# NON-DISSIPATIVE ENTROPY SATISFYING DISCONTINUOUS RECONSTRUCTION SCHEMES FOR HYPERBOLIC CONSERVATION LAWS.

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Abstract. In this paper, we derive non-dissipative stable and entropy satisfying finite volume schemes for scalar PDEs. It is based on the previous analysis of [20], which deals with general reconstruction schemes. More precisely, we develop discontinuous-in-cell reconstruction schemes, based on a discontinuous reconstruction of the solution in each cell of the mesh at each time step. The intend is to handle well with discontinuous solutions. The schemes satisfy $L^\infty$-stability, decrease of the total variation and of an entropy, and thus are convergent to an entropy solution. A link with other formalisms is established. We propose a detailed numerical study in the cases of advection with constant velocity, of Burgers' equations, and finally for a non-convex flux.

## 1. Introduction

This paper deals with the finite volume discretization of scalar partial differential equations in dimension 1 in space,

$$\partial_t u + \partial_x f(u) = 0 \quad \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}, \tag{1}$$

where $f \in \mathcal{C}^1(\mathbb{R})$, with initial condition

$$u(0,\cdot) = u^0(\cdot) \in L^\infty(\mathbb{R}). \tag{2}$$

For the sake of simplicity and as in [2], we restrict the discussion to the case without sonic points: to fix the ideas, $f'(u) > 0$ for every $u$.

We are interested in an *entropy* solution of this problem, that is to say in a weak solution that satisfies the additional partial differential inequality

$$\partial_t S(u) + \partial_x G(u) \leq 0 \quad \text{for } t \in \mathbb{R}^+, x \in \mathbb{R} \tag{3}$$

for one entropy-entropy flux pair $(S, G)$, i.e. a pair of $C^1(\mathbb{R})$ functions $(S, G)$ such that $S$ is convex and $G' = S'f'$. It is known that this solution is unique if $f$ and $S$ are strictly convex and that this solution is the Krushkov entropy solution, cf. [12, 18] for example. This solution belongs to $L^\infty((0,T) \times \mathbb{R})$ $\forall T \in \mathbb{R}^+$ and is furthermore total variation decreasing, thus $u(t,\cdot) \in BV(\mathbb{R})$ $\forall t \in \mathbb{R}^+$ if $u^0 \in BV(\mathbb{R})$.

We are here concerned with the numerical approximation of these entropy solutions in the standard framework of finite volume schemes. Let $\Delta x \in \mathbb{R}^+_*$

and $\Delta t \in \mathbb{R}_*^+$ be given positive real numbers. We replace equation (1) with the discrete in time and space equation

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(f_{j+1/2}^n - f_{j-1/2}^n) \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z},$$

and replace (2) with the numerical initial condition $(u_j^0)_{j\in\mathbb{Z}}$,

$$u_j^0 = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u^0(x)\,dx \quad \forall j \in \mathbb{Z}.$$

The numerical fluxes $f_{j+1/2}^n$ $(j \in \mathbb{Z}, n \in \mathbb{N})$ have to be computed in such a manner that the numerical approximation

$$\overline{u}_{\Delta x}^{\Delta t}(t,x) = \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{Z}} u_j^n \chi_{[n\Delta t,(n+1)\Delta t)}(t)\chi_{[(j-1/2)\Delta x,(j+1/2)\Delta x)}(x) \tag{4}$$

converges towards the (an) entropy solution of (1,2) as $\Delta t$ and $\Delta x$ tend to 0 (in a norm to be specified). Our previous work [20] presents some general convergence conditions. The present paper uses this analysis to derive some new anti-dissipative convergent schemes. As in [20], we consider reconstruction schemes. The originality of this work is to analyze *discontinuous reconstruction* schemes, which means that the reconstruction of the solution is discontinuous in each cell. We do not intend to obtain a high-order scheme but a scheme that is precise for discontinuous solutions. This seems not to have been the aim in previous works on reconstruction schemes, usually looking for second order or high order approximations. See for example the geometric limiters theory (slope limiters) in [13], in relation with [28] for the flux limiter theory. A general reference on discrete entropy conditions is [14]. The paper [3] focuses on entropy conditions for second order geometric reconstruction schemes. MUSCL schemes and entropy conditions are studied in [8], [5] and [24], and [22] deals with high order algorithms. Let us also recall [26] for a study of convergence and order in general.

The paper organizes as follows.

First (section 2), we recall the framework and the stability and convergence results obtained in [20] for reconstruction schemes. Reconstruction schemes can be decomposed into 3 steps. The first step consists in reconstructing a given constant-in-cell solution; the second step is a resolution of the exact PDE with the reconstructed solution; the last step is a "projection on the mesh", that is to say an $L^2$-projection on the space of constant-in-cell functions.

Equipped with this, we explore a new class of reconstruction schemes: discontinuous reconstruction schemes. Section 3 is thus devoted to the transposition of the former general results to this particular reconstruction schemes. Discontinuous reconstruction schemes are based on the reconstruction of the approximate solution as a discontinuous-in-cell function with one discontinuity in each cell and one constant value on the left and one other constant value on the right of the discontinuity. The position of the discontinuity is not fixed (in particular, it is not *a priori* the middle of the cell). The goal of such a reconstruction is to compute good approximate *discontinuous* solutions and to avoid numerical dissipation. In this aim, we are led, in a first stage, to choose the reconstruction that maximizes the total variation of the solution among reconstructions guaranteeing stability and the decrease of the entropy. For this particular scheme, we prove that shocks are computed exactly. We finally analyze more precisely the cases of linear advection and the case of a convex flux: Burgers' equation. Note that the linear case, which is trivial in some sense (in particular because the exact solution is explicitly known, thus the second step of the algorithm is here trivial), is in fact a key problem: indeed, it carries the difficulty of finding a projection procedure that does not introduce numerical diffusion. Moreover, it is known that in nonlinear systems, the linearly degenerate fields are more subject to numerical diffusion, because the nonlinear shocks are compressive. Thus the case of transport is precisely taken into account in this paper.

For the sake of computing simplicity, we then explore a way to perform the previous computations in an approximate way, replacing the exact computation of the solution in the second step of the algorithm by an approximate one. This simplified procedure has to guarantee the same properties. We compare the results for different (non-dissipative) reconstructions on Burgers' equation and on an equation with a non-convex flux.

These different cases are illustrated with various numerical results.

## 2. Framework

This section is devoted to recalling the main ingredients of [20].

As mentioned in the introduction, we consider finite volume approximations of (1) of the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(f_{j+1/2}^n - f_{j-1/2}^n) \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \tag{5}$$

where $u_j^n$ is ought to represent the value of the solution $u$ in the space cell $C_j = [(j-1/2)\Delta x, (j+1/2)\Delta x)$ at time $t = n\Delta t$. The numerical initial condition $u_j^0$ is given by

$$u_j^0 = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u^0(x)\, dx \quad \forall j \in \mathbb{Z}. \tag{6}$$

We propose to compute the numerical fluxes $(f_{j+1/2}^n)_{n\in\mathbb{N}, j\in\mathbb{Z}}$ using a three-step procedure:

- given a constant-in-cell function, compute a reconstructed function that contains more details,
- compute the exact (entropy) solution at time $\Delta t$ of (1) with the reconstructed function as initial condition,
- "project" this exact solution on the mesh in order to obtain a constant-in-cell function for the following time step.

Note that the last two steps are equivalent to computing the *fluxes* of the exact solution, which shows the finite volume form of the algorithm: see equation (8).

Each of these steps can be represented by an operator: we shall call $\mathcal{R}$, $\mathcal{E}$ and $\mathcal{P}$ respectively the reconstruction, the exact and the projection operators. Here is their precise definition.

**Definition 1.** 1 *Let* $u : \mathbb{R} \longrightarrow \mathbb{R}$ *be a constant-in-cell function.*
$\mathcal{R}u : \mathbb{R} \longrightarrow \mathbb{R}$ *denotes the reconstruction of* $u$*.*

2 *Let* $t \in \mathbb{R}$ *and* $u : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function in* $L^\infty(\mathbb{R})$*.*
$\mathcal{E}(t)u : \mathbb{R} \longrightarrow \mathbb{R}$ *denotes the exact entropy solution at time* $t$ *of equation (1) with initial condition* $u$*.*

3 *Let* $u : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function in* $L^\infty(\mathbb{R})$*.*
$\mathcal{P}u : \mathbb{R} \longrightarrow \mathbb{R}$ *denotes the "projection" of* $u$ *on the mesh:*

$$\mathcal{P}u(x) = \sum_{j\in\mathbb{Z}} u_j \chi_{C_j}(x)$$

*(*$\chi_{C_j}$ *denotes the characteristic function of the cell* $C_j = [(j-1/2)\Delta x, (j+1/2)\Delta x))$ *with*

$$u_j = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx \quad \forall j \in \mathbb{Z}.$$

We now define the approximate solution $\overline{u}^n : \mathbb{R} \longrightarrow \mathbb{R}$ at time step $n$ by

$$\overline{u}^n(x) = \sum_{j\in\mathbb{Z}} u_j^n \chi_{C_j}(x).$$

The reconstruction scheme is defined by

$$\overline{u}^{n+1} = \mathcal{P}\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n. \tag{7}$$

Classically, thanks to a Green formula, the reconstruction scheme can be viewed as a finite volume one by solving the exact computation and the projection in one, taking as numerical fluxes in equation (5)

$$f_{j+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} f\left(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x)\right)\,ds \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z} \tag{8}$$

(the equivalent formulations (7) and (5,8) will be alternatively used).

In this paper, we essentially focus on the reconstruction operator (first step above, operator $\mathcal{R}$). At first we assume to be able to compute the exact solution and the projection. This of course strongly depends on the reconstruction itself and on the flux of the PDE; it will be done in the case of linear advection and Burgers' equation for discontinuous-in-cell reconstructions. For arbitrary fluxes, we propose at the end of the paper to replace the exact computation by an approximate one and apply this procedure to a non-convex flux.

The Godunov scheme corresponds to $\mathcal{R}u = u$ (no reconstruction). If $\mathcal{R}u$ is an affine-in-cell function for every constant-in-cell function $u$, the resulting scheme is a MUSCL scheme; a precise study of such a MUSCL scheme has been performed in [3].

We now briefly recall the principal results shown in [20] for general reconstructions. These results will be applied to the particular case of discontinuous reconstructions in section 3.

2.1. **Conservativity.** We consider only conservative reconstructions such that

$$\frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathcal{R}u(x)\,dx = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx = \mathcal{P}u(j\Delta x). \tag{9}$$

The exact operator $\mathcal{E}(t)$ and the projection $\mathcal{P}$ being conservative, the scheme defined by (7) is consequently conservative.

2.2. **$L^\infty$-decrease and decrease of the total variation.** We here recall an $L^\infty$ stability result extracted form [20]. Let us introduce the notations

$$\begin{cases} m = \inf_{j\in\mathbb{Z}} u_j^0, \\ M = \sup_{j\in\mathbb{Z}} u_j^0. \end{cases} \tag{10}$$

**Proposition 1.** *Assume that the CFL (Courant-Friedrichs-Lewy) condition $\max_{u\in[m,M]} f'(u)\Delta t \leq \Delta x$ is fulfilled. Assume that the reconstructed solution $\mathcal{R}\overline{u}^n$ verifies, $\forall n \in \mathbb{N}$, $\forall j \in \mathbb{Z}$,*

$$\min(u_{j-1}^n, u_j^n) \leq \frac{1}{\Delta x}\int_{(j-1/2-\theta)\Delta x}^{(j+1/2-\theta)\Delta x} \mathcal{R}\overline{u}^n(x)\,dx \leq \max(u_{j-1}^n, u_j^n) \qquad \forall\theta\in[0,1]. \tag{11}$$

*Then, the scheme given by (5, 8), or (7), is $L^\infty$-decreasing and Total Variation Decreasing (TVD).*

Recall that in the whole paper, $f'$ is assumed to be positive.

The proof relies on a stability property of $\frac{1}{\Delta x}\chi_{[-\Delta x/2,\Delta x/2]} * u(t,x) = \frac{1}{\Delta x}\int_{x-\Delta x/2}^{x+\Delta x/2} u(t,x)\,dx$ and is done in [20]. This result gives a constraint on the reconstruction for the scheme to be stable.

2.3. **Numerical entropy inequalities.** It is known that the TVD property does not imply the convergence toward the entropy solution of (1,2). To ensure this, a usual criterion is the existence of *numerical entropy fluxes.* Here is the definition from [3].

**Definition 2.** *Let $(S, G)$ be an entropy-entropy flux pair. It is said that scheme (5, 8) has discrete entropy fluxes relatively to $(S, G)$ if for every $(u_j^n)_{j\in\mathbb{Z}}$ there exists $(G_{j+1/2}^n)_{j\in\mathbb{Z}}$ such that*

- *$G_{j+1/2}^n$ is consistent with $G$ (in the classical sense of finite volume);*

•

$$S_j^{n+1} \leq S_j^n - \frac{\Delta t}{\Delta x}\left(G_{j+1/2}^n - G_{j-/2}^n\right) \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z} \tag{12}$$

$$\text{with} \quad S_j^n = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{R}\overline{u}^n(x))\,dx. \tag{13}$$

It seems reasonable, the exact resolution being used, to take, as entropy flux $G_{j+1/2}^n$, the exact flux (similarly to eq. (8))

$$G_{j+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} G\left(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x)\right)\,ds \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z}. \tag{14}$$

This choice will be done in the following.

With these numerical entropy fluxes specified, equation (12) acts like a new constraint on the reconstruction procedure.

**Remark 1.** *Thus the fluxes $f_{j+1/2}^n$ depend on the values $u_j^n$ at time step $n$ and on quantities*

$$S_j^{n-1} - \frac{\Delta t}{\Delta x}\left(G_{j+1/2}^{n-1} - G_{j-1/2}^{n-1}\right).$$

*To compute the fluxes by "following" a supplementary unknown, the entropy, is already the key ingredient of the non-dissipative schemes of* [2]*. The non-dissipative reservoir scheme of* [1] *also relies on the computation of additional variables (not entropies in this case, but a reservoir and a CFL counter).*

## 3. Discontinuous-in-cell reconstruction schemes

We now propose a new kind of reconstruction schemes that enters the stable schemes derived in [20]. These schemes are derived to capture discontinuous solutions of linear and non-linear scalar equations. For this purpose, we reconstruct the unknown inside each cell as a discontinuous function with one and only one discontinuity separating two constant values (in order the analysis to be simple). The adopted framework is thus different from the classical one of reconstruction schemes whose goal is to elaborate some high order approximate solutions: [3], [8], [13], [30] using a piecewise affine reconstruction, [4], [22] using piecewise polynomials (of desired order) reconstructions. The discontinuous reconstruction schemes are comparable to the so-called "sub-cell resolution schemes" developed in [17] by Harten. The difference is that we here use only a first order reconstruction (which is not coupled with a polynomial smooth one) and that this reconstruction is done in every cell. Furthermore, we derive some explicit stability and entropy conditions on this reconstruction. From an other point of view, such discontinuous reconstructions aiming at preserving discontinuities are related to moving mesh methods. Indeed, putting a discontinuity in the reconstructed solution can be viewed as putting an additional cell. We refer to [15] and [25] for this type of algorithms.

Let us describe the reconstruction operator $\mathcal{R}$. Given a value $u_j^n$, we reconstruct it as a function taking the value $u_{j,l}^n$ on an interval of length $d_j^n \Delta x \in [0, \Delta x]$ on the left side of the cell and the value $u_{j,r}^n$ on an interval of length $(1-d_j^n)\Delta x \in [0, \Delta x]$ on the right side of the cell (see figure 1):

(15)
$$\mathcal{R}\overline{u}^n(x) = \begin{cases} u_{j,l}^n \text{ for } x \in [(j-1/2)\Delta x, (j-1/2+d_j^n)\Delta x), \\ u_{j,r}^n \text{ for } x \in [(j-1/2+d_j^n)\Delta x, (j+1/2)\Delta x), \end{cases} \qquad \forall j \in \mathbb{Z}.$$

where $d_j^n \in [0,1]$, and $u_{j,l}^n$ and $u_{j,r}^n$ are to be specified. In all the following, $\mathcal{R}\overline{u}^n(x)$ is of the form (15). The conservativity, stability and entropy requirements will provide some constraints on the 3 parameters just introduced.

3.1. **Conservativity.** This section is the translation of section 2.1 in terms of discontinuous-in-cell reconstructions. The conservativity constraint (9) reads

$$d_j^n u_{j,l}^n + (1-d_j^n) u_{j,r}^n = u_j^n. \tag{16}$$

Thus among the 3 parameters, only 2 are free.

3.2. $L^\infty$ **and total variation-decreasing.** We here examine a simple way to adapt results of section 2.2 to discontinuous-in-cell reconstructions. We find in theorem 1 a condition that ensures the stability. In the following, $\lfloor a, b \rceil$ denotes $\mathrm{conv}(\{a\},\{b\})$, i.e. $[a,b]$ if $a \le b$, $[b,a]$ if $b \le a$.

**Theorem 1.** *Assume that* $\max_{u\in[m,M]} f'(u)\Delta t \le \Delta x$. *Assume that the reconstruction operation is conservative (eq. (16)) and that* $\mathcal{R}\overline{u}^n(x)$ *verifies*

$$u_{j,l}^n \in \lfloor u_{j-1}^n, u_j^n \rceil \quad \textit{and} \quad u_{j,r}^n \in \lfloor u_j^n, u_{j+1}^n \rceil \quad \forall j \in \mathbb{Z} \quad \textit{(see fig. 1)}. \tag{17}$$

*Then, the scheme (7, 15) is* $L^\infty$*-decreasing and TVD. As* $\Delta t$ *and* $\Delta x$ *converge to 0, the numerical solution* $\overline{u}_{\Delta x}^{\Delta t}$ *(see equation (4)) converges in* $L^\infty((0,+\infty), L^1(\mathbb{R}))$*, up to a subsequence, toward a weak solution of the initial value problem (1-2).*

**Remark 2.** *Condition (17) together with (16) implies that there is no reconstruction on a local extremum:*

$$(u_{j+1}^n - u_j^n)(u_j^n - u_{j-1}^n) \le 0 \Longrightarrow u_{j,l}^n = u_{j,r}^n = u_j^n$$

*(and* $d_j^n$ *is indifferent).*

**Remark 3.** *The result in theorem 1 is not trivial. Indeed, the condition there proposed for the reconstruction does not at all imply that the total variation of the reconstruction is smaller than the one of the initial function u. Indeed, consider the monotone case* $u_j^n < u_{j+1}^n$ $\forall j \in \mathbb{Z}$ *and the* *reconstruction*

$$\left\{\begin{array}{l} u_{j,l}^n = u_{j-1}^n, \\ u_{j,r}^n = u_{j+1}^n, \\ d_j^n = \frac{u_{j+1}^n - u_j^n}{u_{j+1}^n - u_{j-1}^n}, \end{array}\right.$$

*(this choice will be studied in section 3.4). In this case, the total variation of the reconstructed solution is three times greater than the one of* $u$*. What the theorem states is that after an exact resolution and a projection, the total variation will be reduced enough.*

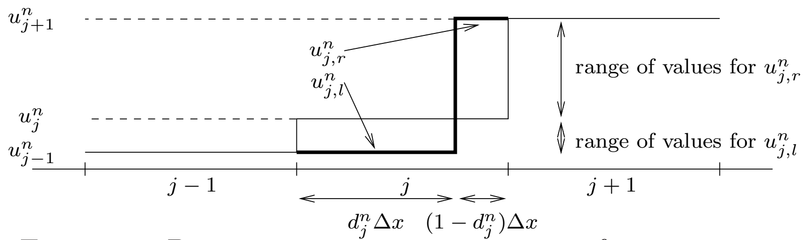


Figure 1. Discontinuous reconstruction satisfying conservativity and stability requirements.

**Proof .** *The principle of the proof is to show that condition (17) implies (11). Let us examine the convolution of the discontinuous reconstruction by* $\frac{1}{\Delta x}\chi_{[-\Delta x/2,\Delta x/2]}$ *and let us denote it* $[\mathcal{R}\overline{u}^n]_{\Delta x}$*:*

$$[\mathcal{R}\overline{u}^n]_{\Delta x}((j-\theta)\Delta x) = \frac{1}{\Delta x}\int_{(j-1/2-\theta)\Delta x}^{(j+1/2-\theta)\Delta x} \mathcal{R}\overline{u}^n(x)\,dx, \theta \in [0,1].$$

*It is a convex combination of the 4 local values of the reconstruction* $u_{j-1,l}^n$*,* $u_{j-1,r}^n$*,* $u_{j,l}^n$ *and* $u_{j,r}^n$*:*

$$[\mathcal{R}\overline{u}^n]_{\Delta x}((j-\theta)\Delta x) = \alpha_1 u_{j-1,l}^n + \alpha_2 u_{j-1,r}^n + \alpha_3 u_{j,l}^n + \alpha_4 u_{j,r}^n \tag{18}$$

*where the coefficients are given by*

$$\left\{\begin{array}{l} \alpha_1 = \max(0, d_{j-1}^n + \theta - 1), \\ \alpha_2 = \min(\theta, 1 - d_{j-1}^n), \\ \alpha_3 = \min(d_j^n, 1-\theta), \\ \alpha_4 = \max(0, 1 - d_j^n - \theta). \end{array}\right.$$

*Let us assume that* $u_{j-1}^n \le u_j^n$*. Then,* $u_{j-1}^n \le u_{j-1,r}^n$ *and the conservativity equation,*

$$d_{j-1}^n u_{j-1,l}^n + (1-d_{j-1}^n)u_{j-1,r}^n = u_{j-1}^n,$$

*implies that* $u_{j-1,l}^n \le u_{j-1}^n$*, so that* $u_{j-1,l}^n \le u_{j-1,r}^n$ *and* $u_{j-1,l}^n \le u_{j,l}^n$*, and finally* $u_{j-1,l}^n \le u_{j,r}^n$*. In conclusion,* $u_{j-1,l}^n \le \min(u_{j-1,r}^n, u_{j,l}^n, u_{j,r}^n)$*, so that* $[\mathcal{R}\overline{u}^n]_{\Delta x}((j-\theta)\Delta x)$ *verifies, by equation (18),*

$$[\mathcal{R}\overline{u}^n]_{\Delta x}((j-\theta)\Delta x) \ge [\mathcal{R}\overline{u}^n]_{\Delta x}((j-1)\Delta x) = u_{j-1}^n \quad \forall \theta \in [0,1].$$

*One can prove, on the same manner, that* $[\mathcal{R}\overline{u}^n]_{\Delta x}((j-\theta)\Delta x) \leq u_j^n \ \forall\theta \in [0,1]$. *A similar result is proved under the assumption* $u_{j-1}^n \geq u_j^n$, *leading to the inequalities (11). Proposition 1 ends the proof of stability. The convergence toward a weak solution of the problem is a classical result combining compactness in* $L^1$ *and the Lax-Wendroff theorem, see* [21].

### 3.3. **Numerical entropy inequalities.**

3.3.1. *Decrease of one strictly convex entropy.* Let $(S, G)$ be an entropy-entropy flux pair with $S$ strictly convex. We consider the entropy condition (12), where $S_j^n$ is simply given by

$$S_j^n = d_j^n S(u_{j,l}^n) + (1 - d_j^n) S(u_{j,r}^n).$$

Thus, the three parameters $d_j^n$, $u_{j,l}^n$ and $u_{j,r}^n$ are to be chosen such that they allow the existence of numerical entropy fluxes with equation (12). Some particular choices will be done in sections 3.4, 3.8 and 3.9.

The next sections are devoted to the choice of schemes that satisfy the previous conditions, for some particular scalar equations: advection equation, Burgers' equation and, finally, a non-convex example.

3.4. **A non-entropy scheme for linear advection with constant velocity.** Here is considered the linear flux $f(u) = au$ where $a > 0$ is a given constant. The weak solution to the Cauchy problem is unique and given by $u(t, x) = u^0(x - at)$, so that no entropy condition is needed. Thus, in a first step, we propose a scheme that does not ensure the decreasing of any entropy. The second scheme we propose is entropy satisfying for the quadratic entropy.

The ideas underlying the present study is to derive some non-dissipative schemes. In the classical "upwind" scheme, the dissipation is due to the projection operation $\mathcal{P}$: there is no reconstruction procedure (to say it differently, $u_{j,l}^n = u_{j,r}^n = u_j^n$ and the value of $d_j^n$ does not matter, $\forall n \in \mathbb{N}$, $\forall j \in \mathbb{Z}$). One natural idea is then to choose the reconstruction in such a manner that it maximizes the variation inside each cell, under the conservativity and stability constraints (16) and (17). This defines all the parameters of the discontinuous reconstruction as

$$\left.\begin{aligned} &u_{j,l}^n = u_{j-1}^n, \\ &u_{j,r}^n = u_{j+1}^n, \\ &d_j^n = \frac{u_{j+1}^n - u_j^n}{u_{j+1}^n - u_{j-1}^n}. \end{aligned}\right\} \quad \text{if } (u_{j+1}^n - u_j^n)(u_j^n - u_{j-1}^n) > 0,$$
$$\left.\begin{aligned} &u_{j,l}^n = u_{j,r}^n = u_j^n, \\ &d_j^n \text{ is indifferent} \end{aligned}\right\} \quad \text{if } (u_{j+1}^n - u_j^n)(u_j^n - u_{j-1}^n) \leq 0. \tag{19}$$

The following result establishes a link with other theories.

**Proposition 2.** *Assume that the CFL condition* $a\Delta t/\Delta x \leq 1$. *The scheme (7, 15, 19) with* $f(u) = au$ *and* $a > 0$ *is equivalent to the Ultra-bee limiter (cf.* [29]*) and to the limited downwind scheme (cf.* [9]*).*

**Proof .** *We prove the equivalence with the limited downwind scheme. Equivalence of the Ultra-bee limiter and the limited downwind scheme is proved in* [9]*. For this task, we recall the limited downwind scheme. It is based on formulation (5) ($f(u) = au$, $a > 0$) and relies on the intervals* $I^n_{j+1/2} = [ab^n_{j+1/2}, aB^n_{j+1/2}]$ *with*

$$b^n_{j+1/2} = \max(m^n_{j+1}, M^n_j + \frac{\Delta x}{a\Delta t}(u^n_j - M^n_j)),$$
$$B^n_{j+1/2} = \min(M^n_{j+1}, m^n_j + \frac{\Delta x}{a\Delta t}(u^n_j - m^n_j)),$$

*where*

$$m^n_j = \min(u^n_{j-1}, u^n_j),$$
$$M^n_j = \max(u^n_{j-1}, u^n_j).$$

*These intervals are shown (in* [9]*) to be such that if* $f^n_{j+1/2} \in I^n_{j+1/2}$ $\forall j \in \mathbb{Z}$ *and if* $a\Delta t/\Delta x \leq 1$, $u^{n+1}_j \in [m^n_j, M^n_j]$ $\forall j \in \mathbb{Z}$*. This condition is known to be sufficient to have the TVD property, by a classical argument of incremental analysis of Le Roux in* [23] *and Harten in* [16]*. The limited downwind scheme is obtained taking* $f^n_{j+1/2}$ *as close as possible to* $u^n_{j+1}$ *in* $I^n_{j+1/2}$*. It leads to the formula*

$$f^n_{j+1/2} = \begin{cases} ab^n_{j+1/2} & \text{if } u^n_{j+1} < b^n_{j+1/2}, \\ au^n_{j+1} & \text{if } b^n_{j+1/2} \leq u^n_{j+1} \leq B^n_{j+1/2}, \\ aB^n_{j+1/2} & \text{if } B^n_{j+1/2} < u^n_{j+1}. \end{cases}$$

*Let us now compute the flux* $f^n_{j+1/2}$ *with the discontinuous reconstruction scheme (7, 15, 19) with formula (8), assuming* $(u^n_{j+1} - u^n_j)(u^n_j - u^n_{j-1}) > 0$ *(in the other case, the equivalence of the two schemes is obvious because they both degenerate to the upwind scheme* $f^n_{j+1/2} = au^n_j$*). For the sake of simplicity, we assume that the data is locally increasing:* $u^n_{j,l} = u^n_{j-1} < u^n_j < u^n_{j+1} = u^n_{j,r}$*. Then:*

- *If* $a\Delta t \leq (1 - d^n_j)\Delta x$, $f^n_{j+1/2} = au^n_{j,r} = au^n_{j+1}$*. On the other hand, one has* $d^n_j = \frac{u^n_{j+1} - u^n_j}{u^n_{j+1} - u^n_{j-1}}$*, so that* $a\Delta t \leq (1 - d^n_j)\Delta x$ *is equivalent to* $a\frac{\Delta t}{\Delta x}(u^n_{j+1} - u^n_{j-1}) \leq (u^n_j - u^n_{j-1})$ *(the data is increasing), and* $a\frac{\Delta t}{\Delta x}u^n_{j+1} \leq (u^n_j - u^n_{j-1}) + a\frac{\Delta t}{\Delta x}u^n_{j-1}$ *and finally to* $u^n_{j+1} \leq \frac{\Delta x}{a\Delta t}(u^n_j - u^n_{j-1}) + u^n_{j-1}$*, so that*

$$B^n_{j+1/2} = \min(M^n_{j+1}, m^n_j + \frac{\Delta x}{a\Delta t}(u^n_j - m^n_j))$$
$$= \min(u^n_{j+1}, u^n_{j-1} + \frac{\Delta x}{a\Delta t}(u^n_j - u^n_{j-1})) = u^n_{j+1}.$$

*Thus* $f^n_{j+1/2}$ *and the limited downwind flux are the same under the same condition* $b^n_{j+1/2} \leq u^n_{j+1} \leq B^n_{j+1/2}$ *(condition* $b^n_{j+1/2} \leq u^n_{j+1}$ *being automatically verified because* $b^n_{j+1/2} = u^n_j < u^n_{j+1}$*).*

- *If* $a\Delta t > (1-d_j^n)\Delta x$, $f_{j+1/2}^n = (1-d_j^n)\frac{\Delta x}{\Delta t}u_{j,r}^n + (a-(1-d_j^n)\frac{\Delta x}{\Delta t})u_{j,l}^n$. *Once more, let us replace* $d_j^n$ *by* $\frac{u_{j+1}^n-u_j^n}{u_{j+1}^n-u_{j-1}^n}$. *The value of the flux is then* $f_{j+1/2}^n = \frac{u_j^n-u_{j-1}^n}{u_{j+1}^n-u_{j-1}^n}\frac{\Delta x}{\Delta t}u_{j+1}^n + (a - \frac{u_j^n-u_{j-1}^n}{u_{j+1}^n-u_{j-1}^n}\frac{\Delta x}{\Delta t})u_{j-1}^n$, *which finally leads to* $f_{j+1/2}^n = a(u_{j-1}^n + \frac{\Delta x}{a\Delta t}(u_j^n - u_{j-1}^n))$. *Furthermore, the hypothesis that the data is increasing implies that* $B_{j+1/2}^n = \min(u_{j+1}^n, u_{j-1}^n + \frac{\Delta x}{a\Delta t}(u_j^n - u_{j-1}^n))$, *and hypothesis* $a\Delta t > (1-d_j^n)\Delta x$ *tells that this minimum is* $u_{j-1}^n + \frac{\Delta x}{a\Delta t}(u_j^n - u_{j-1}^n)$, *so that* $f_{j+1/2}^n = aB_{j+1/2}^n$. *We conclude that* $f_{j+1/2}^n = aB_{j+1/2}^n$ *if* $B_{j+1/2}^n < u_{j+1}^n$, *which again coincides with the limited downwind scheme.*

*To complete the proof, one would have to consider the case of a decreasing data* $u_{j-1}^n > u_j^n > u_{j+1}^n$. *This can be done in the same manner.*

Let us now recall an interesting property of the limited downwind scheme for advection: the exact advection of step functions (see [9] for the proof, in the limited downwind formalism).

**Proposition 3.** *Let* $(u_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ *be computed with scheme (7, 15, 19) with* $f(u) = au$ *and* $a > 0$, *and* $a\frac{\Delta t}{\Delta x} \leq 1$. *Let us assume that* $\exists n \in \mathbb{N}$ *such that* $(u_j^n)_{j\in\mathbb{Z}}$ *verifies:* $\exists \alpha^n \in [0, 1)$ *such that* $\forall j \in \mathbb{Z}$,

$$u_{3j+1}^n = u_{3j}^n \text{ and } u_{3j+2}^n = \alpha^n u_{3j+1}^n + (1-\alpha^n)u_{3j+3}^n.$$

*Then*

- *either* $0 \leq \alpha^n + a\frac{\Delta t}{\Delta x} < 1$ *and for all* $j$

$$u_{3j+1}^{n+1} = u_{3j}^{n+1} = u_{3j}^n \text{ and } u_{3j+2}^{n+1} = (\alpha^{n+1})u_{3j+1}^{n+1} + (1-\alpha^{n+1})u_{3j+3}^{n+1}$$

  *with* $0 \leq \alpha^{n+1} = \alpha^n + a\frac{\Delta t}{\Delta x} < 1$;
- *or* $1 \leq \alpha^n + a\frac{\Delta t}{\Delta x} < 2$ *and for all* $j$

$$u_{3j+2}^{n+1} = u_{3j+1}^{n+1} = u_{3j+1}^n \text{ and } u_{3j+3}^{n+1} = (\alpha^{n+1})u_{3j+2}^{n+1} + (1-\alpha^{n+1})u_{3j+4}^{n+1}$$

  *with* $0 \leq \alpha^{n+1} = \alpha^n + a\frac{\Delta t}{\Delta x} - 1 < 1$.

This means that the set of step functions is preserved by the scheme and advected with the right velocity. In particular, this proposition applies for discontinuous functions such as Heaviside profiles. See [9] for extension to compressible gas dynamics and [10] for extension to multi-fluids computations. We again refer to [9] for numerical results and for a conjecture of non-diffusion in infinite time. The drawback of this scheme is that it is not entropy decreasing. This is particularly striking (and harmful) for non-linear equations, as noticed in [19]. The next sections are devoted to schemes verifying the entropy conditions above. The following schemes can thus be viewed as entropy modifications of the Ultra-bee (or limited downwind) scheme.

We propose two different entropy schemes. The first one, detailed in section 3.5, is based on the ideas of the limited downwind scheme: it consists in maximizing the variation of the reconstructed solution in each cell, under the conservativity (16), the stability (17) and the entropy (12) constraints. The result, as shown below in the numerical results section, is a non-dissipative scheme that computes exact shocks (located on only one cell) and that produces some staircase in smooth profiles such as rarefaction waves. In order to avoid these staircases, we then propose another way of reconstruction that preserves both exact shocks and regularity inside rarefaction waves, in section 3.6.

3.5. **A non-dissipative entropy scheme that maximizes the total variation.** The limited downwind scheme has been shown to be equivalent to the discontinuous reconstruction scheme with parameters defined by (19), and this can be viewed as choosing the parameters such that they maximize $|u^n_{j,r} - u^n_{j,l}|$ under constraints (16) and (17). This is not an entropy satisfying choice. We here will add some entropy condition for one particular strictly convex entropy. We here propose to maximize $|u^n_{j,r} - u^n_{j,l}|$ under constraints (16), (17) and (12) with numerical entropy fluxes given by (14), $\forall n \in \mathbb{N}$, $\forall j \in \mathbb{Z}$. The choice to maximize the total variation is motivated by the observation that numerical diffusion leads to the diminution of the total variation. Thus, maximizing the total variation during the reconstruction step appears as a natural way to avoid numerical diffusion. Another choice is proposed in section 3.6.

The numerical entropy fluxes are defined by

$$G^n_{j+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x))\,ds \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z}.$$

The following of this section is devoted to the computation of parameters $u^n_{j,l}$, $u^n_{j,r}$ and $d^n_j$ under the assumption $(u^n_{j+1} - u^n_j)(u^n_j - u^n_{j-1}) > 0$. For every $n \in \mathbb{N}, j \in \mathbb{Z}$, let us define $\Sigma^n_j = S^{n-1}_j - \frac{\Delta t}{\Delta x}(G^{n-1}_{j+1/2} - G^{n-1}_{j-1/2})$. So $\Sigma^n_j$ is the upper limit of $S^n_j$, because the numerical entropy inequality reads $S^n_j - \Sigma^n_j \leq 0$. Let us define

$$B^n_j = \left\{ (u_l, u_r, d) \in \mathbb{R}^2 \times [0,1] \text{ s.t. } \left\{ \begin{array}{l} du_l + (1-d)u_r = u^n_j, \\ u_l \in \lfloor u^n_{j-1}, u^n_j \rceil, \\ u_r \in \lfloor u^n_j, u^n_{j+1} \rceil, \\ dS(u_l) + (1-d)S(u_r) \leq \Sigma^n_j \end{array} \right. \right\}.$$

The retained reconstruction $(u^n_{j,l}, u^n_{j,r}, d^n_j)$ is the solution of the maximization problem

$$|u^n_{j,r} - u^n_{j,l}| = \max_{(u_l,u_r,d)\in B^n_j} |u_r - u_l|. \tag{20}$$

**Lemma 1.** *One has $\Sigma^n_j \geq S(u^n_j)$ and the maximization problem has a solution.*

**Proof .** *By definition,*

$$\begin{aligned}\Sigma_j^n &= S_j^{n-1} - \frac{\Delta t}{\Delta x}(G_{j+1/2}^{n-1} - G_{j-1/2}^{n-1}) \\ &= S_j^{n-1} - \frac{1}{\Delta x}\left(\int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^{n-1}((j+1/2)\Delta x))\,ds \right.\\ &\qquad\qquad \left. - \int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^{n-1}((j-1/2)\Delta x))\,ds\right),\end{aligned}$$

*thus the entropy property of the exact operator implies*

$$\frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^{n-1}(x))\,dx \leq \Sigma_j^n.$$

*Now, by virtue of Jensen's inequality, one recovers* $S(u_j^n) \leq \Sigma_j^n$*. This shows that* $B_j^n$ *is a non-empty set containing at least* $(u_j^n, u_j^n, d)$ *for every* $d \in [0,1]$*.* $B_j^n$ *is a closed non-empty set and the function to maximize is continuous, so that there exists a solution to the maximization problem.*

The interest of numerical entropy inequalities is to ensure the convergence toward an entropy solution.

**Theorem 2.** *Consider a scheme in the form (7, 15). Assume that* $\max_{u\in[m,M]} f'(u)\Delta t \leq \Delta x$*. Assume that the reconstruction operation is conservative (eq. (16)) and that* $\mathcal{R}\overline{u}^n(x)$ *verifies*

$$\left(u_{j,l}^n, u_{j,r}^n, d_j^n\right) \in B_j^n \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z}.$$

*Then, the scheme is* $L^\infty$*-decreasing and TVD and owns numerical entropy fluxes. As* $\Delta t$ *and* $\Delta x$ *converge to 0, the numerical solution* $\overline{u}_{\Delta x}^{\Delta t}$ *converges in* $L^\infty((0,+\infty), L^1(\mathbb{R}))$*, up to a subsequence, toward a weak solution of the initial value problem (1-2) that verifies the entropy inequality*

$$\partial_t S(u) + \partial_x G(u) \leq 0 \quad \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}.$$

The proof is a classical consequence of the Lax-Wendroff theorem. Remark that if $f$ and $S$ are strictly convex, the entropy solution is unique, thus the whole sequence of numerical approximations converges to the Krushkov solution.

As for the limited downwind scheme in the linear case, we now state a result of exact computation of pure shock solutions.

**Proposition 4.** *Assume that the initial condition is* $u^0(x) = u_L$ $\forall x \in (-\infty, -\Delta x/2]$*,* $u^0(x) = u_R$ $\forall x \in (-\Delta x/2, +\infty)$*, discretized as* $u_j^0 = u_L$ $\forall j < 0$*,* $u_j^0 = u_R$ $\forall j \geq 0$*, such that* $u^0(t, x - \sigma t)$ *is a shock solution of (1) verifying the entropy inequality (3) for a given entropy* $S$*. Then the scheme (7, 15, 20) is exact in the sense that* $\forall n \in \mathbb{N}$*,* $\forall j \in \mathbb{Z}$*,* $u_j^n = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u^0(t, x - \sigma n\Delta t)\,dx$*.*

**Proof .** *There is no reconstruction for the first iterate. Assume for simplicity that* $\sigma\Delta t \le \Delta x$*, which is less strong than a usual CFL condition. After one iterate with time step* $\Delta t$*, the exact solution is* $\mathcal{E}(\Delta t)\overline{u}^0 = u^0(t, x-\sigma\Delta t)$*, so that* $u_0^1 = \sigma\frac{\Delta t}{\Delta x}u_L + (1-\sigma\frac{\Delta t}{\Delta x})u_R$*, and, because of the entropy inequality,* $\Sigma_0^1 \ge \sigma\frac{\Delta t}{\Delta x}S(u_L) + (1-\sigma\frac{\Delta t}{\Delta x})S(u_R)$*. Thus the solution of (20) is*

$$\begin{cases} u_{1,l}^1 = u_L, \\ u_{1,r}^1 = u_R, \\ d_1^1 = \sigma\frac{\Delta t}{\Delta x}. \end{cases}$$

*The reconstructed solution is the exact solution* $u^0(t, x-\sigma\Delta t)$ *and the shock will be propagated for each time step. This proves the result.*

For the sake of simplicity, we now focus on the entropy

$$S(u) = \frac{u^2}{2}.$$

**Lemma 2.** *Assume that* $(u_{j+1}^n - u_j^n)(u_j^n - u_{j-1}^n) > 0$ *and that* $\Sigma_j^n > S(u_j^n)$*, and let* $(u_{j,l}^n, u_{j,r}^n, d_j^n)$ *be such that*

$$|u_{j,r}^n - u_{j,l}^n| = \max_{(u_l,u_r,d)\in B_j^n} |u_r - u_l|.$$

*Then, either* $u_{j,l}^n = u_{j-1}^n$ *or* $u_{j,r}^n = u_{j+1}^n$*. Moreover,* $S_j^n = d_j^n S(u_{j,l}^n) + (1-d_j^n)S(u_{j,r}^n) = \max(\Sigma_j^n, \delta S(u_{j-1}^n) + (1-\delta)S(u_{j+1}))$ *with* $\delta = (u_{j+1}^n - u_j^n)/(u_{j+1}^n - u_{j-1}^n)$*.*

**Proof .** *First, remark that maximizing* $|u_{j,r}^n - u_{j,l}^n|$ *is equivalent to maximizing* $(u_{j,r}^n - u_{j,l}^n)^2$*. This function to maximize being strictly convex, any solution lies on the boundary of* $B_j^n$*. If* $\Sigma_j^n \ge \delta S(u_{j-1}^n) + (1-\delta)S(u_{j+1}^n)$*, with* $\delta = (u_{j+1}^n - u_j^n)/(u_{j+1}^n - u_{j-1}^n)$*, then the unique solution is trivially* $u_{j,l}^n = u_{j-1}^n$*,* $u_{j,r}^n = u_{j+1}^n$ *(and the lemma is proved). Thus let us now assume that* $\Sigma_j^n < \delta S(u_{j-1}^n) + (1-\delta)S(u_{j+1}^n)$*, which means that the entropy constraint plays a role. We claim that the entropy of the reconstructed solution is* $d_j^n S(u_{j,l}^n) + (1-d_j^n)S(u_{j,r}^n) = \Sigma_j^n$*, that is to say, the maximal value. Indeed, let* $(u_l, u_r, d) \in B_j^n$ *be such that* $dS(u_l) + (1-d)S(u_r) < \Sigma_j^n$*. From* $(u_l, u_r, d) \in \partial B_j^n$ *one has that* $u_l \in \partial\lfloor u_{j-1}^n, u_j^n\rceil$ *or* $u_r \in \partial\lfloor u_j^n, u_{j+1}^n\rceil$*. Assume the first case (the second case could be treated symmetrically):* $u_l \in \partial\lfloor u_{j-1}^n, u_j^n\rceil$*, namely,* $u_l = u_{j-1}^n$ *or* $u_l = u_j^n$*. The case* $u_l = u_j^n$ *leads to* $u_r = u_j^n$ *and is uninteresting (trivially not the solution of the maximization problem), thus we consider* $u_l = u_{j-1}^n$*. Note that* $u_r \ne u_{j+1}^n$ *because we assumed that* $\Sigma_j^n < \delta S(u_{j-1}^n) + (1-\delta)S(u_{j+1}^n)$*, and* $u_r \ne u_j^n$ *because* $u_l \ne u_j^n$*. By continuity, there exists* $\varepsilon > 0$ *such that* $(u_l, u_r+\varepsilon, d_\varepsilon) \in B_j^n$ *and* $(u_l, u_r-\varepsilon, d_{-\varepsilon}) \in B_j^n$*. And either* $|u_r+\varepsilon-u_l| > |u_r-u_l|$ *or* $|u_r-\varepsilon-u_l| > |u_r-u_l|$*, which proves that* $(u_l, u_r, d)$ *is not a solution. Thus we proved the second claim in the lemma.*

*The maximization problem in $B_j^n$ reduces to the maximization problem in*

$$D_j^n = \left\{ (u_l, u_r, d) \in \mathbb{R}^2 \times [0,1] \ s.t. \ \left\{ \begin{array}{l} du_l + (1-d)u_r = u_j^n, \\ u_l \in \lfloor u_{j-1}^n, u_j^n \rceil \ \text{and} \ u_l \neq u_j^n, \\ u_r \in \lfloor u_j^n, u_{j+1}^n \rceil \ \text{and} \ u_r \neq u_j^n, \\ dS(u_l) + (1-d)S(u_r) = \Sigma_j^n \end{array} \right. \right\}.$$

*Now, using the expression of the entropy, one has*

$$\begin{aligned} 2S_j^n = \frac{u_{j,r}^n - u_j^n}{u_{j,r}^n - u_{j,l}^n} {u_{j,l}^n}^2 + \frac{u_j^n - u_{j,l}^n}{u_{j,r}^n - u_{j,l}^n} {u_{j,r}^n}^2 &= (u_j^n - u_{j,l}^n)(u_{j,r}^n - u_j^n) + u_j^{n2} \\ &= (u_j^n - u_{j,l}^n)(u_{j,r}^n - u_{j,l}^n) - (u_j^n - u_{j,l}^n)^2 + u_j^{n2}. \end{aligned}$$

*Thus*

$$u_{j,r}^n - u_{j,l}^n = \frac{2S_j^n + (u_j^n - u_{j,l}^n)^2 - u_j^{n2}}{u_j^n - u_{j,l}^n}.$$

*This gives an explicit form of $D_j^n$ (always in the case where $\Sigma_j^n < \delta S(u_{j-1}^n) + (1-\delta)S(u_{j+1}^n)$):*

$$D_j^n = \left\{ (u_l, u_r, d) \in \mathbb{R}^2 \times [0,1] \ s.t. \ \left\{ \begin{array}{l} du_l + (1-d)u_r = u_j^n, \\ u_l \in \lfloor u_{j-1}^n, u_j^n \rceil \ \text{and} \ u_l \neq u_j^n, \\ u_r \in \lfloor u_j^n, u_{j+1}^n \rceil \ \text{and} \ u_r \neq u_j^n, \\ u_r = u_l + \frac{2\Sigma_j^n + (u_j^n - u_l)^2 - u_j^{n2}}{u_j^n - u_l} \end{array} \right. \right\}.$$

*Once again thanks to the strict convexity of the function to maximize, any solution is attained on $\partial D_j^n$: $u_{j,l}^n = u_{j-1}^n$ or $u_{j,r}^n = u_{j+1}^n$. See figure 2.*

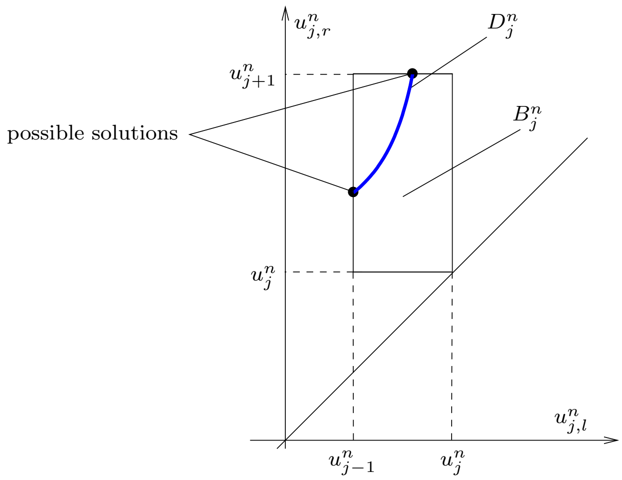


FIGURE 2. Constraints set.

**Effective computations**

It remains to describe briefly the scheme. This is now only a matter of computation. Without the entropy constraint, the parameter $d_j^n$ solution to the maximization problem is $d_j^n = \frac{u_{j+1}^n - u_j^n}{u_{j+1}^n - u_{j-1}^n}$ if $(u_{j+1}^n - u_j^n)(u_j^n - u_{j-1}^n) > 0$. With constraint (12), the maximization is more intricate, but lemma 2 tells that only two possibilities have to be tested.

The reconstruction of the solution at the time step $n$ in the cell $C_j$ follows the following algorithm. Assume that $u_{j-1}^n$, $u_j^n$, $u_{j+1}^n$ and $\Sigma_j^n = S_j^{n-1} - \frac{\Delta t}{\Delta x}(G_{j+1/2}^{n-1} - G_{j-1/2}^{n-1})$ are known.

- If $(u_{j+1}^n - u_j^n)(u_j^n - u_{j-1}^n) \leq 0$, take $u_{j,l}^n = u_{j,r}^n = u_j^n$, and $d_j^n$ is indifferent (case without reconstruction).
- Else, we have to compute (before comparing the resulting local variations) $(u_{j,l}^n, u_{j,r}^n, d_j^n)$ in the two cases: $u_{j,l}^n = u_{j-1}^n$ and $u_{j,r}^n = u_{j+1}^n$. Straightforward and uninteresting computations lead to the following algorithm.
  - For the case $u_{j,l}^n = u_{j-1}^n$, compute first
  $$l_j^n = 1 - \frac{(u_j^n - u_{j-1}^n)^2}{2\Sigma_j^n - 2u_{j-1}^n u_j^n + {u_{j-1}^n}^2},$$
  who is a candidate to be $d_j^n$, and
  $$v_{j,r}^n = \frac{u_j^n - l_j^n u_{j-1}^n}{1 - l_j^n},$$
  who is a candidate to be $u_{j,r}^n$. Then, either $v_{j,r}^n \in \lfloor u_j^n, u_{j+1}^n \rceil$ and all the constraints are satisfied, or $v_{j,r}^n \notin \lfloor u_j^n, u_{j+1}^n \rceil$ and then we take finally
  $$v_{j,r}^n = u_{j+1}^n$$
  and
  $$l_j^n = \frac{u_{j+1}^n - u_j^n}{u_{j+1}^n - u_{j-1}^n},$$
  the entropy constraint is here inactive.
  - For the case $u_{j,r}^n = u_{j+1}^n$, compute first
  $$m_j^n = \frac{(u_{j+1}^n - u_j^n)^2}{2\Sigma_j^n - 2u_{j+1}^n u_j^n + {u_{j+1}^n}^2},$$
  who is another candidate to be $d_j^n$, and
  $$w_{j,l}^n = \frac{u_j^n - (1-m)u_{j+1}^n}{m},$$
  who is a candidate to be $u_{j,l}^n$. Then, either $w_{j,l}^n \in \lfloor u_{j-1}^n, u_j^n \rceil$ and all the constraints are satisfied, or $w_{j,l}^n \notin \lfloor u_{j-1}^n, u_j^n \rceil$ and then we take finally
  $$w_{j,l}^n = u_{j-1}^n$$

and

$$m_j^n = \frac{u_{j+1}^n - u_j^n}{u_{j+1}^n - u_{j-1}^n},$$

the entropy constraint is here inactive.

- Now, compare the two possible reconstructions in term of total variation:
  - either $|v_{j,r}^n - u_{j-1}^n| > |u_{j+1}^n - w_{j,l}^n|$ and
  $$u_{j,l}^n = u_{j-1}^n, \quad u_{j,r}^n = v_{j,r}^n, \quad d_j^n = l_j^n,$$
  - or $|v_{j,r}^n - u_{j-1}^n| \leq |u_{j+1}^n - w_{j,l}^n|$ and
  $$u_{j,l}^n = w_{j,l}^n, \quad u_{j,r}^n = u_{j+1}^n, \quad d_j^n = m_j^n.$$

The algorithm is complete.

**Remark 4.** *By Jensen's inequality, quantities* $2\Sigma_j^n - 2u_{j-1}^n u_j^n + {u_{j-1}^n}^2$ *and* $2\Sigma_j^n - 2u_{j+1}^n u_j^n + {u_{j+1}^n}^2$ *involved in the definitions of possible* $d_j^n$ *are non-negative and smaller than 1. They unfortunately can be* 0*, but this means that* $\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^{n-1} = u_j^n = u_{j-1}^n$ *in* $[(j-1/2)\Delta x, (j+1/2)\Delta x]$*, which naturally leads to the choice of* $u_{j,l}^n = u_{j,r}^n = u_j^n$*, and the value of* $d_j^n$ *then does not matter. This has to be taken in consideration carefully on the numerical point of view.*

3.6. **A more regularizing entropy satisfying choice.** The algorithm previously described, maximizing the total variation of the reconstruction (under some constraints), presents the drawback of creating stair-cases (see the sections devoted to numerical experiments in the following). We here propose, among all possibilities, another reconstruction, which is justified *a posteriori* by the numerical results it produces. With the same definitions as in the algorithm above, let us follow

- either $|v_{j,r}^n - u_{j-1}^n| > |u_{j+1}^n - w_{j,l}^n|$ and
$$u_{j,l}^n = w_j, n_l, \quad u_{j,r}^n = u_{j+1}^n, \quad d_j^n = m_j^n,$$
- or $|v_{j,r}^n - u_{j-1}^n| \leq |u_{j+1}^n - w_{j,l}^n|$ and
$$u_{j,l}^n = u_{j-1}^n, \quad u_{j,r}^n = v_{j,r}^n, \quad d_j^n = l_j^n.$$

This reconstruction consists in taking either $u_{j,l}^n = u_{j-1}^n$ or $u_{j,r}^n = u_{j+1}^n$, the choice being the one leading to the *smallest* total variation *of the two*. This produces more regularized solutions without any spreading of discontinuities. Of course proposition 4 remains true with this new scheme.

3.7. **Application to advection equation with constant velocity.** The schemes developed above are here used to solve the advection equation with unit velocity $\partial_t u + \partial_x u = 0$, with periodic boundary conditions. The entropy flux is $G(u) = u^2/2$. The test-case is Harten's one (cf. [17]) in $[-1, 1]$:

$$u^0(x) = \begin{cases} 2x + 2 - \sin(3\pi(x - 1/2))/6 & \text{if } -1 \le x < -1/2, \\ (1/2 - x)\sin(3/2\pi(x - 1/2)^2) & \text{if } -1/2 \le x < 1/6, \\ |\sin(2\pi(x - 1/2))| & \text{if } 1/6 \le x < 5/6, \\ 2x - 2 - \sin(3\pi(x - 1/2))/6 & \text{if } 5/6 \le x < 1. \end{cases}$$

Here are the results after 50 periods ($t = 100$) with $\Delta t = 0.8\Delta x$, with 50 cells and 200 cells. We compare the results of the classical Minmod limiter (cf. [12, 28, 29]), the limited downwind scheme (Ultra-bee scheme, or discontinuous reconstruction scheme without entropy constraint) the self-adaptive anti-diffusive scheme of [2] and the two entropy decreasing reconstructions proposed in the paper.

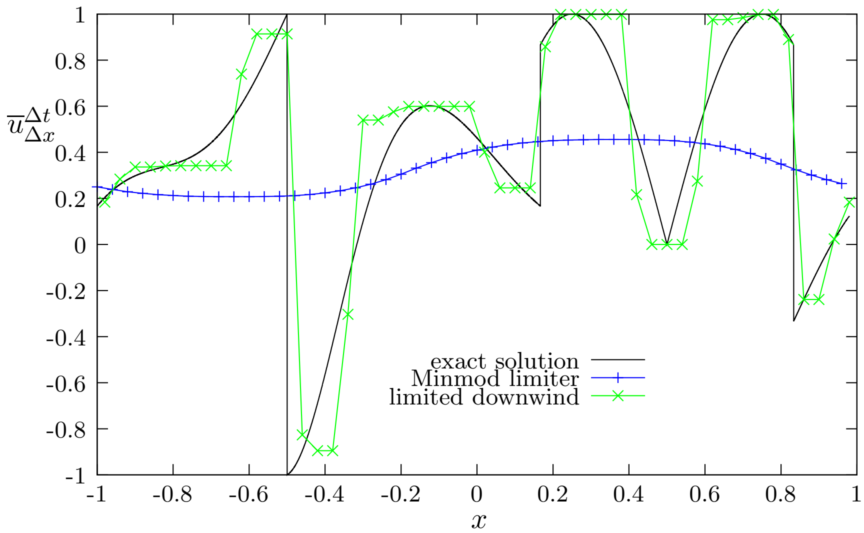


FIGURE 3. With 50 cells.

The results show that discontinuities are kept by the two proposed versions of discontinuous reconstruction schemes (as stated). The behavior in smooth region is nevertheless different. Small steps are created during the first time iterates by the first discontinuous reconstruction scheme, and are then "perfectly" advected. This behavior is similar to the one of the limited downwind scheme, that is to say the Ultra-bee scheme (as noticed in [9]), but here with much better accuracy. Indeed, the length of the steps does not exceed 2 cells. The results with the self-adaptive anti-diffusive scheme of [2] have a similar shape in certain regions, see for example figures 7 and 8. The second reconstruction provides more smoothness but has the light drawback of deteriorating a little the solution in very long time, as is seen on figure 6 near $x = -0.8$ and $x = -0.2$. For those results in the linear case, the first reconstruction provides better results in long time. Nevertheless,

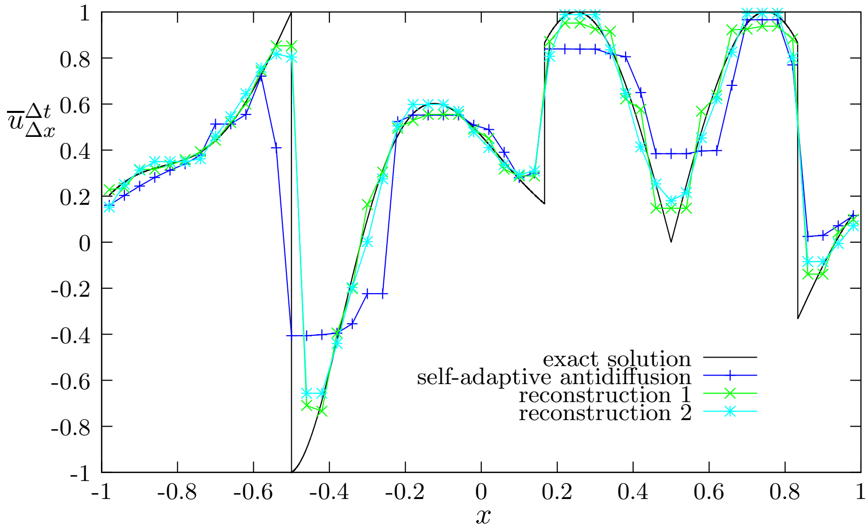


FIGURE 4. With 50 cells.

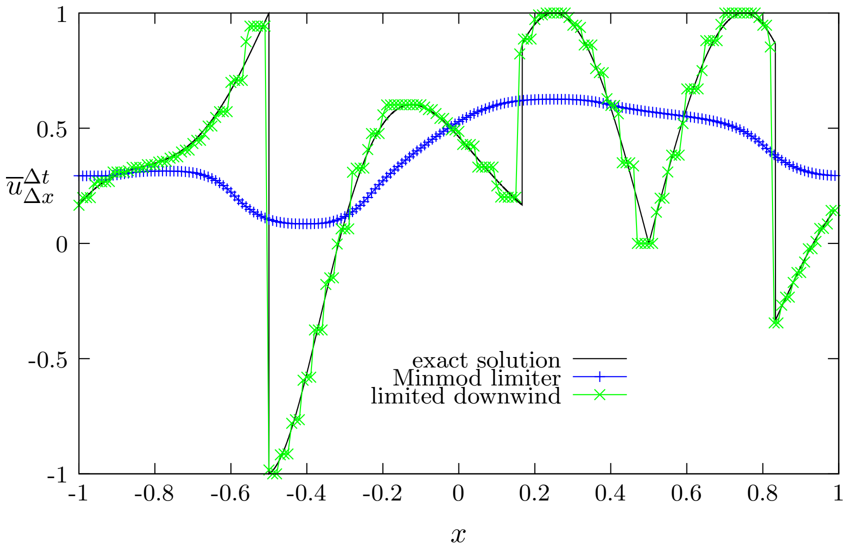


FIGURE 5. With 200 cells.

one shall see in the following that it creates much more steps in rarefaction waves of non-linear equations.

At last, let us mention that the reconstruction schemes here presented are of order of accuracy 1, (recall that Bouchut's self-adaptive scheme is of order 2), as shown on table 3.7, presenting the $L^1$ error for the advection test with sine initial condition $u^0(x) = \sin(2\pi x)$ on $[0, 1]$ with periodic boundary conditions.

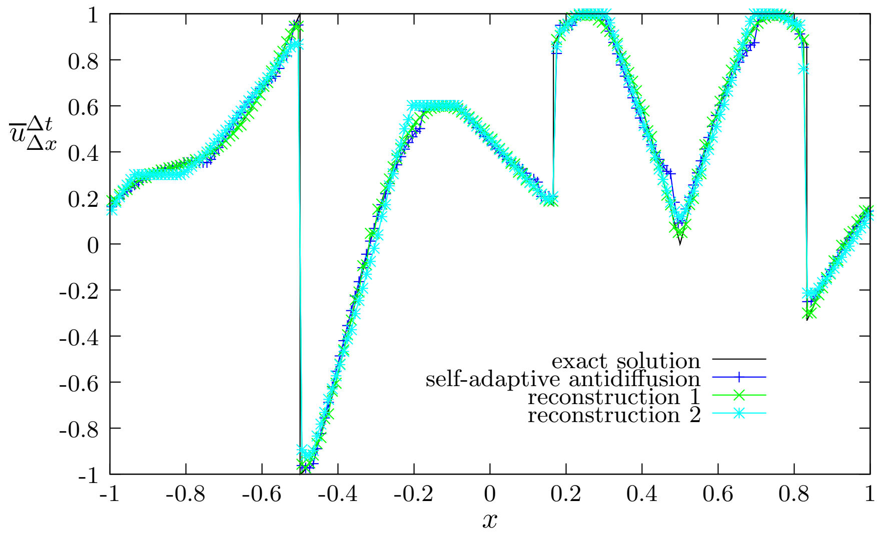


FIGURE 6. With 200 cells.

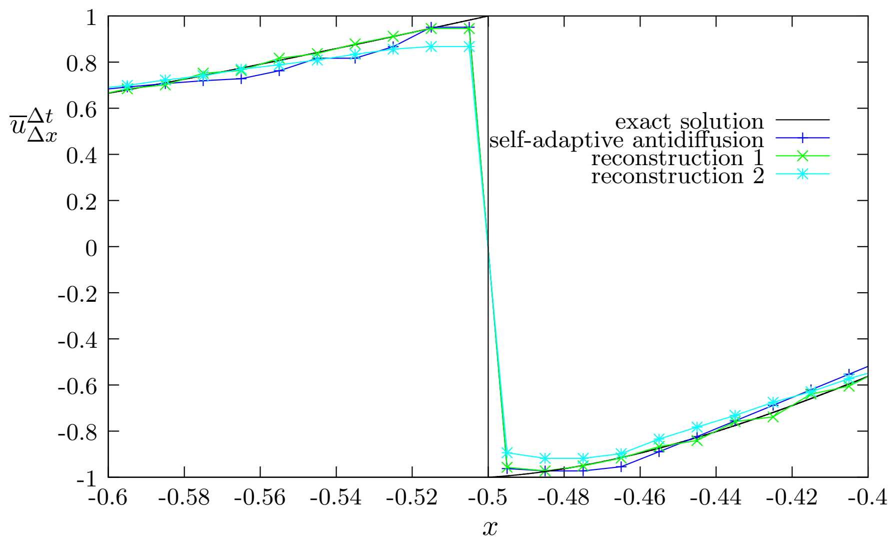


FIGURE 7. Zoom of figure 6.

| Number of cells | First reconstruction | Second reconstruction |
|---|---|---|
| 16 | 0.05 | 0.048 |
| 32 | 0.035 | 0.027 |
| 64 | 0.017 | 0.0095 |
| 128 | 0.0068 | 0.0039 |
| 256 | 0.0039 | 0.002 |
| 512 | 0.0017 | 0.001 |
| 1024 | 0.0009 | 0.00042 |

FIGURE 9. Error with different meshes with a sine initial condition, after one revolution.

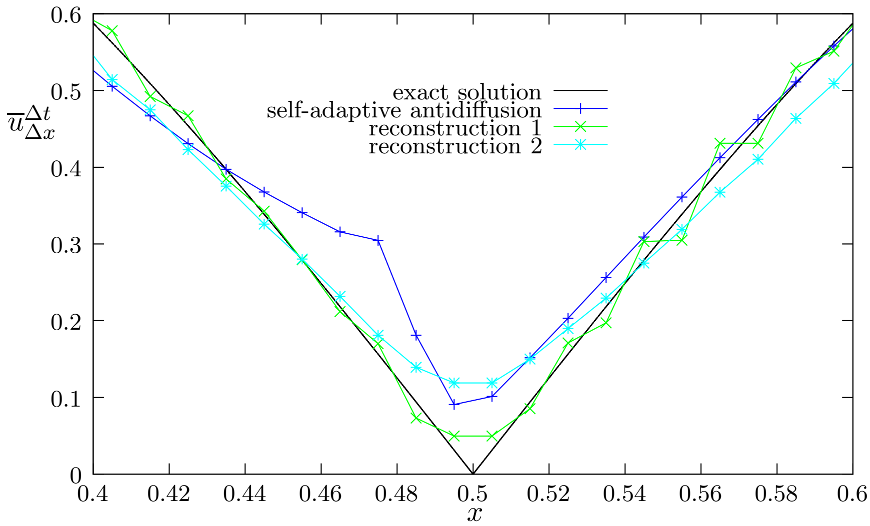


FIGURE 8. Another zoom of figure 6.

3.8. **Application to Burgers' equation.** The limited downwind scheme, here reinterpreted as the discontinuous reconstruction scheme (7, 15, 19) with $f(u) = \frac{u^2}{2}$, has been shown to produce non-entropy shocks for Burgers' equation in [19]. In this reference, a modification of the limited downwind scheme is proposed that leads to entropy inequalities, but with too much dissipation in rarefaction waves. We will see in the present section that the entropy conditions derived above are large enough to obtain non-dissipative entropy schemes. The entropy flux is $G(u) = \frac{u^3}{3}$.

All the tests below are computed with periodic boundary conditions.

We first present results combining a shock and a rarefaction wave, obtained with $1+\chi_{[0.1,0.6]}$ as initial condition. The final time is 0.2. The exact solution is

$$u(0.2, x) = \begin{cases} 1 \text{ if } x \leq 0.3, \\ 1 + 5(x - 0.3) \text{ if } 0.3 \leq x \leq 0.5, \\ 2 \text{ if } 0.5 \leq x \leq 0.9. \end{cases}$$

Numerical computations have been done with a Courant number

$$\max_j u_j^n \Delta t/\Delta x = 0.3.$$

We shall compare the results given by the Minmod limiter, the self-adaptive anti-diffusive entropy scheme of [2] and the two reconstruction schemes discussed above. Figures 10 and 11 present results at time $t = 0.2$ with 50 and 200 cells. We then observe the long-time behavior of the numerical solutions on figures 12 and 13. The initial condition is the same as above, but the final time is now $t = 100$. Both test-cases show the anti-dissipative behavior of the proposed reconstruction algorithms: the shock is not dissipated at all. What is remarkable is that the stair-cases effect with the first reconstruction, that maximizes the total variation in the reconstruction, does not disagree with the (stated) convergence toward the entropy solution. This is numerically verified on the results with 200 cells. In particular, we notice that the stair-cases due to the first reconstruction, similar at the first glance to the ones of the Ultra-bee limiter or limited downwind scheme (cf [29], [9]), are here controlled (to fix the ideas, the length of the stair-cases is of 3 or 4 cells, thus converges to 0 when refining the mesh). We note that the second proposed reconstruction gives excellent results, both for the shock and the rarefaction wave. It allows to obtain more smooth numerical solutions that are comparable to those of [2].

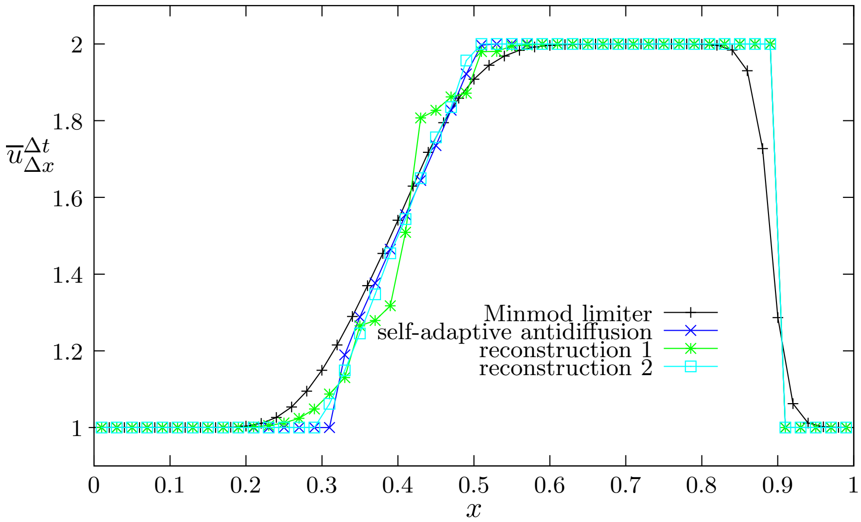


FIGURE 10. Final time 0.2, 50 cells.

3.9. **Approximate resolution and application to other scalar equations.** It may be difficult to compute the exact fluxes with such discontinuous reconstructed initial conditions, because there is no positive lower bound on $1 - d_j^n$, so that there is no CFL condition ensuring *a priori* that $f_{j+1/2}^n = f(u_{j,r}^n)$: the wave coming from the reconstructed discontinuity *inside* the cell $C_j$ can travel through the cell interface at $(j+1/2)\Delta x$ in arbitrarily small time (see figure 14). In the previous computations, the *exact* flux was computed, taking into account the variation in time of the exact solution at the point $(j+1/2)\Delta x$. This is tractable in the case of Burgers' equation, but not easy for a general non-linear flux. We propose one way to compute easily *approximate* fluxes with the discontinuous reconstruction. The reconstruction procedure is the same as in the preceding. To compute an approximate flux $f_{j+1/2}^n$ (and the associate entropy flux $G_{j+1/2}^n$), we decompose the time step in two stages: the first stage corresponds to a

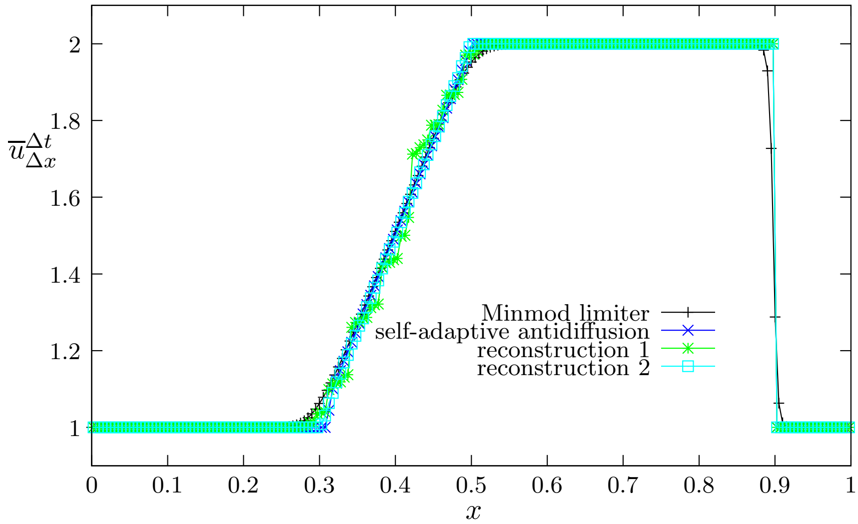


FIGURE 11. Final time 0.2, 200 cells.

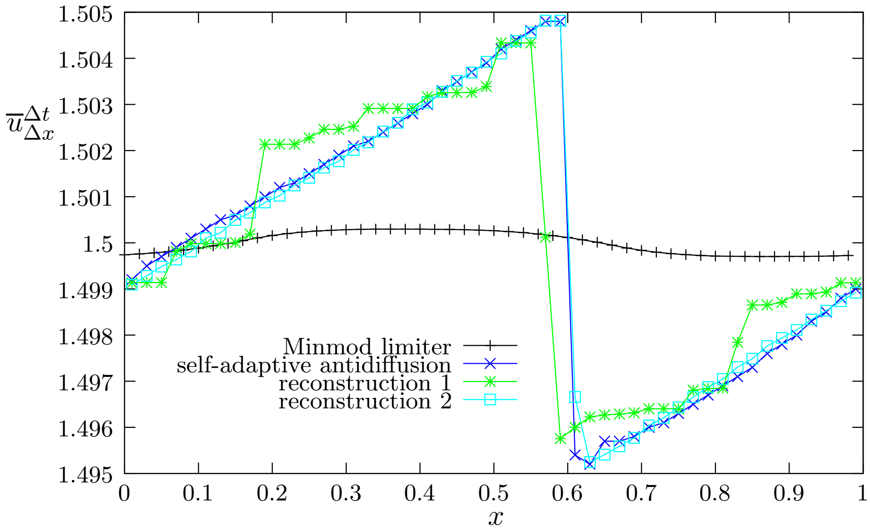


FIGURE 12. Final time 100, 50 cells.

time step $\Delta t^n_{j+1/2}$ which is sufficiently small to prevent the wave generated inside the cell $C_j$ (at $(j - 1/2 + d^n_j)\Delta x$) from attaining $(j + 1/2)\Delta x$. The second stage, with time step $\Delta t - \Delta t^n_{j+1/2}$, is preceded by a projection of the solution on the right half-cell $[j\Delta x, (j + 1/2)\Delta x]$.

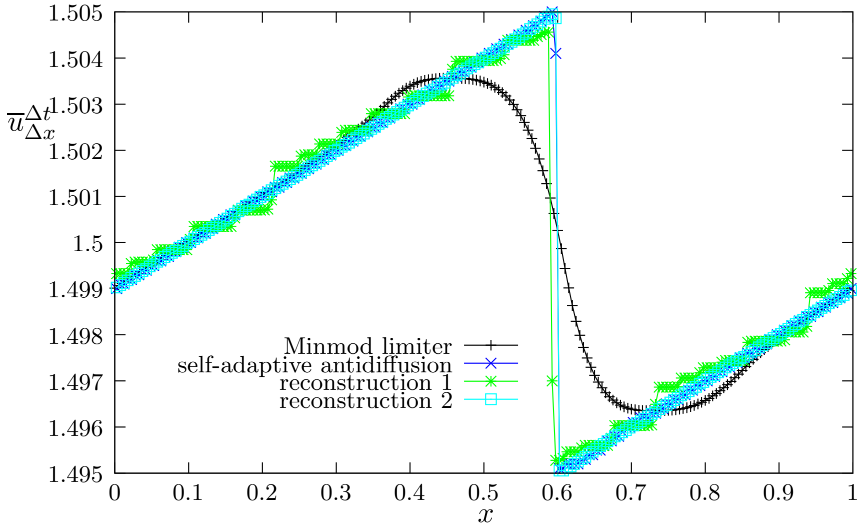


FIGURE 13. Final time 100, 200 cells.

Let us describe precisely the procedure. We consider a time step $\Delta t$ verifying

$$\frac{\Delta t}{\Delta x} \max_{u \in [\inf_{j\in\mathbb{Z}} u_j^0, \sup_{j\in\mathbb{Z}} u_j^0]} f'(u) \leq 1/2, \tag{21}$$

corresponding to a Courant number 1/2 (the usual upper bound is 1). The procedure to compute the flux $f_{j+1/2}^n$ is the following.

- Perform the reconstruction of $\overline{u}^n$ as in the rest of the paper.
- If $d_j^n \leq 1/2$, take $f_{j+1/2}^n = f(u_{j,r}^n)$ and $G_{j+1/2}^n = G(u_{j,r}^n)$ (these fluxes correspond to the exact solution): the CFL condition above indeed guarantees that the discontinuity in the cell $C_j$ does not pass through the cell edge $(j+1/2)\Delta x$.
- If $d_j^n > 1/2$, then the maximal time step computed above (eq. (21)) may be to large to prevent the wave generated inside the cell $C_j$ (resulting from the reconstruction) from attaining $(j+1/2)\Delta x$ (and the exact computation of the flux is hard to perform); thus compute a maximal *local* time step $\Delta t_{j+1/2}^n$ ensuring that the wave coming from the point $(j-1/2+d_j^n)\Delta x$ does not attain $(j+1/2)\Delta x$. That is to say, compute $\Delta t_{j+1/2}^n$ such that

  $$\Delta t_{j+1/2}^n \sigma_j^n = (1-d_j^n)\Delta x.$$

  where $\sigma_j^n$ is the maximal wave (shock or rarefaction) velocity for the Riemann problem $(u_{j,l}^n, u_{j,r}^n)$ inside the cell $C_j$ (recall that $\sigma_j^n \geq 0$).

– If $\Delta t^n_{j+1/2} \geq \Delta t$ (in this case $\Delta t$ is sufficiently large to prevent from interaction), take $f^n_{j+1/2} = f(u^n_{j,r})$ and $G^n_{j+1/2} = G(u^n_{j,r})$, this corresponds to the exact resolution.
– Else, that is to say, if $\Delta t^n_{j+1/2} < \Delta t$ (which does occur for example if $d^n_j \approx 1$) the flux $f^n_{j+1/2}$ will be the addition of the flux $f(u^n_{j,r})$ during $\Delta t^n_{j+1/2}$ and a "residual" flux for the residual time $\Delta t - \Delta t^n_{j+1/2}$. This is done in the following way. We consider the half-cell $[j\Delta x, (j+1/2)\Delta x]$ (which "contains" the discontinuity of the reconstruction). The "projection" of the reconstructed solution $\overline{u}^n$ on this half-cell is

$$2\left((d^n_j - 1/2)u^n_{j,l} + (1-d^n_j)u^n_{j,r}\right).$$

During the time $\Delta t^n_{j+1/2}$, the exact right flux for this half-cell (for the reconstructed solution) is $f(u^n_{j,r})$ while the left flux is $f(u^n_{j,l})$. Thus we can compute the mean value $u^{n+1,-}_{j+1/2,-}$ in the right half-part of the cell $C_j$ at time $t^n + \Delta t^n_{j+1/2}$:

$$u^{n+1,-}_{j+1/2,-} = 2\left((d^n_j - 1/2)u^n_{j,l} + (1-d^n_j)u^n_{j,r}\right) - \frac{\Delta t^n_{j+1/2}}{\Delta x/2}\left(f(u^n_{j,r}) - f(u^n_{j,l})\right).$$

Then it is possible to compute the exact fluxes associated with this new initial condition for a time step $\Delta t - \Delta t^n_{j+1/2} < \Delta t$ because $\Delta t$ is such that the wave generated at the middle of the cell does not attain $(j+1/2)\Delta x$ (see figure 14). The numerical flux and numerical entropy flux between time $t^n + \Delta t^n_{j+1/2}$ and time $t^{n+1} = t^n + \Delta t$ are respectively $f(u^{n+1,-}_{j+1/2,-})$ and $G(u^{n+1,-}_{j+1/2,-})$. Thus the global fluxes are taken as

$$f^n_{j+1/2} = \frac{1}{\Delta t}\left(\Delta t^n_{j+1/2} f(u^n_{j,r}) + (\Delta t - \Delta t^n_{j+1/2}) f(u^{n+1,-}_{j+1/2,-})\right).$$

and

$$G^n_{j+1/2} = \frac{1}{\Delta t}\left(\Delta t^n_{j+1/2} G(u^n_{j,r}) + (\Delta t - \Delta t^n_{j+1/2}) G(u^{n+1,-}_{j+1/2,-})\right).$$

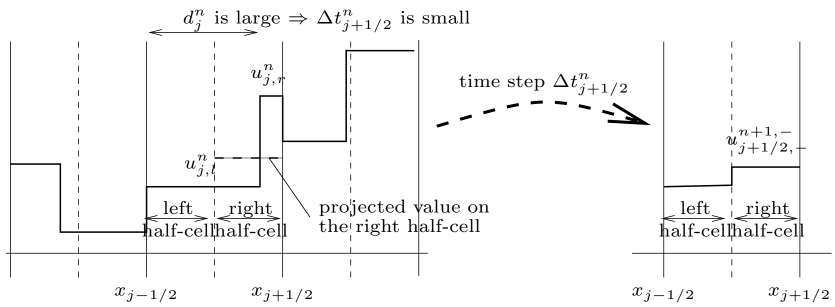


FIGURE 14. Approximate resolution with the help of a finer mesh.

We notice that this procedure consists in, when necessary, projecting on a half-cell the exact solution, which reduces the total variation and the entropy. Thus the approximate solution is $L^\infty$-decreasing, TVD, and owns entropy fluxes, as in theorem 2.

What is remarkable in this approximate resolution is that the projection on half-cells does not introduce more numerical diffusion, as it could be expected (see the numerical results). The difference between the solutions obtained with the exact and the approximate operators is very tiny.

**Remark 5.** *The computation of wave velocity $\sigma_j^n$ can be achieved in the following manner. First compute the shock speed associated with discontinuity inside $C_j$,* via *Rankine-Hugoniot jump relations:*

$$s_j^n = \frac{f(u_{j,r}^n) - f(u_{j,l}^n)}{u_{j,r}^n - u_{j,l}^n}.$$

*Then, if $-s_j^n(S(u_{j,r}^n) - S(u_{j,l}^n)) + G(u_{j,r}^n) - G(u_{j,l}^n) \leq 0$, the shock is an admissible shock and thus one takes $\sigma_j^n = s_j^n$, else*

$$\sigma_j^n = \max_{u \in \lfloor u_{j,l}^n, u_{j,r}^n \rceil} f'(u).$$

Let us now just present some results. First, we compare the above approximate schemes with the previous ones based on exact resolution of the equation with the reconstructed condition. We then present some results obtained for another flux than Burgers' one.

For initial condition $1+\chi_{[0.1,0.6]}$ and time 0.2 with a Courant number 0.3, results with first and second reconstruction are reported on figures 15, 16, 17 and 18.

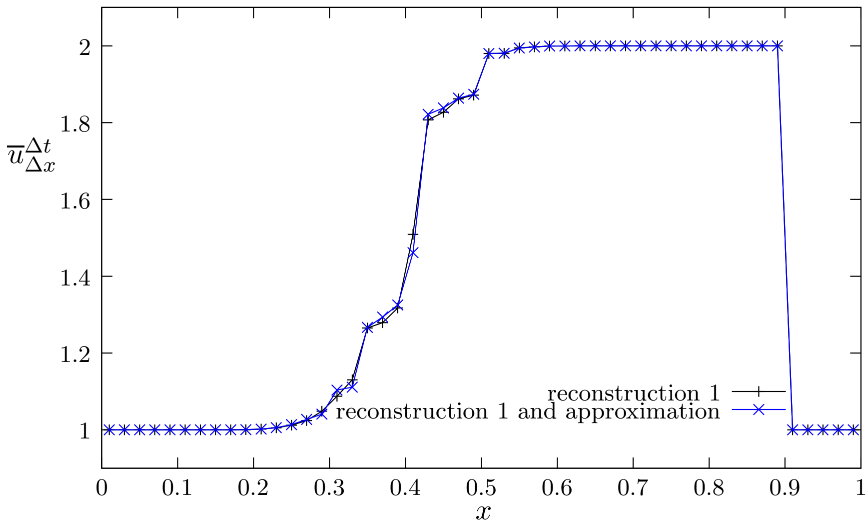


FIGURE 15. First reconstruction, 50 cells.

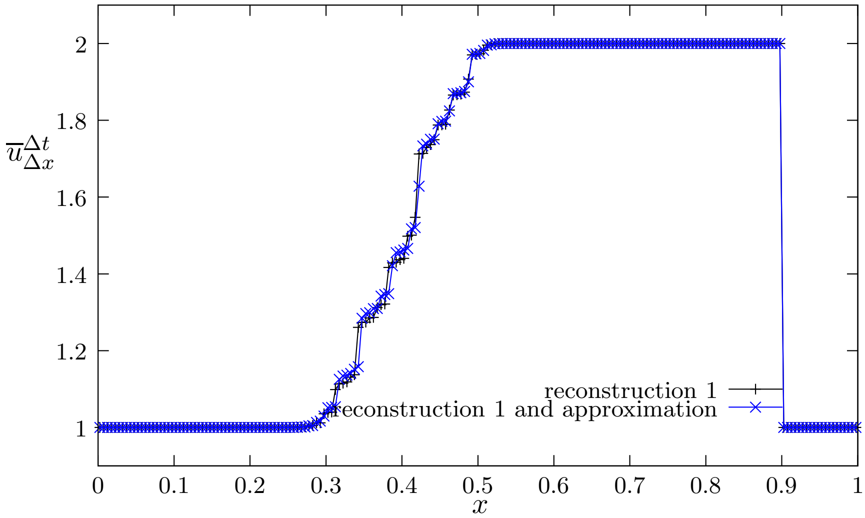


FIGURE 16. First reconstruction, 200 cells.

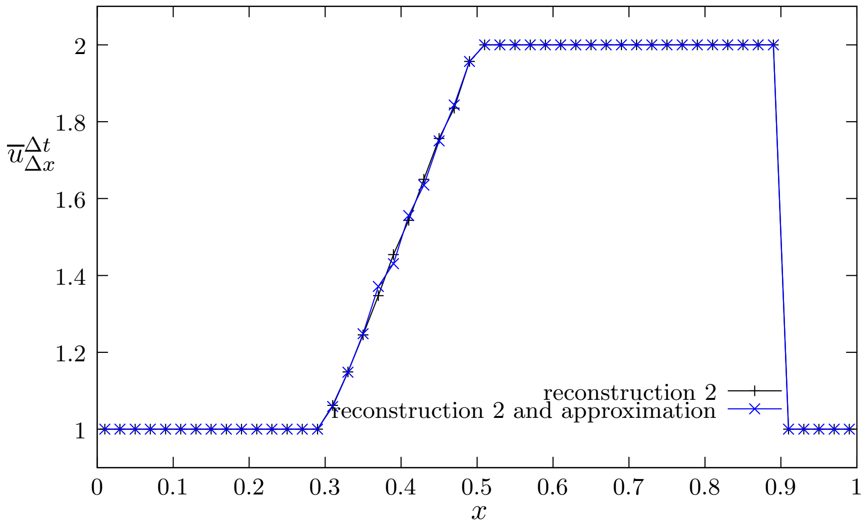


FIGURE 17. Second reconstruction, 50 cells.

The schemes with approximate resolution give results that are very close to the exact resolution scheme. It is quite surprising that the second part of the resolution step, which is a pure projection (one half-cells), does not bring numerical diffusion, or, to say it differently, that the reconstruction procedure manages to reduce that much the numerical diffusion due to the projection part of the algorithm.

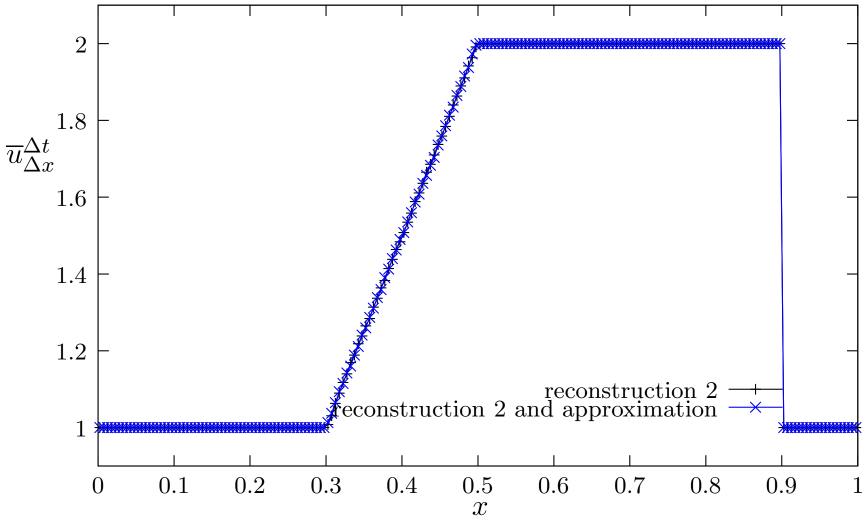


FIGURE 18. Second reconstruction, 200 cells.

Equipped with this simple algorithm, we are able to consider various scalar equations. We now will consider the following flux:

$$f(x) = \begin{cases} \dfrac{1}{5-4x} & \text{if } x \in [0,1], \\ \dfrac{4}{5-x} & \text{if } x \in (1,5), \end{cases}$$

as in [2]. This flux is not convex, thus the numerical decrease of one entropy does not guarantee convergence toward the entropy solution (this entropy criterion does not select a unique solution). The considered entropy is $S(u) = u^2/2$ so that the entropy flux is

$$G(u) = \begin{cases} \dfrac{\log(|5-4x|)}{4} + \dfrac{5}{4(5-4x)} + 4\log(4) + \dfrac{15}{4} & \text{if } x \in [0,1], \\ 4\log(|5-x|) + \dfrac{20}{5-x} & \text{if } x \in (1,5). \end{cases}$$

The initial condition is $2\chi_{[0.1,0.4]}$ and the final time is $5/8$, for a Courant number of 0.3. We compare results with a reference solution computed with the Godunov scheme and 10000 cells (thus close to the Krushkov solution). The schemes used are a self-adaptive entropy scheme of [2], and the first and the second reconstructions. We see this 3 different schemes provide numerical approximations converging toward different weak solutions of the PDE.

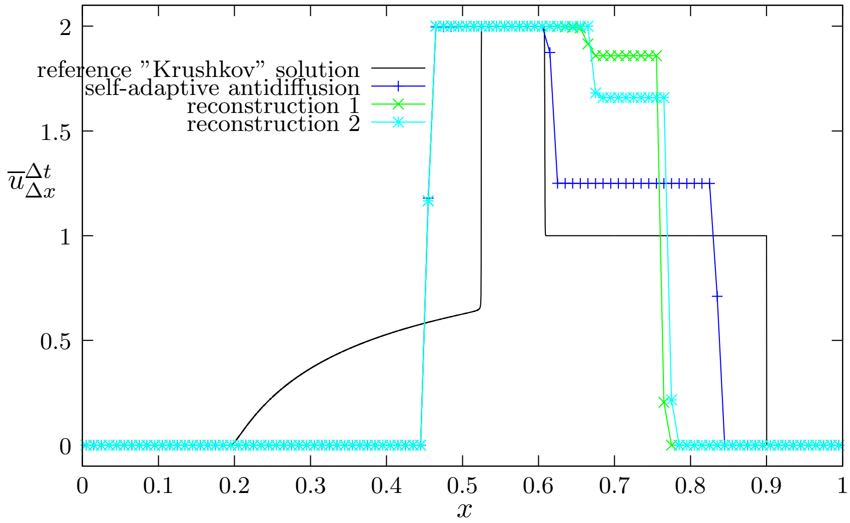


FIGURE 19. Different weak solutions with different algorithms.

## 4. Perspectives

In this paper, a new kind of reconstruction schemes is proposed: the class of discontinuous-in-cell reconstructions. At each time step, the reconstruction procedure is followed by the resolution of the considered equation (with the reconstructed function as initial condition). This resolution can be either exact or approximate. A stability and entropy analysis is done. Then, some numerical results are reported, for the advection equation, Burgers' equation, and, finally, for a non-convex scalar equation, with 2 different reconstructions.

The aim of discontinuous reconstruction schemes is not to obtain high order approximations of solutions, but actually to compute sharp numerical discontinuities. This is shown to be achieved in the numerical results. The use of such reconstruction schemes for linear and nonlinear systems is being studied, using ideas developed in [9] for the Euler system in dimension 1. Another interesting perspective would be to derive a multidimensional reconstruction to perform multidimensional computations. It is done in the case of linear transport on unstructured meshes in [11].

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