} \quad U^\varepsilon(\xi_0^-) > U^\varepsilon(\xi_0^+). \tag{3.9}$$

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We now state a result characterizing admissible jumps across the particle.

Proposition 3.3. *Let $\mathcal{U}_0(\bar{U}, \lambda, v) \subset \mathbb{R}$ be the set defined by*

$$\mathcal{U}_0(\bar{U}, \lambda, v) = \begin{cases} \{\bar{U} - \lambda\} & \text{if } \bar{U} < v, \\ [2v - \bar{U} - \lambda, v] & \text{if } v \leqslant \bar{U} \leqslant v + \lambda, \\ \{\bar{U} - \lambda\} \cup [2v - \bar{U} - \lambda, 2v - \bar{U} + \lambda] & \text{if } \bar{U} > v + \lambda. \end{cases} \tag{3.10}$$

If U^ε is a solution of (3.6)-(3.7) then one has

$$U^\varepsilon(\varepsilon/2) \in \mathcal{U}_0(U_L, \lambda, h'(t)) \quad \text{for all } t \in \mathbb{R}_+. \tag{3.11}$$

Conversely, for any $U_R \in \mathcal{U}_0(U_L, \lambda, h'(t))$ there exists a unique solution U^ε of (3.6)-(3.7) such that $U^\varepsilon(\varepsilon/2) = U_R$.

Proof. Consider a solution U^ε of (3.6)-(3.7). From (3.8) and from the fact that the discontinuities are decreasing, we deduce that U^ε is non-increasing. This fact and (3.9) imply in particular that U^ε admits at most one discontinuity.

  i. Assume that U^ε has no discontinuity in $[-\varepsilon/2, \varepsilon/2]$. Then, if $U_L > h'(t) + \lambda$ or if $U_L < h'(t)$, from (3.8) we have $(U^\varepsilon)' = -\lambda(H^\varepsilon)'$ and thus $U^\varepsilon(\varepsilon/2) = U_L - \lambda$. Else, i.e. if $h'(t) \leqslant U_L \leqslant h'(t) + \lambda$, we have $U^\varepsilon = h'(t)$ in an interval I and $(U^\varepsilon)' = -\lambda(H^\varepsilon)'$ outside I. Therefore, $U^\varepsilon(\varepsilon/2) \in [U_L - \lambda, h'(t)]$.

  ii. Assume that U^ε has a discontinuity at $\xi = \xi_0 \in [-\varepsilon/2, \varepsilon/2]$. Then (3.9) yields $U_L > h'(t)$ and $(U^\varepsilon)' = -\lambda(H^\varepsilon)'$ on $[-\varepsilon/2, \varepsilon/2] \setminus \{\xi_0\}$. We have $U^\varepsilon(\xi_0^-) = U_L - \alpha\lambda$ with $\alpha \in [0, 1]$ such that

$$U_L - \alpha\lambda > h'(t). \tag{3.12}$$

  Then from (3.9) we have $U^\varepsilon(\xi_0^+) = 2h'(t) - U_L + \alpha\lambda$ and from (3.8), we get $U^\varepsilon(\varepsilon/2) = 2h'(t) - U_L + (2\alpha - 1)\lambda$. If $h'(t) < U_L \leqslant h'(t) + \lambda$ then (3.12) implies that $\alpha < \frac{U_L - h'(t)}{\lambda}$ and thus that $U^\varepsilon(\varepsilon/2) \in [2h'(t) - U_L - \lambda, U_L - \lambda)$. If $U_L > h'(t) + \lambda$ then (3.12) holds for any $\alpha \in [0, 1]$ and thus $U^\varepsilon(\varepsilon/2) \in [2h'(t) - U_L - \lambda, 2h'(t) - U_L + \lambda]$.

Gathering both cases, we get (3.11) which concludes the proof of the proposition. $\quad\square$

Remark 1. We notice that the set $\mathcal{U}_0$ does not depend on ε neither on w^ε, which indicates a strong stability of the following construction of the Riemann solutions.

3.3 Definition of solution for Problem (1.1)

We now propose a definition of solution for Problem (1.1), using the entropy inequalities (3.2) in the spirit of Kružkov [22] and the definition of the non-conservative product provided by Proposition 3.3.

Definition 3.1. Assume that $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ and $v_0 \in \mathbb{R}$. A pair $(u, h) \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}) \times \mathcal{C}^1(\mathbb{R}_+)$ is called an *entropy solution* of the problem (1.1) if it satisfies the two following relations:

D1 For any $\kappa \in \mathbb{R}$ and for any non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R})$, one has

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u - \kappa| \, \partial_t \varphi \, dx \, ds + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(\text{sgn}(u - \kappa) \frac{u^2 - \kappa^2}{2} \right) \partial_x \varphi \, dx \, ds$$
$$+ m \int_{\mathbb{R}_+} |h' - \kappa| \, \partial_t(\varphi(h)) \, ds + \int_{\mathbb{R}} |u_0(x) - \kappa| \varphi(0, x) dx + m|h'(0) - \kappa| \, \varphi(0, h(0)) \geqslant 0. \tag{3.13}$$

D2 The left and right traces $u(t, h(t)^\pm)$ of u satisfy:

$$u(t, h(t)^+) \in \mathcal{U}_0(u(t, h(t)^-), \lambda, h'(t)). \tag{3.14}$$

Remark 2. Let us define $\Sigma = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}, x = h(t)\}$. If u verifies (3.13), it is an entropy solution of the Burgers equation in $(\mathbb{R}_+ \times \mathbb{R}) \setminus \Sigma$. Therefore, following Vasseur [31], u admits traces on each side of Σ; in particular, relation (3.14) is meaningful.

The entropy criterion should permit to select a unique solution. However the uniqueness of solution will not be studied here.

Actually, Definition 3.1 is difficult to handle in the purpose of constructing explicit solutions. One may notice that, surprisingly, the ODE which governs the velocity of the particle in (1.1) explicitly appears neither in **D1** nor in **D2**. Actually, it is included in **D1**, as stated in the following Proposition.

Proposition 3.4. *Assume that $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ and $v_0 \in \mathbb{R}$. Consider a pair $(u,h) \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}) \times \mathcal{C}^1(\mathbb{R}_+)$ such that* **D2** *holds. Then, (u,h) satisfies* **D1** *if and only if it satisfies* **D1a** *and* **D1b**:

D1a *For any $\kappa \in \mathbb{R}$ and for any non-negative function $\varphi \in \mathcal{C}_c^\infty((\mathbb{R} \times \mathbb{R}) \setminus \Sigma)$, one has*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u - \kappa| \, \partial_t \varphi \, dx \, ds + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(sgn(u - \kappa) \frac{u^2 - \kappa^2}{2} \right) \partial_x \varphi \, dy \, ds + \int_{\mathbb{R}} |u_0(x) - \kappa| \varphi(0,x) dx \geqslant 0,$$

(3.15)

where $\Sigma = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}, x = h(t)\}$.

D1b *The trajectory of the particle h is solution of*

$$\begin{cases} mh''(t) = (u_-(t) - u_+(t)) \left(\dfrac{u_-(t) + u_+(t)}{2} - h'(t) \right), & t > 0, \\ (h(0), h'(0)) = (0, v_0). \end{cases}$$

(3.16)

where $u_-(t)$ and $u_+(t)$ denote the left and right traces of $u(t,x)$ at $x = h(t)$, i.e. $u(t, h(t)^-)$ and $u(t, h(t)^+)$.

See Appendix B for the proof.

It is worth noting that this proposition provides an alternative definition of solution (which will be used in the construction of Riemann solutions).

Remark 3. The first equation of (3.16) implies in particular that the velocity of the particle is relaxed toward the mean of the right and left values of u at $h(t)$. The instantaneous rate of relaxation is $(u_-(t) - u_+(t))/m \geqslant 0$ in every configurations.

4 Case of a particle with a given constant velocity

In this section, the particle velocity h' is *given* and is assumed to be *constant*. This hypothesis drastically simplifies the Riemann problem and guarantees the *self-similarity* of solutions. Though this problem is not a sub-case of Problem (1.1), its solutions will be the cornerstone for the construction of the Riemann solutions to (1.1).

4.1 Definition of the solution

Let $v_0 \in \mathbb{R}$ and let $h(t) = v_0 t$ be the position of the particle. The problem under study in this section is:

$$\begin{cases} \partial_t u + \partial_x(u^2/2) - \lambda(v_0 - u)\, \partial_x w = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ \partial_t w + v_0 \partial_x w = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases} \\ w(0,x) = H(x), & x \in \mathbb{R}. \end{cases}$$

(4.1)

We directly have $w(t,x) = H(x - v_0 t)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Since the trajectory of the particle is known, Proposition 3.4 leads us to the following characterization:

D1a The function u is an entropy weak solution of the Burgers equation in $(\mathbb{R}_+ \times \mathbb{R}) \setminus \Sigma_{v_0}$, where $\Sigma_{v_0} = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}, x = v_0 t\}$. More precisely, for any $\kappa \in \mathbb{R}$ and for any non-negative function $\varphi \in \mathcal{C}_c^\infty((\mathbb{R} \times \mathbb{R}) \setminus \Sigma_{v_0})$, one has

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u-\kappa|\, \partial_t \varphi \, dx \, ds + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(\mathrm{sgn}(u-\kappa) \frac{u^2 - \kappa^2}{2} \right) \partial_x \varphi \, dy \, ds + \int_{\mathbb{R}} |u_0(x) - \kappa| \varphi(0, x) dx \geqslant 0.$$

D2 The left and right traces $u(t, (v_0 t)^\pm)$ of u satisfy

$$u(t, (v_0 t)^+) \in \mathcal{U}_0(u(t, (v_0 t)^-), \lambda, v_0). \tag{4.2}$$

Following the classical approach for Riemann problems (see [29] and [15] for instance), we look for a solution u with bounded variation, self-similar, composed by constant states separated by waves. These waves can be either a classical Burgers wave (satisfying the Lax entropy condition) or the wave associated with the particle (satisfying (4.2)). Using the classical self-similar parametrization

$$\mathbb{U}(x/t) = u(t, x),$$

we can write Relation (4.2) as

$$\mathbb{U}(v_0^+) \in \mathcal{U}_0(\mathbb{U}(v_0^-), \lambda, v_0). \tag{4.3}$$

We now characterize the set $\mathcal{U}_-(u_L, v_0)$ (respectively $\mathcal{U}_+(u_R, v_0)$) of states $\mathbb{U}(v_0^-)$ (resp. $\mathbb{U}(v_0^+)$) that can be joined to u_L (resp. u_R) via a simple wave of the Burgers equation. In order to do this, we denote by $\mathbb{W}(x/t; u_a, u_b)$ the self-similar solution of the Riemann problem

$$\begin{cases} \partial_t u + \partial_x(u^2/2) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = \begin{cases} u_a & \text{if } x < 0, \\ u_b & \text{if } x > 0, \end{cases} \end{cases} \tag{4.4}$$

and we set

$$\mathcal{U}_-(u_L, v_0) = \{\mathbb{W}(v_0^-; u_L, \overline{u}), \overline{u} \in \mathbb{R}\} \tag{4.5}$$

$$\mathcal{U}_+(u_R, v_0) = \{\mathbb{W}(v_0^+; \overline{u}, u_R), \overline{u} \in \mathbb{R}\}. \tag{4.6}$$

Lemma 4.1. *For any $v_0 \in \mathbb{R}$, one has*

$$\mathcal{U}_-(u_L, v_0) = \begin{cases} (-\infty, v_0] & \text{if } u_L \leqslant v_0, \\ \{u_L\} \cup (-\infty, 2v_0 - u_L) & \text{if } u_L > v_0, \end{cases} \tag{4.7}$$

and

$$\mathcal{U}_+(u_R, v_0) = \begin{cases} [v_0, +\infty) & \text{if } u_R \geqslant v_0, \\ \{u_R\} \cup (2v_0 - u_R, +\infty) & \text{if } u_R < v_0. \end{cases} \tag{4.8}$$

Proof. We perform the computation for $\mathcal{U}_-(u_L, v_0)$, the set $\mathcal{U}_+(u_R, v_0)$ can be obtained by symmetry. We obtain the set $\mathcal{U}_-(u_L, v_0)$ by giving the explicit formula of $\mathbb{W}(\xi; u_L, \overline{u})$ which depends on the value of $\overline{u} \in \mathbb{R}$.

- Let us first assume that $u_L \leqslant v_0$,

  - if $\overline{u} < u_L$, then the solution develops a shock with velocity $\sigma := (u_L + \overline{u})/2$:

    $$\mathbb{W}(\xi; u_L, \overline{u}) = \begin{cases} u_L & \text{if } \xi < \sigma, \\ \overline{u} & \text{if } \xi > \sigma. \end{cases}$$

    Since $\sigma < v_0$, the above formula implies that $\mathbb{W}(v_0; u_L, \overline{u}) = \overline{u}$, and thus $(-\infty, u_L) \subset \mathcal{U}_-(u_L, , v_0)$.

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– if $\overline{u} \geqslant u_L$, the solution develops a rarefaction:

$$\mathbb{W}(\xi; u_L, \overline{u}) = \begin{cases} u_L & \text{if } \xi \leqslant u_L, \\ \xi & \text{if } u_L \leqslant \xi \leqslant \overline{u}, \\ \overline{u} & \text{if } \xi \geqslant \overline{u}. \end{cases}$$

In particular $\mathbb{W}(v_0; u_L, \overline{u}) = v_0$ if $v_0 \leqslant \overline{u}$ and $\mathbb{W}(v_0; u_L, \overline{u}) = \overline{u}$ if $v_0 > \overline{u}$, which implies $[u_L, v_0] \subset \mathcal{U}_-(u_L, v_0)$.

Gathering both cases, we conclude that $\mathcal{U}_-(u_L, v_0) = (-\infty, v_0]$ if $u_L \leqslant v_0$.

• Let us now assume that $u_L > v_0$,

– if $\overline{u} < u_L$, the solution develops a shock with velocity $\sigma = (u_L + \overline{u})/2$:

$$\mathbb{W}(\xi; u_L, \overline{u}) = \begin{cases} u_L & \text{if } \xi < \sigma, \\ \overline{u} & \text{if } \xi > \sigma. \end{cases}$$

The above relation yields $\mathbb{W}(v_0; u_L, \overline{u}) = \overline{u}$ if $\overline{u} < 2v_0 - u_L$ and $\mathbb{W}(v_0; u_L, \overline{u}) = u_L$ if $\overline{u} > 2v_0 - u_L$. If $\overline{u} = 2v_0 - u_L$, then $\mathbb{W}(v_0^-; u_L, \overline{u}) = u_L$. We conclude that $\{u_L\} \cup (-\infty, 2v_0 - u_L) \subset \mathcal{U}_-(u_L, v_0)$.

– if $\overline{u} \geqslant u_L$, the solution develops a rarefaction:

$$\mathbb{W}(\xi; u_L, \overline{u}) = \begin{cases} u_L & \text{if } \xi \leqslant u_L, \\ \xi & \text{if } u_L \leqslant \xi \leqslant \overline{u}, \\ \overline{u} & \text{if } \xi \geqslant \overline{u}, \end{cases}$$

and we get $\mathbb{W}(v_0; u_L, \overline{u}) = u_L$.

Thus $\mathcal{U}_-(u_L, v_0) = \{u_L\} \cup (-\infty, 2v_0 - u_L)$.

$\square$

We have also the following result (which will be used for the resolution of the Riemann problem).

Lemma 4.2. *Let us consider the set*

$$\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) = \{u_+ \in \mathcal{U}_0(u_-, \lambda, v_0) \; ; \; u_- \in \mathcal{U}_-(u_L, v_0)\}.$$

Then,

• *If $u_L \leqslant v_0 + \lambda$ then $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) = (-\infty, v_0]$.*

• *If $u_L > v_0 + \lambda$ then $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) = \{u_L - \lambda\} \cup (-\infty, 2v_0 - u_L + \lambda]$.*

Moreover for any $u_+ \in \mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0)$, there exists a unique $u_- \in \mathcal{U}_-(u_L, v_0)$ such that $u_+ \in \mathcal{U}_0(u_-, \lambda, v_0)$.

Proof. We consider three cases:

Case 1: $u_L \leqslant v_0$. Then $\mathcal{U}_-(u_L, v_0) = (-\infty, v_0]$. Moreover,

$$\mathcal{U}_0((-\infty, v_0]) = (-\infty, v_0 - \lambda), \quad \mathcal{U}_0(\{v_0\}) = [v_0 - \lambda, v_0].$$

We deduce that $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) = (-\infty, v_0]$.

Case 2: $v_0 < u_L \leqslant v_0 + \lambda$. Then $\mathcal{U}_-(u_L, v_0) = (-\infty, 2v_0 - u_L) \cup \{u_L\}$. Moreover,

$$\mathcal{U}_0((-\infty, 2v_0 - u_L)) = (-\infty, 2v_0 - u_L - \lambda), \quad \mathcal{U}_0(\{u_L\}) = [2v_0 - u_L - \lambda, v_0].$$

We deduce that $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) = (-\infty, v_0]$.

Case 3: $u_L > v_0 + \lambda$. Then $\mathcal{U}_-(u_L, v_0) = (-\infty, 2v_0 - u_L) \cup \{u_L\}$. Moreover,

$$\mathcal{U}_0((-\infty, 2v_0 - u_L)) = (-\infty, 2v_0 - u_L - \lambda), \quad \mathcal{U}_0(\{u_L\}) = \{u_L - \lambda\} \cup [2v_0 - u_L - \lambda, 2v_0 - u_L + \lambda].$$

We deduce that $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) = (-\infty, 2v_0 - u_L + \lambda] \cup \{u_L - \lambda\}$.

$\square$

4.2 Resolution of the Riemann problem

We are now in position to solve the Riemann problem (4.1).

Theorem 4.3. *Let us consider the Riemann problem* (4.1). *For every pair* $(u_L, u_R) \in \mathbb{R}^2$, *there exists a unique solution composed of constant states joined by simple waves, which fulfills* $\overline{\textbf{D1a}}$ *and* $\overline{\textbf{D2}}$. *It is given by* $u(t, x) = \mathbb{U}(x/t)$ *with* $\mathbb{U}$ *described by the formulas below.*

    i. If $u_L \leqslant v_0$ *and* $u_R \geqslant v_0$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant u_L, \\ \xi & \text{if } u_L < \xi \leqslant u_R, \\ u_R & \text{if } u_R < \xi. \end{cases} \tag{I}$$

    ii. If $u_L < v_0$ *and* $u_R < v_0$ *and* $u_R > v_0 - \lambda$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant u_L, \\ \xi & \text{if } u_L < \xi \leqslant v_0, \\ u_R & \text{if } v_0 < \xi. \end{cases} \tag{II}$$

    iii. If $u_R < v_0 - \lambda$ *and* $u_R \geqslant u_L - \lambda$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant u_L, \\ \xi & \text{if } u_L < \xi \leqslant u_R + \lambda, \\ u_R + \lambda & \text{if } u_R + \lambda < \xi \leqslant v_0, \\ u_R & \text{if } v_0 < \xi. \end{cases} \tag{III}$$

    iv. If $u_R < u_L - \lambda$ *and* $u_R < 2v_0 - u_L - \lambda$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant \frac{u_L + u_R + \lambda}{2}, \\ u_R + \lambda & \text{if } \frac{u_L + u_R + \lambda}{2} < \xi \leqslant v_0, \\ u_R & \text{if } v_0 < \xi. \end{cases} \tag{IV}$$

    v. If $u_L \geqslant v_0$ *and* $u_R \leqslant v_0$ *and* $u_R \geqslant 2v_0 - u_L - \lambda$ *and* $u_R \leqslant 2v_0 - u_L + \lambda$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant v_0, \\ u_R & \text{if } v_0 < \xi. \end{cases} \tag{V}$$

    vi. If $u_L > v_0$ *and* $u_L \leqslant v_0 + \lambda$ *and* $u_R > v_0$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant v_0, \\ \xi & \text{if } v_0 < \xi \leqslant u_R, \\ u_R & \text{if } u_R < \xi. \end{cases} \tag{VI}$$

    vii. If $u_L > v_0 + \lambda$ *and* $u_R \geqslant u_L - \lambda$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant v_0, \\ u_L - \lambda & \text{if } v_0 < \xi \leqslant u_L - \lambda, \\ \xi & \text{if } u_L - \lambda < \xi \leqslant u_R, \\ u_R & \text{if } u_R < \xi. \end{cases} \tag{VII}$$

    viii. If $u_R < u_L - \lambda$ *and* $u_R > 2v_0 - u_L + \lambda$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant v_0, \\ u_L - \lambda & \text{if } v_0 < \xi \leqslant \frac{u_L + u_R - \lambda}{2}, \\ u_R & \text{if } \frac{u_L + u_R - \lambda}{2} < \xi. \end{cases} \tag{VIII}$$

Proof. The proof consists in looking for the pairs $(u_-, u_+) \in \mathcal{U}_-(u_L, v_0) \times \mathcal{U}_+(u_R, v_0)$ such that

$$u_+ \in \mathcal{U}_0(u_-, \lambda, v_0). \tag{4.9}$$

To achieve this, we have to consider all the possible configurations for the triplet $(u_L, \ u_R, \ v_0) \in \mathbb{R}^3$.

We first notice that $u_+ \in \mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) \cap \mathcal{U}_+(u_R, v_0)$ and that from Lemma 4.2, for any u_+ in this intersection, there exists a unique $u_- \in \mathcal{U}_-(u_L, v_0)$ such that (4.9) holds. Using Lemma 4.2 and (4.8), we easily get

(α). If $u_L \leqslant v_0 + \lambda$ and $u_R \geqslant v_0$ then $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) \cap \mathcal{U}_+(u_R, v_0) = \{v_0\}$.

(β). If $u_L \leqslant v_0 + \lambda$ and $u_R < v_0$ then $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) \cap \mathcal{U}_+(u_R, v_0) = \{u_R\}$.

(γ). If $u_L > v_0 + \lambda$ and $u_R \geqslant v_0$ then $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) \cap \mathcal{U}_+(u_R, v_0) = \{u_L - \lambda\}$.

(δ). If $u_L > v_0 + \lambda$ and $u_R < v_0$ then

$$\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) \cap \mathcal{U}_+(u_R, v_0) = (\{u_L - \lambda\} \cap (2v_0 - u_R, \infty)) \cup (\{u_R\} \cap (-\infty, 2v_0 - u_L + \lambda]).$$

We consider two sub-cases:

(δ1). If $u_R \leqslant 2v_0 - u_L + \lambda$ then $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) \cap \mathcal{U}_+(u_R, v_0) = \{u_R\}$.

(δ2). If $u_R > 2v_0 - u_L + \lambda$ then $\mathcal{U}_0(\mathcal{U}_-(u_L, v_0), \lambda, v_0) \cap \mathcal{U}_+(u_R, v_0) = \{u_L - \lambda\}$.

We can now consider the 8 cases of the theorem:

i. $u_L \leqslant v_0$ and $u_R \geqslant v_0$.
We are in the situation (α) above so $u_+ = v_0$ and a straightforward calculation shows that $u_- = v_0$. In that case, the solution is a rarefaction around the particle:

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant u_L, \\ \xi & \text{if } u_L < \xi \leqslant u_R, \\ u_R & \text{if } u_R < \xi. \end{cases} \tag{I}$$
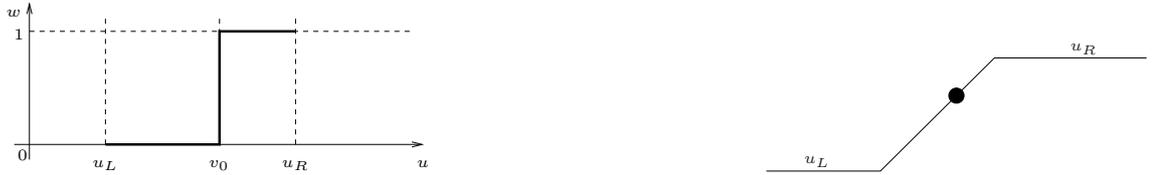
Figure 1: Solution in the case (I), in the (u, w)-coordinates and in the (ξ, u)-coordinates.

ii. $u_L < v_0$ and $u_R < v_0$ and $u_R \geqslant v_0 - \lambda$.
We are in the situation (β) above so $u_+ = u_R$. Using the proof of Lemma 4.2, we get $u_- = v_0$.

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant u_L, \\ \xi & \text{if } u_L < \xi \leqslant v_0, \\ u_R & \text{if } v_0 < \xi. \end{cases} \tag{II}$$

Figure 2: Solution in the case (II), in the (u, w)-coordinates and in the (ξ, u)-coordinates.

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iii. $u_R < v_0 - \lambda$ and $u_R \geqslant u_L - \lambda$. We necessarily have $u_L < v_0$.

We are again in the situation (β) above so $u_+ = u_R$. Using the proof of Lemma 4.2, we get $u_- = u_+ + \lambda = u_R + \lambda$.

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant u_L, \\ \xi & \text{if } u_L < \xi \leqslant u_R + \lambda, \\ u_R + \lambda & \text{if } u_R + \lambda < \xi \leqslant v_0, \\ u_R & \text{if } v_0 < \xi. \end{cases} \tag{III}$$

Figure 3: Solution in the case (III), in the (u, w)-coordinates and in the (ξ, u)-coordinates.

iv. $u_R < u_L - \lambda$ and $u_R < 2v_0 - u_L - \lambda$. First remark that $u_R < v_0 - \lambda$.

We are either in the situation (β) or in the situation $(\delta 1)$ above so $u_+ = u_R$. By considering the three cases $u_L \leqslant v_0$, $v_0 < u_L \leqslant v_0 + \lambda$ and $u_L > v_0 + \lambda$ in the proof of Lemma 4.2, we easily get $u_- = u_+ + \lambda$. Therefore $u_- = u_R + \lambda$.

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant \frac{u_L + u_R + \lambda}{2}, \\ u_R + \lambda & \text{if } \frac{u_L + u_R + \lambda}{2} < \xi \leqslant v_0, \\ u_R & \text{if } v_0 < \xi. \end{cases} \tag{IV}$$

Figure 4: Solution in the case (IV), in the (u, w)-coordinates and in the (ξ, u)-coordinates (the general shape of this solution does not depend on the sub-case considered in the proof; here the second one is considered: $v_0 < u_L \leqslant v_0 + \lambda$).

v. $u_L \geqslant v_0$ and $u_R \leqslant v_0$ and $u_R \geqslant 2v_0 - u_L - \lambda$ and $u_R \leqslant 2v_0 - u_L + \lambda$.

- If $u_R = v_0$, then we are in the situation (α) above so $u_+ = v_0 = u_R$.
- If $u_R < v_0$, then we are either in the situation (β) or in the situation $(\delta 1)$ above so $u_+ = u_R$.

To obtain u_-, we consider again the three cases $u_L = v_0$, $v_0 < u_L \leqslant v_0 + \lambda$ and $u_L > v_0 + \lambda$ in the proof of Lemma 4.2. We obtain in each case $u_- = u_L$.

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant v_0, \\ u_R & \text{if } v_0 < \xi. \end{cases} \tag{V}$$

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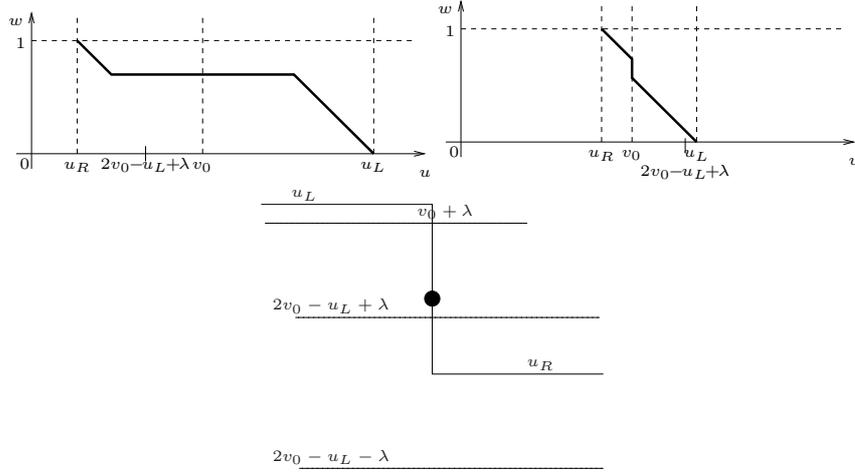
Figure 5: Solution in the case (V), in the (u, w)-coordinates (case $u_R < u_L - \lambda$ and case $u_R > u_L + \lambda$) and in the (ξ, u)-coordinates.

We obtain the result in the cases (VI), (VII) and (VIII) by using the symmetry of the problem with respect to the line $u_R = 2v_0 - u_L$. Indeed, assume that $(u_-, u_+) \in \mathcal{U}_-(u_L, v_0) \times \mathcal{U}_+(u_R, v_0)$ is such that

$$u_+ \in \mathcal{U}_0(u_-, \lambda, v_0).$$

Then we have

$$2v_0 - u_+ \in \mathcal{U}_-(2v_0 - u_R, v_0), \quad 2v_0 - u_- \in \mathcal{U}_+(2v_0 - u_L, v_0),$$

and

$$2v_0 - u_- \in \mathcal{U}_0(2v_0 - u_+, \lambda, v_0).$$

This implies that we can exchange (u_L, u_R) by $(2v_0 - u_R, 2v_0 - u_L)$.

Consequently, we have

vi. If $u_L > v_0$ and $u_L \leqslant v_0 + \lambda$ and $u_R \geqslant v_0$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant v_0, \\ \xi & \text{if } v_0 < \xi \leqslant u_R, \\ u_R & \text{if } u_R < \xi. \end{cases} \tag{VI}$$

Figure 6: Solution in the case (VI), in the (u, w)-coordinates and in the (ξ, u)-coordinates.

vii. If $u_L > v_0 + \lambda$ and $u_R \geqslant u_L - \lambda$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant v_0, \\ u_L - \lambda & \text{if } v_0 < \xi \leqslant u_L - \lambda, \\ \xi & \text{if } u_L - \lambda < \xi \leqslant u_R, \\ u_R & \text{if } u_R < \xi. \end{cases} \tag{VII}$$

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Figure 7: Solution in the case (VII), in the (u, w)-coordinates and in the (ξ, u)-coordinates.

viii. If $u_R < u_L - \lambda$ and $u_R > 2v_0 - u_L + \lambda$,

$$\mathbb{U}(\xi) = \begin{cases} u_L & \text{if } \xi \leqslant v_0, \\ u_L - \lambda & \text{if } v_0 < \xi \leqslant \frac{u_L + u_R - \lambda}{2}, \\ u_R & \text{if } \frac{u_L + u_R - \lambda}{2} < \xi. \end{cases} \tag{VIII}$$

Figure 8: Solution in the case (VIII), in the (u, w)-coordinates and in the (ξ, u)-coordinates.

$\square$

We can illustrate this result by drawing the corresponding regions:
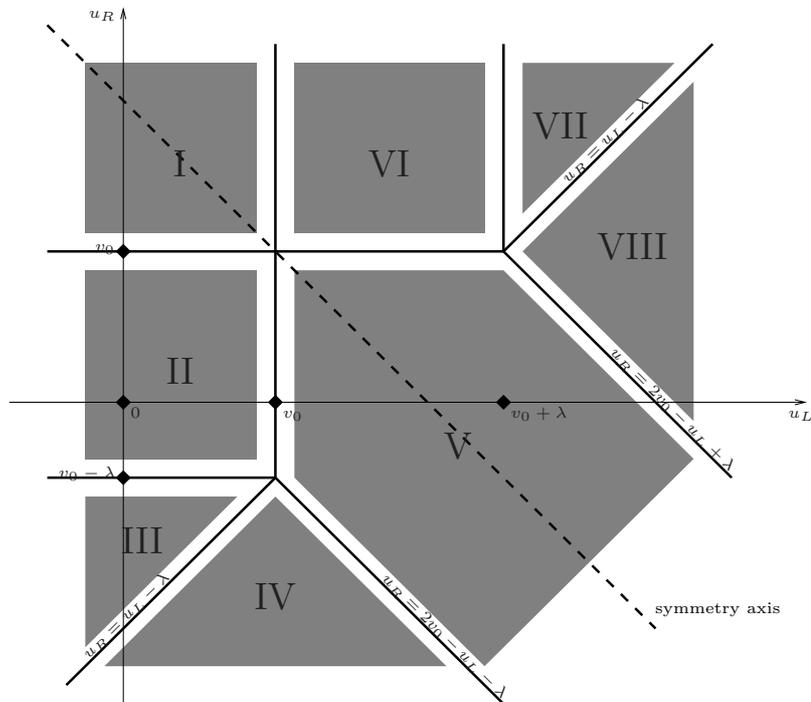
Figure 9: Partition of the (u_L, u_R)-plane.

5 Riemann problem for the complete system

In this section we show the existence of solution for the problem (1.1) when the initial condition is

$$
\begin{cases}
u(0, x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases} \\
(h(0), h'(0)) = (0, v_0).
\end{cases}
\tag{5.1}
$$

We seek for solutions which verify **D1-D2** (or alternatively **D1a-D1b-D2**) as functions composed by constant states separated by waves (shocks, rarefaction waves and discontinuity along the particle trajectory). The main result of this section is the following global existence of such solutions.

Theorem 5.1. *For any $\lambda > 0$ and for any $(u_L, u_R, v_0) \in \mathbb{R}^3$ in the initial condition (5.1), there exists a solution (u, h) of (1.1) in the sense of Definition 3.1.*

Besides, we have the two asymptotic behaviors for the constructed solution (u, h).

Theorem 5.2.   • *Behavior for $t \to +\infty$.*

        – *If $u_L \leqslant u_R$, let us define $v_\infty = \max(u_L, \min(u_R, v_0))$ and u_∞ as the entropy solution of the uncoupled Burgers equation with the initial condition u_0. Then for all $\lambda > 0$,*

$$
\begin{aligned}
&\lim_{t \to +\infty} h'(t) = v_\infty, \\
&\lim_{t \to +\infty} \|u(t, \cdot) - u_\infty(t, \cdot)\|_{L^\infty(\mathbb{R})} = 0.
\end{aligned}
\tag{5.2}
$$

        – *If $u_L > u_R$, let us define $v_\infty = (u_L + u_R)/2$ and u_∞ as the entropy solution of the uncoupled Burgers equation with the initial condition*

$$
u_\infty(0, x) = u_0 \left(x - m \frac{v_0 - v_\infty}{u_L - u_R} \right).
\tag{5.3}
$$

    Then for all $\lambda > 0$,

$$
\begin{aligned}
&\lim_{t \to +\infty} h'(t) = v_\infty, \\
&\lim_{t \to +\infty} \|u(t, \cdot) - u_\infty(t, \cdot)\|_{L^1(\mathbb{R})} = 0.
\end{aligned}
$$

  • *Behavior for $\lambda \to +\infty$: there exists v_∞,*

$$
\begin{cases}
v_\infty = \max(u_L, \min(u_R, v_0)) & \text{if } u_L \leqslant u_R, \\
v_\infty = (u_L + u_R)/2 & \text{if } u_L > u_R,
\end{cases}
$$

  such that for all $t > 0$,

$$
\lim_{\lambda \to \infty} \|u(t, \cdot) - u_\infty(t, \cdot)\|_{L^1(\mathbb{R})} = 0, \qquad \lim_{\lambda \to \infty} (h)'(t) = v_\infty,
\tag{5.4}
$$

  where u_∞ is the entropy solution of the uncoupled Burgers equation with the initial condition u_0.

The sequel is devoted to the proof of the above results. First, we state two crucial lemmas related to condition **D2**.

Lemma 5.3. *Assume that $0 \leqslant \tau_1 < \tau_2$, that $h \in \mathcal{C}^1([\tau_1, \tau_2])$ and that $u_+ \in \mathcal{U}_0(u_-, \lambda, h'(\tau_1))$. Assume also that one of the following conditions is satisfied:*

  (a) *The function h' is non-increasing on $[\tau_1, \tau_2]$ and*

$$
h'(t) \geqslant \frac{u_- + u_+}{2} \quad \forall t \in [\tau_1, \tau_2].
\tag{5.5}
$$

(b) The function h' is non-decreasing on $[\tau_1, \tau_2]$ and

$$h'(t) \leqslant \frac{u_- + u_+}{2} \quad \forall t \in [\tau_1, \tau_2]. \tag{5.6}$$

Then

$$u_+ \in \mathcal{U}_0(u_-, \lambda, h'(t)) \quad \forall t \in [\tau_1, \tau_2].$$

Proof. We prove the lemma only in the first case, the other one can be done in a similar way. We have to consider three cases:

i. Assume that $u_- < h'(\tau_1)$. Then we have $u_+ = u_- - \lambda$ and $u_+ \in \mathcal{U}_0(u_-, \lambda, a)$ for any $a \in \mathbb{R}$.

ii. Assume that $h'(\tau_1) \leqslant u_- \leqslant h'(\tau_1) + \lambda$. Then we have $u_+ \in [2h'(\tau_1) - u_- - \lambda, h'(\tau_1)]$. Since h' is decreasing and since (5.5) holds, we have $u_+ \in [2h'(t) - u_- - \lambda, h'(t)]$ for all $t \in [\tau_1, \tau_2]$. On the other hand, we have $u_- \geqslant h'(t)$ for all $t \in [\tau_1, \tau_2]$ and consequently, $u_+ \in \mathcal{U}_0(u_-, \lambda, h'(t)) \quad \forall t \in [\tau_1, \tau_2]$.

iii. Assume that $u_- > h'(\tau_1) + \lambda$. If $u_+ = u_- - \lambda$ then $u_+ \in \mathcal{U}_0(u_-, \lambda, a)$ for any $a \in \mathbb{R}$. Else we have $u_+ \in [2h'(\tau_1) - u_- - \lambda, 2h'(\tau_1) - u_- + \lambda]$. Since h' is decreasing, $u_+ \geqslant 2h'(t) - u_- - \lambda$ for all $t \in [\tau_1, \tau_2]$. From (5.5), we also get

$$h'(t) + \frac{\lambda}{2} \geqslant \frac{u_- + u_+}{2} + \frac{\lambda}{2} > \frac{u_- + u_+}{2}$$

and thus $u_+ < 2h'(t) - u_- + \lambda$.

$\square$

Lemma 5.4. *Assume that $u_+ \in \mathbb{R}$ and that $\tau \in \mathbb{R}_+^* \cup \{\infty\}$ and consider two functions $u_- \in \mathcal{C}([0, \tau))$ and $h \in \mathcal{C}^1([0, \tau))$ such that u_- and h' are non-increasing. Suppose that $u_-(0) \leqslant u_+ + \lambda$, that $u_+ \in \mathcal{U}_0(u_-(0), \lambda, h'(0))$, that $u_-(t) \geqslant u_+$ for all $t \geqslant 0$ and that*

$$h'(t) \in \left[\frac{u_-(t) + u_+}{2}, u_-(t)\right] \quad \forall t \in [0, \tau). \tag{5.7}$$

Then

$$u_+ \in \mathcal{U}_0(u_-(t), \lambda, h'(t)) \quad \forall t \in [0, \tau).$$

Proof. From (5.7) and from the inequality $u_+ \geqslant u_-(t) - \lambda$, we get

$$h'(t) \geqslant u_-(t) - \frac{\lambda}{2} \geqslant u_-(t) - \lambda.$$

Combining the above equation with (5.7), we deduce that $h'(t) \in [u_-(t) - \lambda, u_-(t)]$ for all $t \in [0, \tau)$. This relation and the definition (3.10) imply

$$\mathcal{U}_0(u_-(t), \lambda, h'(t)) = [2h'(t) - u_-(t) - \lambda, h'(t)] \quad \forall t \in [0, \tau).$$

From (5.7) and from $u_-(t) \geqslant u_+$ for $t \in [0, \tau)$, we deduce $u_+ \leqslant h'(t)$. Finally using again (5.7) and the inequality $u_-(t) \leqslant u_+ + \lambda$, we conclude that $u_+ \geqslant 2h'(t) - u_-(t) - \lambda$. This ends the proof of the lemma. $\square$

To prove Theorem 5.1, we consider the eight cases obtained in the study for a particle with constant and given velocity (Theorem 4.3). The idea is to use here the same eight cases for (u_L, u_R, v_0) to deal with the complete system except that, now, v_0 is the velocity of the particle only at initial time.

Lemma 5.5 (Cases (I) and (V))**.**

Case (I). $u_L \leqslant v_0$ *and* $u_R \geqslant v_0$.
    Then the solution of (1.1) *is given by*

$$u(t,x) = \begin{cases} u_L & \text{if } x < u_L t, \\ x/t & \text{if } u_L t \leqslant x \leqslant u_R t, \\ u_R & \text{if } x > u_R t, \end{cases}$$

and $h(t) = v_0 t$ *for all* $t > 0$.

Case (V). $u_L \geqslant v_0$ *and* $u_R \leqslant v_0$ *and* $u_R \geqslant 2v_0 - u_L - \lambda$ *and* $u_R \leqslant 2v_0 - u_L + \lambda$.
    Then the solution of (1.1) *is given by*

$$u(t,x) = \begin{cases} u_L & \text{if } x < h(t), \\ u_R & \text{if } x > h(t), \end{cases}$$

and

$$h(t) = \frac{u_L + u_R}{2} t + \left(v_0 - \frac{u_L + u_R}{2} \right) \frac{m}{u_L - u_R} (1 - e^{-\frac{u_L - u_R}{m} t}) \tag{5.8}$$

for all $t > 0$.

These two cases are the simplest ones because they are "stable" in the sense that the solution stays in the same area of figure 9 when h' is varying according to its ODE.

For a better understanding of these solutions, their shapes have been drawn in the (x, t)-plane in Figures 10 and 11.
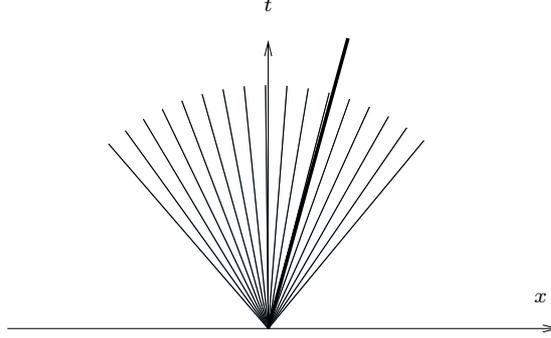
Figure 10: Solution in the case (I).
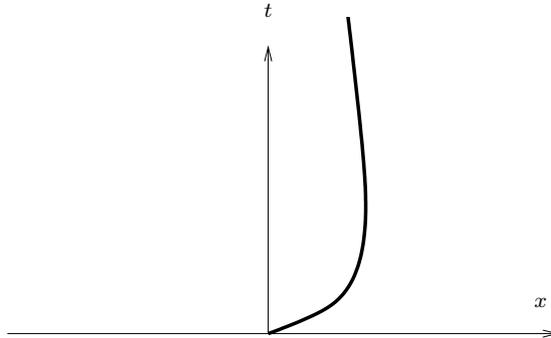
Figure 11: Solution in the case (V).

Proof. We first consider the case (I). Let us consider (u, h) given by the formulas of the lemma. In that case, u is a rarefaction wave so u satisfies **D1a**. Moreover, since $u(t, h(t)^-) = h'(t) = u(t, h(t)^+)$ and since $mh''(t) = 0$ then **D1b** is also satisfied. Finally, the relation $u(t, h(t)^-) = h'(t) =$

$u(t, h(t)^+)$ implies **D2**.

We now deal with the case (V). Property **D1a** is obviously satisfied. We differentiate the formula (5.8) to find that h is solution of (3.16) and thus that **D1b** holds. Besides, from the definition (3.10), we deduce that $u_L \in \mathcal{U}_0(u_R, \lambda, v_0)$. Using (5.8), we notice easily that Condition (a) or (b) of Lemma 5.3 is satisfied, according to whether $v_0 \geqslant \frac{u_L + u_R}{2}$ or not. We can thus apply Lemma 5.3 and we deduce that **D2** is verified. $\qquad \square$

Lemma 5.6 (Case (II)). *$u_L < v_0$ and $u_R < v_0$ and $u_R > v_0 - \lambda$.*
If $u_L \leqslant u_R$ then the solution of (1.1) is given by

$$u(t, x) = \begin{cases} u_L & \text{if } x < u_L t, \\ x/t & \text{if } u_L t \leqslant x \leqslant h(t), \\ u_R & \text{if } x > h(t), \end{cases} \qquad (5.9)$$

and by

$$h''(t) = \frac{1}{m} \left(\frac{h(t)}{t} - u_R \right) \left(\frac{u_R}{2} + \frac{h(t)}{2t} - h'(t) \right) \qquad (5.10)$$

for all $t > 0$.
If $u_L > u_R$ then there exists $t_1 > 0$ such that

1. *$h(t_1) = u_L t_1$ and $h(t) > u_L t$ for all $t \in (0, t_1)$.*

2. *For $t \in [0, t_1)$ the solution of (1.1) is given by (5.9)-(5.10).*

3. *For $t > t_1$ we have a Riemann problem with $u_R < h'(t_1) < u_L$ and with $u_R > u_L - \lambda$, so this new Riemann problem belongs to the case (V) (up to a translation in the (x, t)-plane).*

For the case (II), two sub-cases can appear, they are illustrated in Figure 12.
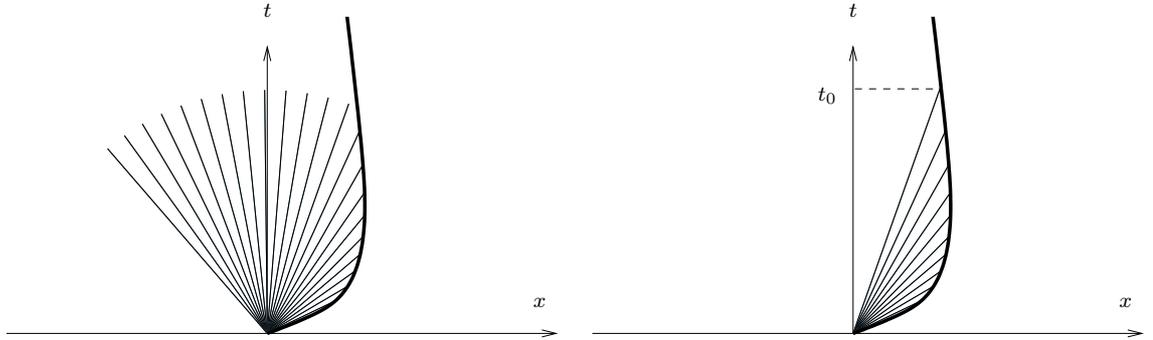
Figure 12: The two sub-cases for the case (II).

Proof. First let us consider the initial value problem

$$\begin{cases} z''(t) = \frac{1}{m} \left(\frac{z(t)}{t} - u_R \right) \left(\frac{u_R}{2} + \frac{z(t)}{2t} - z'(t) \right) \\ z(0) = 0, \ z'(0) = v_0, \end{cases} \qquad (5.11)$$

One can check that the above system has a unique solution $z \in \mathcal{C}^2(\mathbb{R}_+)$. Moreover, since

$$z''(0) = -\frac{1}{m}(v_0 - u_R)^2 < 0,$$

we have $z'' < 0$ in a neighborhood of 0. Assume that z'' admits a zero and denote by $T_1 > 0$ its first zero. Then, we have

$$\frac{z(t)}{t} > u_R, \quad \frac{u_R}{2} + \frac{z(t)}{2t} - z'(t) < 0 \qquad (5.12)$$

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for all $t \in (0, T_1)$.

We first assume that $z(T_1)/T_1 = u_R$, which gives

$$\frac{u_R}{2} + \frac{z(T_1)}{2T_1} - z'(T_1) = \frac{z(T_1)}{T_1} - z'(T_1) \leqslant 0.$$

But on the other hand, the strict concavity of z on $[0, T_1)$ implies that $(z(T_1) - z(0))/T_1 = z(T_1)/T_1 > z'(T_1)$, which contradicts the above inequality.

Second, we assume that

$$\frac{u_R}{2} + \frac{z(T_1)}{2T_1} - z'(T_1) = 0.$$

Let us consider the function Υ defined by

$$\Upsilon(t) = \frac{u_R}{2} + \frac{z(t)}{2t} - z'(t).$$

This function is negative for all $t \in (0, T_1)$ and the strict concavity of z yields

$$\Upsilon'(T_1) = \frac{z'(T_1)T_1 - z(T_1)}{2(T_1)^2} - z''(T_1) = \frac{z'(T_1)T_1 - z(T_1)}{2(T_1)^2} < 0.$$

This inequality contradicts $\Upsilon(T_1) = 0$. Therefore, we deduce from the above study that for all $t \in \mathbb{R}_+$, $z''(t) < 0$. In particular (5.12) holds for all $t \in \mathbb{R}_+$. Moreover, it can be shown that $z' > u_R$ and $z' \to u_R$ as $t \to \infty$. Consequently, if $u_L \leqslant u_R$, then $z(t) > u_L t$ for all $t \in \mathbb{R}_+$. Otherwise, there exists a unique t_1 such that $z'(t_1) = u_L$.

We then define (u, h) by the formulas (5.9) and (5.10) for all $t \in \mathbb{R}_+$ if $u_L \leqslant u_R$ and for all $t \in [0, t_1)$ if $u_L > u_R$. For sake of simplicity, we set $\tau = +\infty$ in the first case and $\tau = t_1$ in the second case.

Here again, **D1a** is verified, since there are a rarefaction wave on the left of the particle and constant states elsewhere. From (5.9), we have $u(t, h(t)^-) = h(t)/t$ for all $t \in (0, \tau)$. Therefore, (5.10) becomes

$$h''(t) = \frac{1}{m}\left(u(t, h(t)^-) - u_R\right)\left(\frac{u(t, h(t)^-) + u_R}{2} - h'(t)\right),$$

which yields **D1b**.

Since h is equal in $[0, \tau)$ to the function z studied at the beginning of this proof, we can check that $u_-(t) = h(t)/t$, h, $u_+ = u_R$ and v_0 satisfy the hypotheses of Lemma 5.4 which gives **D2**.

Finally, it only remains to prove that if $u_L < u_R$, then the initial conditions $(u_L, u_R, h'(t_1))$ of the new Riemann problem belong to the case (V). We have the following inequalities: $u_L > h'(t_1) > u_R$ and $h'(t_1) < u_R + \lambda$, and thus $h'(t_1) < (u_L + u_R + \lambda)/2$. Moreover, $h'(t_1) > (u_R + h(t_1)/t_1)/2$, and by definition of t_1, we deduce that $h'(t_1) > (u_L + u_R - \lambda)/2$. Therefore, at $t = t_1$, we have a Riemann problem located at $(u_L t_1, t_1)$ which satisfies the hypotheses of case (V). $\qquad\square$

Lemma 5.7 (Case (III)). $u_R < v_0 - \lambda$ and $u_R \geqslant u_L - \lambda$.
There exists $t_1 > 0$ such that

1. $h(t_1) = (u_R + \lambda)t_1$ and $h(t) > (u_R + \lambda)t$ for all $t \in (0, t_1)$.

2. For $t \in [0, t_1)$ the solution is given by

$$h(t) = \left(u_R + \frac{\lambda}{2}\right)t + \left(v_0 - u_R - \frac{\lambda}{2}\right)\frac{m}{\lambda}(1 - e^{-\frac{\lambda}{m}t}) \tag{5.13}$$

and

$$u(t, x) = \begin{cases} u_L & \text{if } x < u_L t, \\ x/t & \text{if } u_L t \leqslant x \leqslant (u_R + \lambda)t, \\ u_R + \lambda & \text{if } (u_R + \lambda)t \leqslant x \leqslant h(t), \\ u_R & \text{if } x > h(t), \end{cases} \tag{5.14}$$

We denote $x_1 = h(t_1)$.

If $u_L \leqslant u_R$ then the solution is given by

$$u(t,x) = \begin{cases} u_L & \text{if } x < x_1 + u_L(t - t_1), \\ (x - x_1)/(t - t_1) & \text{if } x_1 + u_L(t - t_1) \leqslant x \leqslant h(t), \\ u_R & \text{if } x > h(t), \end{cases} \qquad (5.15)$$

and with h given by

$$h''(t) = \frac{1}{m}\left(\frac{h(t)}{t - t_1} - u_R\right)\left(\frac{u_R}{2} + \frac{h(t)}{2(t - t_1)} - h'(t)\right) \qquad (5.16)$$

for all $t > t_1$.

If $u_L > u_R$ then there exists $t_2 > t_1$ such that

1. $h(t_2) = u_L t_2$ and $h(t) > u_L t$ for all $t \in (0, t_2)$.

2. For $t \in [t_1, t_2)$ the solution is given by (5.15)-(5.16).

3. For $t > t_2$ we have a Riemann problem with $u_R < h'(t_2) < u_L$ and with $u_R > u_L - \lambda$, so that this new Riemann problem belongs to the case (V).

Here again, this case is divided in two sub-cases, which are depicted in Figure 13.
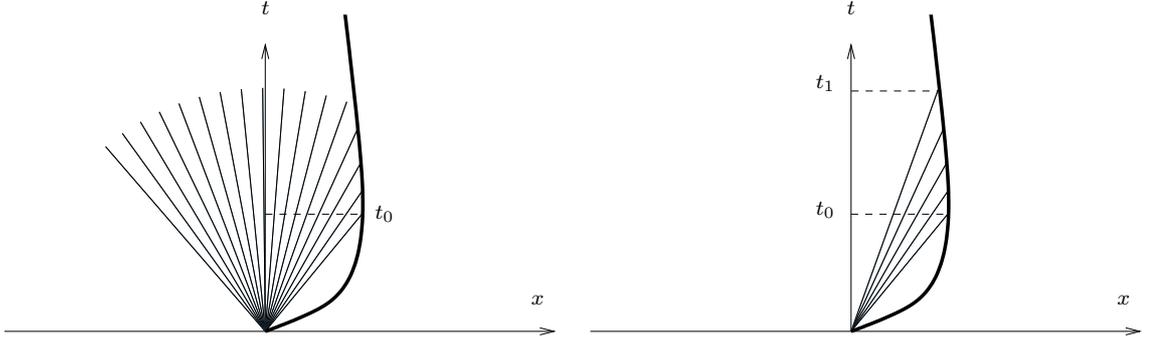
Figure 13: The two sub-cases for the case (III).

Proof. It is clear from the hypotheses on v_0 and u_R that there exists a unique $t_1 > 0$ such that

$$\left(u_R + \frac{\lambda}{2}\right)t_1 + \left(v_0 - u_R - \frac{\lambda}{2}\right)\frac{m}{\lambda}(1 - e^{-\frac{\lambda}{m}t_1}) = (u_R + \lambda)\,t_1.$$

We consider h and u given by (5.13)-(5.14). Relation **D1a** clearly holds in $(0, t_1)$. Moreover, since $u(t, h(t)^-) = u_R + \lambda$ and $u(t, h(t)^+) = u_R$, an easy calculation shows that **D1b** is satisfied. Since $u(t, h(t)^+) = u(t, h(t)^-) - \lambda$, the definition (3.10) implies **D2**.

For $t > t_1$, the situation is similar to the one of case (II) and by following the proof of Lemma 5.6 we can complete the proof of the lemma. $\qquad \square$

Lemma 5.8 (Cases (IV) and (VIII)). *Assume that $u_R < u_L - \lambda$ and that either $u_R < 2v_0 - u_L - \lambda$ or $u_R > 2v_0 - u_L + \lambda$.*

There exist $n \in \mathbb{N}^*$ and $t_0, \ldots, t_n \in \mathbb{R}$ with $0 = t_0 < \ldots < t_n$ such that for $t = t_n$ the solution of (1.1) is equal to the initial condition of a Riemann problem of case (V) and such that, denoting $x_k = h(t_k)$ for all k, for $t \in (t_{k-1}, t_k)$ with $k \leqslant n$, the solution of (1.1) is given either by

$$h(t) = x_{k-1} + \left(u_R + \frac{\lambda}{2}\right)(t - t_{k-1}) + \left(h'(t_{k-1}) - u_R - \frac{\lambda}{2}\right)\frac{m}{\lambda}\left(1 - e^{-\frac{\lambda}{m}(t - t_{k-1})}\right) \qquad (5.17)$$

and

$$u(t,x) = \begin{cases} u_L & \text{if } x < x_{k-1} + \frac{u_L + u_R + \lambda}{2}(t - t_{k-1}), \\ u_R + \lambda & \text{if } x_{k-1} + \frac{u_L + u_R + \lambda}{2}(t - t_{k-1}) \leqslant x \leqslant h(t), \\ u_R & \text{if } x > h(t), \end{cases} \tag{5.18}$$

or by

$$h(t) = x_{k-1} + \left(u_L - \frac{\lambda}{2}\right)(t - t_{k-1}) + \left(h'(t_{k-1}) - u_L + \frac{\lambda}{2}\right)\frac{m}{\lambda}\left(1 - e^{-\frac{\lambda}{m}(t - t_{k-1})}\right) \tag{5.19}$$

and

$$u(t,x) = \begin{cases} u_L & \text{if } x < h(t), \\ u_L - \lambda & \text{if } h(t) \leqslant x \leqslant x_{k-1} + \frac{u_L + u_R - \lambda}{2}(t - t_{k-1}), \\ u_R & \text{if } x > x_{k-1} + \frac{u_L + u_R - \lambda}{2}(t - t_{k-1}). \end{cases} \tag{5.20}$$

Moreover, the formula for the solution of (1.1) alternates at each time t_k between (5.17)-(5.18) and (5.19)-(5.20).

Two illustrations of solutions of the case (IV) are represented in Figure 14.
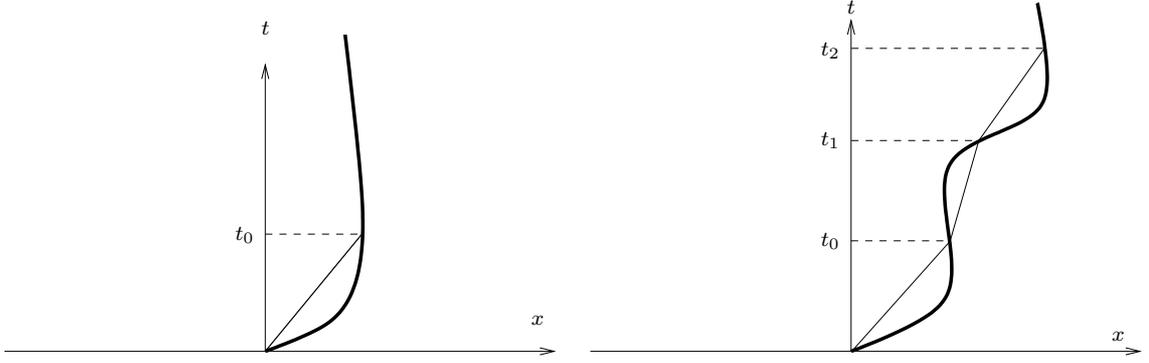
Figure 14: Two examples of solutions for the case (IV).

Proof. Assume that $t_{k-1} \geqslant 0$ and that (h, u) is given by (5.17)-(5.18). It is clear that there exits $t_k > 0$ such that

$$h(t) > x_{k-1} + \frac{u_L + u_R + \lambda}{2}(t - t_{k-1}) \tag{5.21}$$

for $t \in (t_{k-1}, t_k)$, and such that

$$h(t_k) = x_{k-1} + \frac{u_L + u_R + \lambda}{2}(t_k - t_{k-1}). \tag{5.22}$$

It is clear that u satisfies **D1a** and that h satisfies **D1b** in (t_{k-1}, t_k). Moreover, **D2** can be easily deduced from Lemma 5.3.

For $t = t_k$, we obtain a new Riemann problem centered at $(t_k, h(t_k)) = (t_k, x_k)$, with u_L, and u_R as initial data for u and $h'(t_k)$ satisfying

$$u_R + \frac{\lambda}{2} < h'(t_k) < \frac{u_L + u_R + \lambda}{2}.$$

At this stage, two possibilities occur: either this Riemann problem belongs to case (V) if

$$h'(t_k) > \frac{u_L + u_R - \lambda}{2}$$

or it belongs to case (VIII) if

$$h'(t_k) \leqslant \frac{u_L + u_R - \lambda}{2}.$$

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By symmetry, if we start from a Riemann problem in case (VIII), we can obtain a new Riemann problem belonging either to case (V) or to case (IV). Since we know the global solution in case (V) (see Lemma 5.5), we only need to focus on the switches between cases (IV) and (VIII). We are going to show that the number of switches is bounded by

$$N = \frac{3}{2} + \frac{1}{\lambda}\left|h'(0) - \frac{u_L + u_R}{2}\right|. \tag{5.23}$$

In order to prove this assertion, it is sufficient to verify that

$$h'(0) - \frac{u_L + u_R}{2} > \frac{u_L + u_R}{2} - h'(t_1) + \lambda. \tag{5.24}$$

By symmetry and induction, the above relation can be iterated. This implies that at each switch, the difference between $h'(t_i)$ and $(u_L + u_R)/2$ decreases of λ at least. We now prove (5.24). By definition of t_1, we have

$$\left(\frac{\lambda}{m}t_1\right)\frac{u_L - u_R}{2} = \left[h'(0) - \left(u_R + \frac{\lambda}{2}\right)\right]\left(1 - e^{-\frac{\lambda}{m}t_1}\right) \tag{5.25}$$

and

$$h'(t_1) = h'(0) - \left(\frac{\lambda}{m}t_1\right)\frac{u_L - u_R}{2}. \tag{5.26}$$

One may check that for $\alpha > 1$,

$$\alpha(1 - e^{-2(1-\alpha)}) < 2(\alpha - 1) \quad (\alpha > 1). \tag{5.27}$$

Let us note that the hypotheses on case (IV) leads to

$$\frac{2}{u_L - u_R}\left[h'(0) - \left(u_R + \frac{\lambda}{2}\right)\right] > 1,$$

then we can combine it with (5.27). It provides

$$\frac{2}{u_L - u_R}\left[h'(0) - \left(u_R + \frac{\lambda}{2}\right)\right]\left(1 - \exp\left(-\frac{4}{u_L - u_R}\left[h'(0) - \frac{u_L + u_R + \lambda}{2}\right]\right)\right)$$
$$< \frac{4}{u_L - u_R}\left[h'(0) - \frac{u_L + u_R + \lambda}{2}\right].$$

We deduce

$$X\frac{u_L - u_R}{2} > \left[h'(0) - \left(u_R + \frac{\lambda}{2}\right)\right]\left(1 - e^{-X}\right) \tag{5.28}$$

where

$$X = \frac{4}{u_L - u_R}\left[h'(0) - \frac{u_L + u_R + \lambda}{2}\right].$$

By definition of t_1, we get

$$\frac{\lambda}{m}t_1 < X.$$

Then, using (5.26), we finally obtain (5.24). □

Proof of Theorem 5.1. In Lemmas 5.5, 5.6, 5.7, 5.8, we show that starting with a Riemann problem (u_L, u_R, v_0) in case (I), (V), (II), (III), (IV) and (VIII), there exists a global solution of (1.1) in the sense of Definition 3.1.

The remaining cases, i.e. (VI) and (VII), can be treated by symmetry of respectively case (II) and case (III). □

Proof of Theorem 5.2. Let us first study the long time behavior.

- If $u_L \leqslant u_R$, the solution of the Riemann problem involves a rarefaction wave in all cases. If $v_0 \in [u_L, u_R]$ then the solution belongs to the case (I) and $h'(t) = v_0$ for all $t \in \mathbb{R}_+$. If $v_0 > u_R$, this corresponds to the first sub-case of the case (II) or to the first sub-case of the case (III). As shown in the proof of Lemma 5.6, the limit of $h'(t)$ is u_R when $t \to +\infty$. Moreover, the solution u is composed of a rarefaction wave whose slope is $1/t$ connected to u_L. Since the solution u_∞ of the uncoupled Burgers equation also involves a rarefaction wave connected to u_L with the same slope, u and u_∞ exactly match for all $x < u_R t$ and all $x > h(t)$ (see figure 15). Finally, using (5.9) and (5.15), one may see that the maximum of
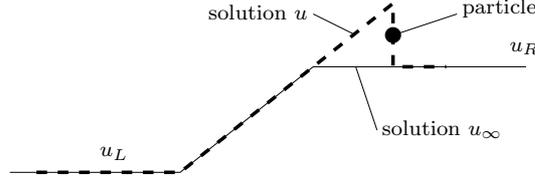
Figure 15: Coupled versus uncoupled solution

$||u(t, \cdot) - u_\infty(t, \cdot)||_{L^\infty(\mathbb{R})}$ is attained at $x = h(t)^-$ and since $\lim_{t \to +\infty} h'(t) = u_R$, we obtain the convergence result (5.2). The case $v_0 < u_L$ is symmetric.

- if $u_L > u_R$, we directly see by the preceding lemmas that there exists a time $T > 0$ such that for any $t \geqslant T$, the solution $u(t, \cdot)$ is composed of a single discontinuity linking u_L to u_R, that is to say the solution belongs to the case (V) (up to a translation in the (x, t)-plane). Differentiating Equation (5.8) provides $\lim_{t \to +\infty} h'(t) = (u_L + u_R)/2$. Hence, the limit solution is composed only by the states u_L and u_R, separated by a shock wave moving with the velocity $(u_L + u_R)/2$. It means there exists $b \in \mathbb{R}$ such that the trajectory of the particle admits as asymptote the straight line $x - b = (u_L + u_R)/2 t$. The constant b can be computed using the conservation of the difference between the total impulsion associated to Problem (1.1) and the total impulsion of the Burgers equation (with initial condition $u_\infty(0, x) = u_0(x - b)$):

$$\frac{d}{dt} \left(\int_\mathbb{R} (u(t, x) - u_\infty(t, x)) dx + m h'(t) \right) = 0.$$

Note that, for all $t > 0$, this integral is well defined since $u(t, .) - u_\infty(t, .)$ is bounded and has a compact support (though u and u_∞ are not integrable). Then, an integration by parts provides

$$b = m \frac{v_0 - (u_L + u_R)/2}{u_L - u_R}$$

and we recover (5.3). Note that due to the presence of a persisting discontinuity with amplitude $u_L - u_R$, the convergence does not hold in $L^\infty(\mathbb{R})$.

The behavior for $\lambda \to \infty$ can be obtained directly from the above proof of theorem 5.1. $\qquad \square$

A Well-posedness of the problem with viscosity

We are going to prove the existence and the uniqueness for problem (3.1). By using the change of variables

$$w(t, y) = u(t, y + h(t)),$$

we obtain the following equations for (w, h):

$$\begin{cases} \partial_t w - \mu \partial_{yy}^2 w + \partial_y \left(\frac{w^2}{2} \right) - h' \partial_y w = \lambda \left(h' - w(0) \right) \delta_0, & (t, y) \in \mathbb{R}_+ \times \mathbb{R}, \\ m h'' = -\lambda \left(h' - w(0) \right), & t \in \mathbb{R}_+, \\ w(0, y) = u_0(y), & y \in \mathbb{R}, \\ (h(0), h'(0)) = (0, v_0). \end{cases} \quad \text{(A.1)}$$

Theorem A.1. *Assume that $u_0 \in \mathbf{H}^1(\mathbb{R})$, $v_0 \in \mathbb{R}$. Then for all $T < \infty$ there exists a unique solution (v, h) of (A.1) such that*

$$w \in \mathbf{L}^2(0, T; \mathbf{H}^2_*(\mathbb{R})) \cap \mathcal{C}([0, T]; \mathbf{H}^1(\mathbb{R})) \cap \mathbf{H}^1(0, T; \mathbf{L}^2(\mathbb{R})), \quad h \in \mathbf{H}^2(0, T). \qquad \text{(A.2)}$$

Here the space $\mathbf{H}^2_*(\mathbb{R})$ is defined by

$$\mathbf{H}^2_*(\mathbb{R}) = \left\{ w \in \mathbf{H}^1(\mathbb{R}); w|_{\mathbb{R}^*_+} \in \mathbf{H}^2(\mathbb{R}^*_+), w|_{\mathbb{R}^*_-} \in \mathbf{H}^2(\mathbb{R}^*_-) \right\}.$$

Proof. The proof is split in three steps:
Step 1, the linearized problem: we consider the following linear system

$$\begin{cases} \partial_t w - \mu \partial^2_{yy} w - \lambda \left(h' - w(0) \right) \delta_0 = f, & (t, y) \in \mathbb{R}_+ \times \mathbb{R}, \\ m h'' + \lambda \left(h' - w(0) \right) = 0, & t \in \mathbb{R}_+, \\ w(0, y) = u_0(y), & y \in \mathbb{R}, \\ (h(0), h'(0)) = (0, v_0). \end{cases} \qquad \text{(A.3)}$$

We can rewrite it under the form

$$\begin{cases} \partial_t w - \mu \partial^2_{yy} w - \lambda \left(\ell - w(0) \right) \delta_0 = f, & (t, y) \in \mathbb{R}_+ \times \mathbb{R}, \\ m \ell' + \lambda \left(\ell - w(0) \right) = 0, & t \in \mathbb{R}_+, \\ w(0, y) = w_0(y), & y \in \mathbb{R}, \\ \ell(0) = v_0, \end{cases} \qquad \text{(A.4)}$$

with

$$w_0(y) = u_0(y).$$

We can solve the last system by using the semi-group theory. More precisely, we use the following classical result:

Proposition A.2. *Let H be an Hilbert space and let $A : D(A) \mapsto H$ be a self-adjoint operator such that $-A$ be m-dissipative. Suppose $F \in \mathbf{L}^2(0, T; H)$ and $z_0 \in D(A^{1/2})$. Then the problem*

$$z' + Az = F, \quad z(0) = z_0 \qquad \text{(A.5)}$$

admits a unique solution

$$z \in \mathbf{L}^2(0, T; D(A)) \cap \mathcal{C}([0, T]; D(A^{1/2})) \cap \mathbf{H}^1(0, T; H).$$

Moreover, there exists a positive constant $C = C(T)$ such that

$$\|z\|_{\mathbf{L}^2(0,T;D(A))} + \|z\|_{\mathcal{C}([0,T];D(A^{1/2}))} + \|z\|_{\mathbf{H}^1(0,T;H)} \leqslant C \left(\|z_0\|_{D(A^{1/2})} + \|F\|_{\mathbf{L}^2(0,T;H)} \right).$$

To apply this result to the system (A.4), we introduce some notation: we put

$$z = (w, \ell), \quad z_0 = (w_0, \ell_0), \quad F = (f, 0) \quad H = \mathbf{L}^2(\mathbb{R}) \times \mathbb{R}$$

and we consider the operator A defined by

$$D(A) = \left\{ (w, \ell) \in \mathbf{H}^2_*(\mathbb{R}) \times \mathbb{R} \; ; \; \mu[\partial_y w](0) = -\lambda(\ell - w(0)) \right\},$$

and

$$A \begin{pmatrix} w \\ \ell \end{pmatrix} = \begin{pmatrix} -\mu \partial^2_{yy} w - \lambda(\ell - w(0))\delta_0 \\ \dfrac{\lambda}{m}(\ell - w(0)) \end{pmatrix},$$

where $[u](0)$ stands for the jump of the function u at 0. With the above notations the system (A.4) can be written under the form (A.5).

Remark 4. If $(w, \ell) \in D(A)$, then the distribution $-\mu \partial_{yy}^2 w - \lambda(\ell - w(0))\delta_0$ belongs to $\mathbf{L}^2(\mathbb{R})$. Indeed, if $\varphi \in \mathcal{D}(\mathbb{R})$, then

$$\langle -\mu \partial_{yy}^2 w - \lambda(\ell - w(0))\delta_0, \varphi \rangle = \mu \int_{\mathbb{R}} \partial_y w \partial_y \varphi \, dy - \lambda(\ell - w(0))\varphi(0)$$

$$= -\mu[\partial_y w \varphi] - \mu \int_{\mathbb{R}_+^*} \partial_{yy}^2 w \varphi \, dy - \mu \int_{\mathbb{R}_-^*} \partial_{yy}^2 w \varphi \, dy - \lambda(\ell - w(0))\varphi(0)$$

$$= -\mu \int_{\mathbb{R}_+^*} \partial_{yy}^2 w \varphi \, dy - \mu \int_{\mathbb{R}_-^*} \partial_{yy}^2 w \varphi \, dy.$$

Consequently, the distribution $-\mu \partial_{yy}^2 w - \lambda(\ell - w(0))\delta_0$ corresponds to the functional

$$-\mu \partial_{yy}^2 \left(w|_{\mathbb{R}_+^*} \right) 1_{\mathbb{R}_+^*} - \mu \partial_{yy}^2 \left(w|_{\mathbb{R}_-^*} \right) 1_{\mathbb{R}_-^*} \in \mathbf{L}^2(\mathbb{R}).$$

To apply Proposition A.2, we first prove the following lemma.

Lemma A.3. *The operator A defined above is self-adjoint and $-A$ is m-dissipative.*

Proof. Suppose $(u, k), (w, \ell) \in D(A)$. The definition of A shows that

$$(A(w, \ell), (u, k)) = \mu \int_{\mathbb{R}} \partial_y w \partial_y u \, dy - \lambda(\ell - w(0))u(0) - m \frac{\lambda}{m}(\ell - w(0))k = \mu \int_{\mathbb{R}} \partial_y w \partial_y u \, dy.$$

Hence A is symmetric and $-A$ is dissipative. To prove that A is m-dissipative (and therefore self-adjoint), it is enough to show that $I + A$ is onto. Suppose $(f, g) \in H$. We want to prove the existence of $(w, \ell) \in D(A)$ such that $(w, \ell) + A(w, \ell) = (f, g)$ i.e.

$$\begin{cases} w - \mu \partial_{yy}^2 w - \lambda(\ell - w(0))\delta_0 = f, \\ m\ell + \lambda(\ell - w(0)) = mg. \end{cases} \tag{A.6}$$

To solve (A.6), we first consider the corresponding variational problem: find $(w, \ell) \in \mathbf{H}^1(\mathbb{R}) \times \mathbb{R}$ such that

$$\int_{\mathbb{R}} wu \, dy + m\ell k + \mu \int_{\mathbb{R}} \partial_y w \partial_y u \, dy + \lambda(\ell - w(0))(k - u(0)) = \int_{\mathbb{R}} fu \, dy + mgk \quad \forall (u, k) \in \mathbf{H}^1(\mathbb{R}) \times \mathbb{R}. \tag{A.7}$$

Using the Lax-Milgram lemma, we get the existence and uniqueness of the preceding problem. Then, taking $u = 0$ and $k = 1$ in (A.7), we get

$$m\ell + \lambda(\ell - w(0)) = mg.$$

so that (A.7) gives

$$\int_{\mathbb{R}} wu \, dy + \mu \int_{\mathbb{R}} \partial_y w \partial_y u \, dy - \lambda(\ell - w(0))u(0) = \int_{\mathbb{R}} fu \, dy \quad \forall u \in \mathbf{H}^1(\mathbb{R}). \tag{A.8}$$

The above relation yields that

$$w - \mu \partial_{yy}^2 w - \lambda(\ell - w(0))\delta_0 = f,$$

in $\mathbf{H}^{-1}(\mathbb{R})$. In particular we deduce that $w \in \mathbf{H}_*^2(\mathbb{R})$ with $\mu[\partial_y w](0) = -\lambda(\ell - w(0))$. $\qquad \square$

From the above lemma and Proposition A.2 we deduce that for any $w_0 \in \mathbf{H}^1(\mathbb{R})$ and $f \in \mathbf{L}^2(0, T; \mathbb{R})$, there exists a unique solution of (A.4) such that

$$w \in \mathbf{L}^2(0, T; \mathbf{H}_*^2(\mathbb{R})) \cap \mathcal{C}([0, T]; \mathbf{H}^1(\mathbb{R})) \cap \mathbf{H}^1(0, T; \mathbf{L}^2(\mathbb{R})), \quad \ell \in \mathbf{H}^1(0, T). \tag{A.9}$$

In particular, the system (A.3) admits a unique solution (w, h) such that w satisfies the above regularity and $h \in \mathbf{H}^2(0, T)$.

Step 2, local in time existence: We consider the application

$$\mathcal{Z} : \mathbf{L}^2(0, T; \mathbf{L}^2(\mathbb{R})) \mapsto \mathbf{L}^2(0, T; \mathbf{L}^2(\mathbb{R}))$$

which assigns to any $f \in \mathbf{L}^2(0, T; \mathbf{L}^2(\mathbb{R}))$ the function $h' \partial_y w - w \partial_y w$ with (w, h) the solution of (A.3) corresponding to f. We can first note that $\mathcal{Z}$ is well defined:

- $h'\partial_y w \in \mathcal{C}([0,T];\mathbf{L}^2(\mathbb{R}))$ with

$$\|h'\partial_y w\|_{\mathcal{C}([0,T];\mathbf{L}^2(\mathbb{R}))} \leqslant C \left(\|w_0\|_{\mathbf{H}^1(\mathbb{R})} + \|f\|_{\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbb{R}))}\right)^2$$

- Since $\partial_y w \in \mathcal{C}([0,T];\mathbf{L}^2(\mathbb{R}))$, since $w \in \mathcal{C}([0,T];\mathbf{H}^1(\mathbb{R}))$ and since the following Sobolev injection holds

$$\mathbf{H}^1(\mathbb{R}) \subset \mathbf{L}^\infty(\mathbb{R}),$$

we get $w\partial_y w \in \mathcal{C}([0,T];\mathbf{L}^2(\mathbb{R}))$ with the estimate

$$\|w\partial_y w\|_{\mathcal{C}([0,T];\mathbf{L}^2(\mathbb{R}))} \leqslant C \left(\|w_0\|_{\mathbf{H}^1(\mathbb{R})} + \|f\|_{\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbb{R}))}\right)^2.$$

This implies that $\mathcal{Z}$ is well defined and moreover that

$$\mathcal{Z}(\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbb{R}))) \subset \mathcal{C}([0,T];\mathbf{L}^2(\mathbb{R}))$$

so that, using a Banach fixed point, we obtain the local in time existence of solutions for (A.1).
Step 3, global in time existence: To get the global in time existence, we need to prove that the $\mathbf{H}^1$-norm of w is bounded on any interval $[0,T]$, $T < \infty$. On the one hand, we have the following energy estimate

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}} w^2 dy + m(h')^2\right) + \mu \int_{\mathbb{R}}(\partial_y w)^2 dy + \lambda(h' - w(0))^2 = 0, \qquad (A.10)$$

so that the norm in $\mathbf{L}^2(\mathbb{R}) \times \mathbb{R}$ of (w,h) is bounded:

$$\int_{\mathbb{R}} w^2 dx + m(h'(t))^2 + \mu \int_0^t \int_{\mathbb{R}}(\partial_y w)^2 \, dy \, ds \leqslant \int w_0^2 dx + m(h'(0))^2 =: C_0^2. \qquad (A.11)$$

On the other hand, by multiplying the first equation of (A.1) by $\partial_t w$ and the second equation of (A.1) by h'', we get after some integration by parts

$$\int_0^t \int_{\mathbb{R}} |\partial_t w|^2 \, dy \, ds + \int_0^t m|h''|^2 \, ds + \frac{\mu}{2}\int_{\mathbb{R}} |\partial_y w(t)|^2 \, dy$$
$$+ \int_0^t \int_{\mathbb{R}} (w - h')(\partial_y w)(\partial_t w) \, dy \, ds + \lambda\frac{(h'(t) - w(t,0))^2}{2}$$
$$= \frac{\mu}{2}\int_{\mathbb{R}} |\partial_y w(0)|^2 \, dy + \lambda\frac{(h'(0) - w(0,0))^2}{2}. \qquad (A.12)$$

Using the Cauchy-Schwarz inequality, we have

$$\left|\int_0^t \int_{\mathbb{R}} (w\partial_y w - h'\partial_y w)\partial_t w \, dy \, ds\right| \leqslant \frac{1}{2}\int_0^t \int_{\mathbb{R}} |\partial_t w|^2 \, dy \, ds + \int_0^t \int_{\mathbb{R}} (|w|^2 + |h'|^2) \, |\partial_y w|^2 \, dy \, ds.$$

Hence, from the energy estimate,

$$\left|\int_0^t \int_{\mathbb{R}} (w\partial_y w - h'\partial_y w)\partial_t w \, dy \, ds\right| \leqslant \frac{1}{2}\int_0^t \int_{\mathbb{R}} |\partial_t w|^2 \, dy \, ds + C_0 \int_0^t \int_{\mathbb{R}} |\partial_y w|^2 \, dy \, ds$$
$$+ \int_0^t \int_{\mathbb{R}} |w|^2 \, |\partial_y w|^2 \, dy \, ds. \qquad (A.13)$$

To estimate the last integral, we use the Cauchy-Schwarz inequality and the Sobolev injection $\mathbf{H}^{1/2} \subset \mathbf{L}^4$ and we get

$$\begin{aligned}
\int_{\mathbb{R}} |w|^2 \, |\partial_y w|^2 \, dy \, ds &\leqslant \left(\int_{\mathbb{R}} |w|^4 \, dy\right)^{1/2} \left(\int_{\mathbb{R}} |\partial_y w|^4 \, dy\right)^{1/2} \\
&\leqslant C\|w\|_{\mathbf{L}^2(\mathbb{R})}\|w\|_{\mathbf{H}^1(\mathbb{R})}^2 \left(\|\partial_y w\|_{\mathbf{H}^1(\mathbb{R}_+^*)} + \|\partial_y w\|_{\mathbf{H}^1(\mathbb{R}_-^*)}\right) \\
&\leqslant CC_0\|w\|_{\mathbf{H}^1(\mathbb{R})}^3 + \frac{CC_0}{\mu}\|w\|_{\mathbf{H}^1(\mathbb{R})}^2\|\partial_t w + w\partial_y w - h'\partial_y w\|_{\mathbf{L}^2(\mathbb{R})}.
\end{aligned}$$

It follows from the above inequality

$$\int_{\mathbb{R}} |w|^2 \, |\partial_y w|^2 \, dy \, ds \leqslant CC_0 \left(1 + \frac{C_0}{\mu}\right) \|w\|_{\mathbf{H}^1(\mathbb{R})}^3 + C\frac{C_0^2}{\mu^2}\|w\|_{\mathbf{H}^1(\mathbb{R})}^4 + \frac{1}{2}\|\partial_t w\|_{\mathbf{L}^2(\mathbb{R})}^2 + \frac{1}{2}\|w\partial_y w\|_{\mathbf{L}^2(\mathbb{R})}^2.$$

Combining this inequality with (A.13) implies

$$\left|\int_0^t \int_{\mathbb{R}} (w\partial_y w - h'\partial_y w)\, \partial_t w \, dy \, ds\right| \leqslant \frac{3}{4}\int_0^t \int_{\mathbb{R}} |\partial_t w|^2 \, dy \, ds$$
$$+ CC_0 \int_0^t \|w\|_{\mathbf{H}^1(\mathbb{R})}^2 \left(1 + \frac{C_0^2}{\mu^2} + \frac{C_0^2}{\mu^2}\|w\|_{\mathbf{H}^1(\mathbb{R})}^2\right) \, ds. \quad \text{(A.14)}$$

It follows from (A.12) and (A.14) that

$$\mu\|w(t)\|_{\mathbf{H}^1(\mathbb{R})}^2 \leqslant \mu\|w(0)\|_{\mathbf{H}^1(\mathbb{R})}^2 + \mu C_0^2 + \lambda \frac{(h'(0) - w(0,0))^2}{2}$$
$$+ CC_0 \int_0^t \|w\|_{\mathbf{H}^1(\mathbb{R})}^2 \left(1 + \frac{C_0^2}{\mu^2} + \frac{C_0^2}{\mu^2}\|w\|_{\mathbf{H}^1(\mathbb{R})}^2\right) \, ds.$$

Applying the Grönwall lemma to the preceding inequality gives

$$\|w(t)\|_{\mathbf{H}^1(\mathbb{R})}^2 \leqslant \left(\|w(0)\|_{\mathbf{H}^1(\mathbb{R})}^2 + C_0^2 + \lambda\frac{(h'(0) - w(0,0))^2}{2\mu}\right)$$
$$\exp\left(\frac{CC_0}{\mu}\int_0^t \left(1 + \frac{C_0^2}{\mu^2} + \frac{C_0^2}{\mu^2}\|w\|_{\mathbf{H}^1(\mathbb{R})}^2\right) \, ds\right).$$

The above estimates yields the global in time existence for the solutions of (A.1). By achieving a change of variables, we deduce the global in time existence of the solutions of (3.1).

The proof of the uniqueness uses classical arguments and we only sketch the proof: assume that $(w^{(1)}, h^{(1)})$ and $(w^{(2)}, h^{(2)})$ are two solutions of (A.1). Then,

$$w = w^{(1)} - w^{(2)} \quad \text{and} \quad h = h^{(1)} - h^{(2)}$$

verify the following system

$$\begin{cases} \partial_t w - \mu\partial_{yy}^2 w + (w - h')\partial_y w^{(1)} + (w^{(2)} - h^{(2)'})\partial_y w = \lambda(h' - w(0))\,\delta_0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}, \\ mh'' = -\lambda(h' - w(0)), & t \in \mathbb{R}_+, \\ w(0,y) = 0, & y \in \mathbb{R}, \\ (h(0), h'(0)) = (0,0). \end{cases}$$
$$\text{(A.15)}$$

Multiplying the first equation of the above system by w and the second equation of the above system by h' leads us to the equality

$$\frac{1}{2}\|w(t)\|_{\mathbf{L}^2(\mathbb{R})}^2 + \frac{m}{2}|h'(t)|^2 + \mu\int_0^t \|\partial_y w(s)\|_{\mathbf{L}^2(\mathbb{R})}^2 \, ds + \lambda|h'(t) - w(t,0)|^2$$
$$= -\int_0^t \int_{\mathbb{R}} \left((w - h')\partial_y w^{(1)} + (w^{(2)} - h^{(2)'})\partial_y w\right) w \, dy \, ds. \quad \text{(A.16)}$$

By integrating by part, we find

$$-\int_0^t \int_{\mathbb{R}} (w^{(2)} - h^{(2)'})\partial_y w \, w \, dy \, ds = \int_0^t \int_{\mathbb{R}} \partial_y w^{(2)} \, \frac{(w)^2}{2} \, dy \, ds.$$

Owing to (A.9) and to $\mathbf{H}^{1/2}(\mathbb{R}) \subset \mathbf{L}^4(\mathbb{R})$, we deduce

$$\left|-\int_0^t \int_{\mathbb{R}} (w^{(2)} - h^{(2)'})\partial_y w \, w \, dy \, ds\right| \leqslant \frac{1}{2}\int_0^t \|w(s)\|_{\mathbf{L}^4(\mathbb{R})}^2 \|w^{(2)}(s)\|_{\mathbf{H}^1(\mathbb{R})} \, ds$$
$$\leqslant C\int_0^t \|w(s)\|_{\mathbf{H}^1(\mathbb{R})}\|w(s)\|_{\mathbf{L}^2(\mathbb{R})} \, ds. \quad \text{(A.17)}$$

Similarly, we obtain

$$
\left| \int_0^t \int_{\mathbb{R}} (w)^2 \partial_y w^{(1)} \, dy \, ds \right| \leqslant \int_0^t \|w(s)\|_{\mathbf{L}^4(\mathbb{R})}^2 \|w^{(1)}(s)\|_{\mathbf{H}^1(\mathbb{R})} \, ds \leqslant C \int_0^t \|w(s)\|_{\mathbf{H}^1(\mathbb{R})} \|w(s)\|_{\mathbf{L}^2(\mathbb{R})} \, ds
$$
(A.18)

and

$$
\left| \int_0^t \int_{\mathbb{R}} h' w \partial_y w^{(1)} \, dy \, ds \right| \leqslant \int_0^t |h'(s)| \|w(s)\|_{\mathbf{L}^2(\mathbb{R})} \|w^{(1)}(s)\|_{\mathbf{H}^1(\mathbb{R})} \, ds \leqslant C \int_0^t \|w(s)\|_{\mathbf{L}^2(\mathbb{R})}^2 + |h'(s)|^2 \, ds.
$$
(A.19)

The relations (A.16), the inequalities (A.17)-(A.19) and the Grönwall lemma imply $w = 0$ and $h' = 0$. In particular, we deduce that $h^{(1)} = h^{(2)}$ and $u^{(1)} = u^{(2)}$. $\qquad\square$

B   Proof of Proposition 3.4

This section is devoted to the proof of Proposition 3.4, which states that if **D2** holds, then, (u, h) satisfies **D1** if and only if it satisfies **D1a** and **D1b**.

Proof. Assume that **D1** and **D2** hold. Then the pair (u, h) verifies **D1a**. We have only to show (u, h) satisfies **D1b**, that is to say h is solution of (3.16).

We deduce from (3.15) that

$$
-h'(|u_+ - \kappa| - |u_- - \kappa|)\delta_h + \frac{\mathrm{sgn}(u_+ - \kappa)(u_+^2 - \kappa^2) - \mathrm{sgn}(u_- - \kappa)(u_-^2 - \kappa^2)}{2}\delta_h + m\mathrm{sgn}(h' - \kappa)h''\delta_h \leqslant 0
$$

and thus

$$
-h'(|u_+ - \kappa| - |u_- - \kappa|) + \frac{\mathrm{sgn}(u_+ - \kappa)(u_+^2 - \kappa^2) - \mathrm{sgn}(u_- - \kappa)(u_-^2 - \kappa^2)}{2} + m\mathrm{sgn}(h' - \kappa)h'' \leqslant 0
$$
(B.1)

for any $\kappa \in \mathbb{R}$. Taking $\kappa < \min(h', u_-, u_+)$ in (B.1) leads us to

$$
mh'' \leqslant (u_+ - u_-)\left(h' - \frac{u_- + u_+}{2}\right)
$$

while taking $\kappa > \max(h', u_-, u_+)$ yields

$$
mh'' \geqslant (u_+ - u_-)\left(h' - \frac{u_- + u_+}{2}\right).
$$

Consequently h is solution of (3.16).

Conversely assume that **D1a**, **D1b** and **D2** hold. Showing **D1** is sufficient to verify that (B.1) holds. If $\kappa > \max(h', u_-, u_+)$ or if $\kappa < \min(h', u_-, u_+)$, this is easy to check, and thus we only have to show (B.1) for $\kappa \in [\min(h', u_-, u_+), \max(h', u_-, u_+)]$. First, let us set

$$
I = -h'(t)(|u_+ - \kappa| - |u_- - \kappa|) + \frac{\mathrm{sgn}(u_+ - \kappa)(u_+^2 - \kappa^2) - \mathrm{sgn}(u_- - \kappa)(u_-^2 - \kappa^2)}{2} + m\mathrm{sgn}(h' - \kappa)h''.
$$

We study its sign according to κ. Since (3.16) is satisfies, we can write I as

$$
\begin{aligned}
I = -h'(t)(|u_+ - \kappa| - |u_- - \kappa|) &+ \frac{\mathrm{sgn}(u_+ - \kappa)(u_+^2 - \kappa^2) - \mathrm{sgn}(u_- - \kappa)(u_-^2 - \kappa^2)}{2} \\
&+ \mathrm{sgn}(h' - \kappa)(u_+ - u_-)\left(h' - \frac{u_- + u_+}{2}\right).
\end{aligned}
$$

We recall that Relation (3.14) implies that $u_- \geqslant u_+$ and therefore it is sufficient to know the relation between h' and u_- and u_+.

  i. $u_+ \leqslant u_- \leqslant h'(t)$

- $\kappa \in [u_-, h']$:

$$I = h'(u_+ - u_-) + \left(\frac{u_-^2 - u_+^2}{2}\right) + (u_- - u_+)\left(\frac{u_- + u_+}{2} - h'(t)\right)$$

$$\leqslant 2(u_- - u_+)\left(\frac{u_- + u_+}{2} - h'(t)\right) \leqslant 0.$$

- $\kappa \in [u_+, u_-]$:

$$I = h'(u_+ + u_- - 2\kappa) + \kappa^2 - \frac{u_+^2 + u_-^2}{2} + (u_- - u_+)\left(\frac{u_- + u_+}{2} - h'(t)\right)$$

The last expression is a polynomial function of degree 2 in the variable κ, which is decreasing on $(-\infty, h']$ and which vanishes at u_+:

$$h'(u_+ + u_- - 2u_+) + u_+^2 - \frac{u_+^2 + u_-^2}{2} + (u_- - u_+)\left(\frac{u_- + u_+}{2} - h'(t)\right) = 0.$$

Consequently, (B.1) holds in that case.

ii. $h'(t) \leqslant u_+ \leqslant u_-$

- $\kappa \in [h', u_+]$:

$$I = -h'(u_+ - u_-) + \left(\frac{u_+^2 - u_-^2}{2}\right) - (u_- - u_+)\left(\frac{u_- + u_+}{2} - h'(t)\right)$$

$$\leqslant 2(u_+ - u_-)\left(\frac{u_- + u_+}{2} - h'(t)\right) \leqslant 0.$$

- $\kappa \in [u_+, u_-]$:

$$I = h'(u_+ + u_- - 2\kappa) + \kappa^2 - \frac{u_+^2 + u_-^2}{2} - (u_- - u_+)\left(\frac{u_- + u_+}{2} - h'(t)\right)$$

The last expression is a polynomial function of degree 2 in the variable κ, which is increasing on $[h', \infty)$ and which vanishes at u_-:

$$h'(u_+ + u_- - 2u_-) + u_-^2 - \frac{u_+^2 + u_-^2}{2} - (u_- - u_+)\left(\frac{u_- + u_+}{2} - h'(t)\right) = 0.$$

Consequently, (B.1) holds in that case.

iii. $u_+ \leqslant h'(t) \leqslant u_-$

- $\kappa \in [u_+, h']$:

$$I = h'(u_+ + u_- - 2\kappa) + \kappa^2 - \frac{u_+^2 + u_-^2}{2} + (u_- - u_+)\left(\frac{u_- + u_+}{2} - h'(t)\right)$$

The last expression is a polynomial function of degree 2 in the variable κ, which is decreasing on $(-\infty, h']$ and which vanishes at u_+. Consequently, (B.1) holds in that case.

- $\kappa \in [h', u_-]$:

$$I = h'(u_+ + u_- - 2\kappa) + \kappa^2 - \frac{u_+^2 + u_-^2}{2} - (u_- - u_+)\left(\frac{u_- + u_+}{2} - h'(t)\right)$$

The last expression is a polynomial function of degree 2 in the variable κ, which is increasing on $[h', \infty)$ and which vanishes at u_-. Consequently, (B.1) holds in that case.

As in [27], we can show that (B.1) and (3.15) with $\varphi \in \mathcal{C}_c^\infty((\mathbb{R}_+ \times \mathbb{R}) \setminus \Sigma)$ imply that (3.15) is satisfied for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$. $\qquad\square$

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